ESSENTIAL NORM AND ESSENTIAL NUMERICAL RADIUS INEQUALITIES FOR HILBERT SPACE OPERATORS

AKRAM MANSOORI, MOHSEN ERFANIAN OMIDVAR*

Abstract. We obtain some sharp inequalities for essential numerical radius \( w_e(\cdot) \) of operators. Moreover, we give some applications of our results in estimation of essential norm, and compare our results with some usual norm known results. In particular, we obtain under suitable conditions if \( T \in \mathcal{B}(\mathcal{H}) \), then

\[
\sup_{\|x\|=1} |\langle Tx, x \rangle| \leq \|T\| \leq \sup_{\|x\|=1} \|Tx\|.
\]

1. Introduction

Let \( \mathcal{B}(\mathcal{H}) \) denote the \( C^* \)-algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \) with inner product \( \langle \cdot, \cdot \rangle \). For \( T \in \mathcal{B}(\mathcal{H}) \)

\[
w(T) = \sup \{ |\langle Tx, x \rangle| : \|x\| = 1 \}, \]
\[
\|T\| = \sup \{ \|Tx\| : \|x\| = 1 \},
\]
\[
|T| = (T^*T)^{\frac{1}{2}},
\]

denote the numerical radius, the usual operator norm and the absolute value of \( T \), respectively. It is well known that, \( w(\cdot) \) defines a norm on \( \mathcal{B}(\mathcal{H}) \). This norm is equivalent to the operator norm. In fact, the following more precise result holds:

\[
\frac{1}{2} \|T\| \leq w(T) \leq \|T\|. \tag{1.1}
\]

for any \( T \in \mathcal{B}(\mathcal{H}) \). For basic properties and applications of the numerical radius, we refer to [3,6,7,8,10].

The inequalities in (1.1) have been improved considerably by in [9,11,14,15], it has been shown that if \( T \in \mathcal{B}(\mathcal{H}) \), then

\[
w(T) \leq \frac{1}{2} \|(|T| + |T^*|)\| \leq \frac{1}{2} (\|T\| + \|T^2\|^{\frac{1}{2}}) \leq \|T\|, \tag{1.2}
\]

\[
\frac{1}{4} \|T^*T + TT^*\| \leq w(T) \leq \frac{1}{2} \|T^*T + TT^*\|. \tag{1.3}
\]

2000 Mathematics Subject Classification. Primary 47A12, secondary 47A05, 47B15.

Key words and phrases. Essential norm, Essential numerical radius, Essential numerical range, Norm inequality.

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Communicated by S.S. Dragomir.
Moreover if $T$ is invertible, then
\[ \|T\| \leq \sqrt{2} w(T). \]
If $T = A + iB$ is the cartesian decomposition of $T$, then $A$ and $B$ are self-adjoint and $T^*T - TT^* = 2(A^2 + B^2)$. Thus, the inequality in (1.3) can be written as
\[ \frac{1}{2} \| A^2 + B^2 \| \leq w^2(T) \leq \| A^2 + B^2 \|. \]

Let $T \in \mathcal{B}(\mathcal{H})$, the essential norm of $T$ is defined by
\[ \| T \|_e = \| T + \mathcal{K}(\mathcal{H}) \| = \| \pi(T) \| = \inf\{ \| T + K \| ; K \in \mathcal{K}(\mathcal{H}) \} \tag{1.4} \]
where $\mathcal{K}(\mathcal{H})$ is the set of compact operators on complex Hilbert space $\mathcal{H}$.

The essential numerical range $W_e(T)$ of an operator $T$ on $\mathcal{H}$ is given by
\[ W_e(T) = \bigcap_{K \in \mathcal{K}(\mathcal{H})} W(T + K), \]
and note that the most important properties of the essential numerical range are that it is convex and its closure contains the spectrum of the operator (see [4]).

The following properties of $W_e(T)$ are also well known:
\begin{enumerate}
  \item[(I)] $W_e(T + K) = W_e(T)$ for all $K \in \mathcal{K}(\mathcal{H})$.
  \item[(II)] $W_e(T^*) = W_e(T)$.
  \item[(III)] If $a, b \in \mathbb{C}$, $W_e(aT + bI_H) = aW_e(T) + b$.
  \item[(IV)] If $U \in \mathcal{B}(\mathcal{H})$ is a unitary, then $W_e(UTU^*) = W_e(T)$.
\end{enumerate}

The essential numerical radius of $T$ is defined by
\[ w_e(T) = \sup\{ |\lambda| ; \lambda \in W_e(T) \} \]
hence $w_e(\cdot)$ defines a norm on the Calkin algebra.

The purpose of this paper is to establish some essential norm inequalities. We also define a new concept for essential numerical radius. Based on this, we obtain some related inequalities. Essential norm inequalities and a related essential numerical radius inequality for the sum of two operators are also given.

### 2. Essential norm

Recently, Kriete and Moorhouse in [13] showed the following definition of the essential norm. We state it for the sake of convenience.

**Definition 2.1.** Let $T \in \mathcal{B}(\mathcal{H})$, then
\[ \| T \|_e = \sup \left( \limsup_{n \to \infty} \| Tf_n \| \right) \tag{2.1} \]

where $E$ is the collection of all sequences $\{f_n\}$ of unit vector in $\mathcal{H}$ which tend to zero weakly.

**Remark.** Since $\mathcal{H}$ is reflexive, so for every $T \in \mathcal{B}(\mathcal{H})$, $\| T \|_e = \| T^* \|_e$ (see [2]).

To prove our generalized inequality, we need the following basic lemma.

**Lemma 2.1.** Let $T, S \in \mathcal{B}(\mathcal{H})$, then
\begin{enumerate}
  \item[(i)] $\| TS \|_e \leq \| T \|_e \| S \|_e$.
  \item[(ii)] If $0 \leq S \leq T$, then $\| S \|_e \leq \| T \|_e$.
  \item[(iii)] $\| T \|_e^2 = \| T^*T \|_e$.
\end{enumerate}
(iv) \[ \|T\|_e^3 = \|TT^*T\|_e. \]

(v) If \( T \) is a normal operator, then \[ \|T\|^2 = \|T^2\|_e. \]

Proof. (i) For \( K_1, K_2 \in \mathcal{K} (\mathcal{H}) \), we have
\[ \|TS\|_e \leq \|TS + TK_2 + K_1S + K_1K_2\| \quad \text{(by (1.4))} \]
\[ \leq \|T + K_1\| \|S + K_2\|. \]
Take infimum on \( K_1, K_2 \), we get
\[ \|TS\|_e \leq \|T\|_e \|S\|_e. \]

(ii) Let \( T - S \geq 0 \), then \( T - S = C^*C \) for some \( C \in \mathcal{B} (\mathcal{H}) \), therefore
\[ T - S + \mathcal{K} (\mathcal{H}) = C^*C + \mathcal{K} (\mathcal{H}) \]
\[ = (C^* + \mathcal{K} (\mathcal{H})) (C + \mathcal{K} (\mathcal{H})) \]
\[ = (C + \mathcal{K} (\mathcal{H}))^* (C + \mathcal{K} (\mathcal{H})) \geq 0. \]
Which gives the inequality
\[ \|S\|_e \leq \|T\|_e. \]

(iii) Let \( \{f_n\} \) be unit vector in \( \mathcal{H} \) which tend to zero weakly, then
\[ \|Tf_n\|^2 = \langle Tf_n, Tf_n \rangle = \langle T^*Tf_n, f_n \rangle \leq \|(T^*T) f_n\|. \]
By taking \( \sup_{\{f_n\} \in \mathcal{E}} \left( \lim_{n \to \infty} \sup \right) \) on both sides of the inequality we have
\[ \|T\|^2 \leq \|T^*T\|_e. \]
We can also state that
\[ \|T^*T\|_e \leq \|T\|_e^2 \quad \text{(by Lemma 2.1 (i) and Remark 2.1)} \]
The proof is complete. (iv) It is not hard to see that
\[ \|TT^*T\|_e \leq \|T\|^3 \quad \text{(by Lemma 2.1 (i) and Remark 2.1)} \]
On the other hand
\[ \|T\|^4 = \left( \|T\|^2 \right)^2 = \|T^*T\|^2 \]
\[ = \|TT^*T\|_e \quad \text{(by Lemma 2.1 (iii))} \]
\[ \leq \|T^*\|_e \|TT^*\|_e \quad \text{(by Lemma 2.1 (i))} \]
\[ = \|T\|_e \|TT^*T\|_e \quad \text{(by Remark 2.1)} \]
which is exactly the desired result.

(v) Let \( \{f_n\} \) be unit vector in \( \mathcal{H} \) which tend to zero weakly. Then by using normality of the operator \( T \)
\[ \|T^2 f_n\| = \|T (T f_n)\| = \|T^* (T f_n)\| \geq \langle T^*T f_n, f_n \rangle = \langle T f_n, T f_n \rangle = \|T f_n\|^2 \]
by taking \( \sup_{\{f_n\} \in \mathcal{E}} \left( \lim_{n \to \infty} \sup \right) \) on both sides of inequality we get
\[ \|T^2\|_e \geq \|T\|_e^2. \]
On the other hand by this Lemma (i)
\[ \|T^2\|_e \leq \|T\|_e^2 \]
\[ \|T^2\|_e \leq \|T\|_e^2 \]
The proof is complete. \( \square \)
Remark. When $T$ is non normal, the inequality in Lemma 2.1(v) is not true, this can be seen from the example $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

3. Essential numerical radius

We will often use the following well-known lemma later.

**Lemma 3.1.** (see [4, corollary 5.2]). Each of the following conditions is necessary and sufficient in order that

(i) $\lambda \in \mathcal{W}_e(T)$;
(ii) $\langle Tf_n, f_n \rangle \to \lambda$ for some sequence $\{f_n\}$ of unit vectors such that $f_n \to 0$ weakly;
(iii) $\langle Tf_n, f_n \rangle \to \lambda$ for some orthonormal sequence $\{f_n\}$.

To achieve our main results, we state for easy reference the following definition of essential numerical radius, that will be applied below.

**Definition 3.1.** Let $T \in \mathcal{B}(\mathcal{H})$ and $E$ is the collection of all sequences $\{f_n\}$ of unit vector in $\mathcal{H}$ which tend to zero weakly then the essential numerical radius of $T$ is defined by

$$w_e(T) = \sup_{\{f_n\} \in E} \left\{ \limsup_{n \to \infty} |\langle Tf_n, f_n \rangle| \right\}.$$

**Proposition 3.2.** Let $T \in \mathcal{B}(\mathcal{H})$, then

(i) $w_e(T^2) \leq w_e(T)$.
(ii) If $T$ is a self-adjoint operator on $\mathcal{H}$, then $\|T\|_e = w_e(T)$.

Proof. (i) Let $T \in \mathcal{B}(\mathcal{H})$ and $\{f_n\}, \{g_n\}$ be unit vector in $\mathcal{H}$ which tend to zero weakly we verify (with the help of parallelogram identity) the following inequality

$$|2 \langle (Tf_n, g_n) + \langle Tg_n, f_n \rangle \rangle | = | \langle T(f_n + g_n), f_n + g_n \rangle - \langle T(f_n - g_n), f_n - g_n \rangle |$$

$$\leq w_e(T) \left( \|f_n + g_n\|^2 + \|f_n - g_n\|^2 \right)$$

$$= 2w_e(T) \left( \|f_n\|^2 + \|g_n\|^2 \right)$$

$$= 4w_e(T).$$

Taking $Tf_n \neq 0$ and $g_n = \|Tf_n\|^{-1}Tf_n$ we conclude that

$$\|Tf_n\|^2 + \langle T^2f_n, f_n \rangle \leq 2w_e(T)\|Tf_n\|$$

Then

$$0 \leq 2w_e(T)\|Tf_n\| - \|Tf_n\|^2 - |\langle T^2f_n, f_n \rangle |$$

$$= - (w_e(T) - \|Tf_n\|^2 + w_e^2(T) - |\langle T^2f_n, f_n \rangle |$$

$$\leq w_e^2(T) - |\langle T^2f_n, f_n \rangle |$$

Hence

$$|\langle T^2f_n, f_n \rangle | \leq w_e(T)$$

by taking $\sup_{\{f_n\} \in E} \left( \limsup_{n \to \infty} \right)$ on both sides of inequality we have

$$w_e(T^2) \leq w_e^2(T).$$
(ii) Due to the fact that $|\langle Tf_n, f_n \rangle| \leq \|Tf_n\|\|f_n\|$ and $\|f_n\| = 1$, we deduce
\[ w_e(T) \leq \|T\|_e. \]

On the other hand, let $M = \sup_{\{f_n\} \in E} \left\{ \limsup_{n \to \infty} |\langle Tf_n, f_n \rangle| \right\}$, where $f_n, g_n \in \mathcal{H}$, then
\[
\langle T(f_n + g_n), f_n + g_n \rangle - \langle T(f_n - g_n), f_n - g_n \rangle \\
= 4 \Re \langle Tf_n, g_n \rangle.
\]
Therefore,
\[
\Re \langle Tf_n, g_n \rangle \leq \frac{M}{4} \left( |f_n + g_n|^2 + |f_n - g_n|^2 \right) \\
= \frac{M}{2} \left( \|f_n\|^2 + \|g_n\|^2 \right).
\]
Now, suppose $\|f_n\| = 1$ and $Tf_n \neq 0$, if we get $g_n = \frac{Tf_n}{\|Tf_n\|}$, then
\[
\Re \langle Tf_n, g_n \rangle = \Re \left( Tf_n, \frac{Tf_n}{\|Tf_n\|} \right) = \|Tf_n\|
\]
and by
\[
\Re \langle Tf_n, g_n \rangle \leq \frac{M}{2} (\|f_n\|^2 + \|Tf_n\|^2) = M.
\]
Consequently, $\|T\|_e \leq w_e(T)$. \hfill \qed

As a consequence of this result we get

**Corollary 3.3.** Let $T, S \in B(\mathcal{H})$, then

(i) $w_e(TS) \leq 4 w_e(T)w_e(S)$.

(ii) If $T, S \in B(\mathcal{H})$ and $TS = ST$. Then $w_e(TS) \leq 2 w_e(T)w_e(S)$.

**Proof.** (i) It is not hard to see that
\[
w_e(TS) \leq \|TS\|_e \leq \|T\|_e\|S\|_e \leq 4 w_e(T)w_e(S).
\]

(ii) First, we may assume that $w_e(T) = w_e(S) = 1$ and show that $w_e(TS) \leq 2$. Since the essential numerical radius is a norm, by triangle inequality and sub additivity of $w_e$ we have
\[
w_e(TS) = w_e \left( \frac{1}{4} \left[ (T + S)^2 - (T - S)^2 \right] \right) \\
\leq \frac{1}{4} \left[ w_e(T + S)^2 + w_e(T - S)^2 \right] \\
\leq \frac{1}{4} \left[ (w_e(T + S))^2 + (w_e(T - S))^2 \right] \quad \text{(by Proposition 3.1(i))} \\
\leq \frac{1}{4} \left[ (w_e(T) + w_e(S))^2 + (w_e(T) - w_e(S))^2 \right] \\
= 2. \hfill \qed
We will need the following observation. If $S$ is a subset of the complex plane let $\text{co}(S)$ denote the convex hull of $S$.

**Lemma 3.4.** If $T = A \oplus B$ on $\mathcal{H} \oplus \mathcal{H}$, then $W_e(T) = \text{co}(W_e(A) \cup W_e(B))$.

With the above lemma of essential numerical range in [1], we obtain the result of essential numerical radius.

**Theorem 3.5.** If $T = A \oplus B$ on $\mathcal{H} \oplus \mathcal{H}$, then $w_e(T) = \max \{w_e(A), w_e(B)\}$.

**Proof.** Since $W_e(A) \subseteq W_e(A \oplus B)$ and $W_e(B) \subseteq W_e(A \oplus B)$, we have $w_e(A) \leq w_e(A \oplus B)$ and $w_e(B) \leq w_e(A \oplus B)$, and therefore

\[
\max \{w_e(A), w_e(B)\} \leq w_e(A \oplus B).
\]

On the other hand

\[
w_e(A \oplus B) = \sup \{|\lambda| : \lambda \in W_e(A \oplus B)\}
\]

\[
= \sup \left\{ |t_1 x_1 + t_2 x_2| : t_1 + t_2 = 1, x_1 \in W_e(A), x_2 \in W_e(B) \right\}
\]

\[
\leq \sup \left\{ |t_1 x_1 + t_2 x_2| : t_1 + t_2 = 1, x_1 \in W_e(A), x_2 \in W_e(B) \right\}
\]

\[
\leq \max \{w_e(A), w_e(B)\}.
\]

\[\square\]

Utilizing the Cartesian decomposition for operators and same strategies used in [12], one can obtain the following inequality.

**Theorem 3.6.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

\[
\frac{1}{4} \|T^* T + TT^*\|_e \leq w_e^2(T) \leq \frac{1}{2} \|T^* T + TT^*\|_e.
\]

As a particular case of interest, we can state that:

**Corollary 3.7.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

\[
\frac{1}{2} \|T\|_e \leq w_e(T) \leq \|T\|_e.
\]

**Proof.** The desired result follows from the chain of the inequalities

\[
\frac{1}{4} \|T\|^2_e \leq \frac{1}{4} \|T^* T + TT^*\|_e \leq w_e^2(T) \leq \frac{1}{2} \|T^* T + TT^*\|_e \leq \|T\|^2_e.
\]

The first inequality in (3.1) follows from the fact that $0 \leq T^* T \leq T^* T + TT^*$ and lemma(2.1(ii)), and the last inequality in (3.1) follows by the triangle inequality essential norm in Calkin algebra and lemma(2.1(iii))

\[\square\]

**Corollary 3.8.** Let $T \in \mathcal{B}(\mathcal{H})$ with the Cartesian decomposition $T = A + iB$, then

\[
\frac{1}{2} \|A^2 + B^2\|_e \leq w_e^2(T) \leq \|A^2 + B^2\|_e.
\]

(3.2)
Remark

Finally in this section, we obtain the relation between essential norm and numerical radius. In general the inequality \( w(T) \leq \|T\|_e \) is not true. For example if \( T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), then \( \|T\|_e = 0 \). In the following we show that \( w(T) \leq \|T\|_e \) is true under suitable conditions.

We need the next lemma, which will be very crucial for our purpose. For the reader’s convenience, we state it again (see [5, lemma 2.1]).

Lemma 3.9. Let \( T \in B(\mathcal{H}) \), then there exists an orthonormal sequence \( \{f_n\} \) such that \( \|Tf_n\| \to \|T\|_e \). Furthermore, if \( P \) is an infinite rank projection and \( TP = T \), then we can choose \( f_n \) so that the additional condition \( Pf_n = f_n \) for all \( n \) is satisfied.

Theorem 3.10. Let \( T \in B(\mathcal{H}) \), if there exists an infinite rank projection \( P \) such that \( TP = T \), then \( w(T) \leq \|T\|_e \).

Proof. As shown in the proof of Lemma 4.4, there exists an orthonormal sequence \( \{f_n\} \) such that \( \|Tf_n\| \leq \|T\|_e \) for all \( n \), then for \( x = \sum_{n=1}^{\infty} \langle x, f_n \rangle f_n \) we have

\[
|\langle Tx, x \rangle| = \left| \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle x, f_n \rangle \langle x, f_m \rangle \langle Tf_n, f_m \rangle \right| \\
\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| |\langle x, f_m \rangle| |\langle Tf_n, f_m \rangle| \\
\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| |\langle x, f_m \rangle| |Tf_n| \\
\leq \|T\|_e \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle x, f_n \rangle| |\langle x, f_m \rangle| \\
\leq \|T\|_e \left( \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{m=1}^{\infty} |\langle x, f_m \rangle|^2 \right)^{\frac{1}{2}} \\
= \|T\|_e \|x\|^2. 
\]

(by the Cauchy-Schwarz inequality)

(by the Parseval equation)

Now the result follows by taking the supremum over all unit vectors in \( \mathcal{H} \).

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References


Akram Mansoori
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: aram777@yahoo.com

Mohsen Erfanian Omidvar
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran
E-mail address: erfanian@mshdiau.ac.ir