

## A NEW LOMAX DISTRIBUTION FOR MODELING SURVIVAL TIMES AND TAXES REVENUE DATA SETS

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**ABSTRACT.** In this work, a new compound extension of the Lomax distribution is introduced and studied. The new extension is derived based on a well known compound family called exponentiated generalized G Poisson family of distributions. Some of its properties are derived as well as a numerical analysis for the model variance, skewness and kurtosis is introduced. The new density is expressed as a linear combination of the Lomax densities. Two applications are provided along with some important plots to illustrate the importance and the flexibility of the new Lomax distribution. The method of maximum likelihood is used to estimate the unknown parameters as well as a Monte Carlo simulation study is conducted. The new model provided an adequate fit compared to other related models with smallest values for AIC, BIC, CAIC and HQIC. The new model is much better than many other useful well-known Lomax extensions.

### 1. INTRODUCTION

[21] pioneered the Lomax (Lx) or Pareto II (Pa II) or the shifted Pa II (SPaII) distribution for modeling failure time data in business. The Lx distribution has found wide application in various fields such as engineering, actuarial science, size of cities, medical and biological sciences, income and inequality of wealth, lifetime and reliability modeling. The cumulative distribution function (CDF) of the one parameter Lx model is given as

$$H_{\theta}(x) = 1 - (1 + x)^{-\theta}, \quad (1)$$

where  $\theta$  is the shape parameter. The Lx model in (1) is a special case from the well known Burr type XII (BXII) (see [9]). The relationship between the Burr distribution and the various other distributions, namely, the Lx, the Compound Weibull (CW), the Weibull-Exponential (WE), the logistic (Lc), the log logistic (LLc), the Weibull (W) and the Kappa family (Ka) of distributions is summarized in many articles (see [9], [10] and [11], [12] and [29]). The corresponding probability density function (PDF) of (1) is given by

$$h_{\theta}(x) = \theta (1 + x)^{-(1+\theta)}. \quad (2)$$

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A random variable (rv)  $X$  is said to have the BXII distribution if its CDF is given as

$$H_{a,\theta}(x) = a\theta x^{a-1} (1 + x^a)^{-(1+\theta)},$$

where both  $a$  and  $b$  are shape parameters. For  $a = 1$ , we have the Lx model in (2). Recently, [6] introduced and studied a new compound family of distributions called exponentiated generalized G Poisson (EGGP) family of distributions. The CDF and PDF of the EGGP family given by

$$F_{\alpha,\beta,\lambda,\underline{\phi}}(x) = \frac{1}{\kappa_{(\lambda)}} \left[ 1 - \exp \left( -\lambda \left\{ 1 - \left[ 1 - H_{\underline{\phi}}(x) \right]^\alpha \right\}^\beta \right) \right], \quad (3)$$

where  $\kappa_{(\lambda)} = 1 - \exp(-\lambda)$  and the two additional shape parameters are both greater than zero. The CDF in (3) is called the EGGP family of distributions. The corresponding PDF is

$$f_{\alpha,\beta,\lambda,\underline{\phi}}(x) = \alpha\beta\lambda \frac{h_{\underline{\phi}}(x) \left[ 1 - H_{\underline{\phi}}(x) \right]^{\alpha-1} \left\{ 1 - \left[ 1 - H_{\underline{\phi}}(x) \right]^\alpha \right\}^{\beta-1}}{\kappa_{(\lambda)} \exp \left( \lambda \left\{ 1 - \left[ 1 - H_{\underline{\phi}}(x) \right]^\alpha \right\}^\beta \right)}. \quad (4)$$

For  $\beta = 1$  we have EGP class of distribution and for  $\alpha = 1$  we have GGP class of distribution both of which are embedded in EGGP class. In this paper, we propose and study a new compound extension of Lx distribution using the EGGP family of distributions. Due to Aryal and Yousof (2017), the CDF of the exponentiated generalized Lomax Poisson (EGLxP) distribution can be derived as

$$F_{\alpha,\beta,\lambda,\theta}(x) = \frac{1}{\kappa_{(\lambda)}} \left( 1 - \exp \left\{ -\lambda \left[ 1 - (1+x)^{-\alpha\theta} \right]^\beta \right\} \right). \quad (5)$$

The corresponding PDF can be derived as

$$f_{\alpha,\beta,\lambda,\theta}(x) = \alpha\beta\lambda\theta \frac{(1+x)^{-\alpha\theta-1} \left[ 1 - (1+x)^{-\alpha\theta} \right]^{\beta-1}}{\kappa_{(\lambda)} \exp \left\{ -\lambda \left[ 1 - (1+x)^{-\alpha\theta} \right]^\beta \right\}}. \quad (6)$$

The hazard rate function (HRF) can be calculated by  $f_{\alpha,\beta,\lambda,\theta}(x) / [1 - F_{\alpha,\beta,\lambda,\theta}(x)]$ . For  $\beta = 1$ , we have the exponentiated Lomax Poisson (ELxP) distribution. For  $\alpha = 1$ , we have the generalized Lomax Poisson (GLxP) distribution. The EGLxP density may be unimodal and right-skewed (see Figure 1) whereas the EGLxP HRF can be upside down then upside down or increasing or decreasing (see Figure 2).

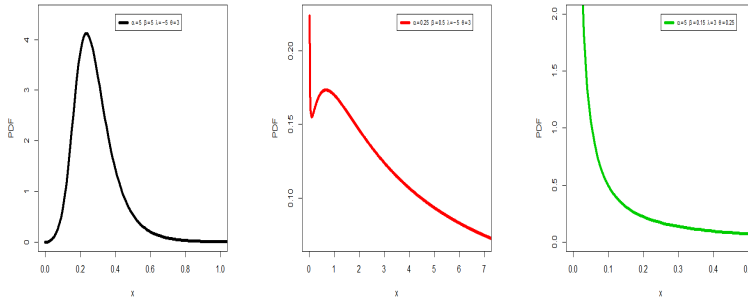


Figure 1: Plots of the new PDF for selected parameter values.

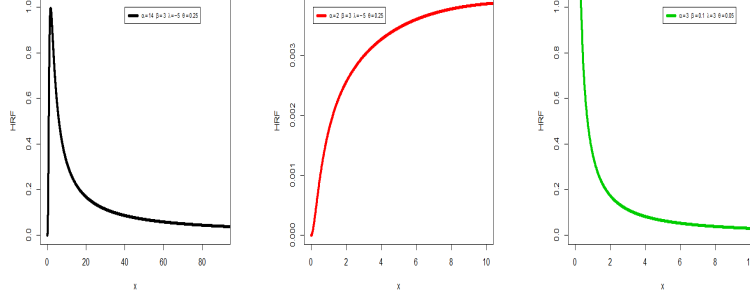


Figure 2: Plots of the new HRF for selected parameter values.

Using the power series expansion of  $\exp(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!}$  and the series expansion

$$(1 - \xi)^{a-1} = \sum_{w=0}^{\infty} \frac{(-1)^w \Gamma(a)}{w! \Gamma(a-w)} \xi^w,$$

the PDF in (6) can be expressed as

$$f_{\alpha, \beta, \lambda, \theta}(x) = \sum_{r=0}^{\infty} \zeta_r g_{\theta(1+r)}(x), \quad (7)$$

where

$$\zeta_r = \alpha \beta \sum_{k=0}^{\infty} \frac{(-1)^r \Gamma(k+1)}{r! \Gamma(k+1-r)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}}{i! \lambda^{-i-1} \kappa_{(\lambda)}(1+r)} \binom{\beta(1+i)-1}{j} \binom{\alpha(1+j)-1}{k}$$

and

$$g_{\theta(1+r)}(x) = a [\theta(1+r)] (1+x)^{-[\theta(1+r)]-1},$$

is the BXII density with parameter  $\theta(1+r)$ . By integrating (6), we obtain the mixture representation of  $f_{\alpha, \beta, \lambda, \theta}(x)$  as

$$F_{\alpha, \beta, \lambda, \theta}(x) = \sum_{r=0}^{\infty} \zeta_r G_{\theta(1+r)}(x), \quad (8)$$

where

$$G_{\theta(1+r)}(x) = 1 - (1+x)^{-\theta(1+r)}$$

is the CDF of the BXII model with parameter  $\theta(1+r)$ . Equation (7) reveals that the EGLxP density function is a linear combination of BXII densities. Thus, some structural properties of the new model such as the ordinary, incomplete moments and generating function can be immediately obtained from well-established properties of the Lx distribution. The properties of Lx distribution have been studied by many authors in recent years, see [8], [17], [23], [3], [24], [30], [20], [4], [18], [16], [31] and [19], among others.

## 2. MATHEMATICAL PROPERTIES

**2.1. Ordinary and incomplete moments.** The  $n$ th ordinary moment of  $X$  is given by

$$\mu'_n = \mathbf{E}(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx.$$

Using (7), we obtain

$$\mu'_n = \theta \sum_{r=0}^{\infty} \zeta_k(1+r) B((1+r)\theta - n, 1+n) |_{[n < (\theta+r\theta)]}. \quad (9)$$

Setting  $r = 1$  in (9), we have the mean of  $X$

$$\mathbf{E}(X) = \mu'_1 = \theta \sum_{r=0}^{\infty} \zeta_k(1+r) B((1+r)\theta - 1, 2) |_{[1 < (\theta+r\theta)]}.$$

The expression in (9) can be computed numerically  $\forall n < (\theta + r\theta)$ . The variance ( $V(X)$ ) skewness ( $S(X)$ ) and kurtosis ( $K(X)$ ) can be calculated from the ordinary moments using the well-known relationships (see Table 1).

The  $s$ th incomplete moments, say  $\mathbf{I}_s(t)$ , is given by

$$\mathbf{I}_s(t) = \int_{-\infty}^t x^s f(x) dx.$$

Using Equation (7), we obtain

$$\mathbf{I}_s(t) = \theta \sum_{r=0}^{\infty} \zeta_k(1+r) B(t; (1+r)\theta - s, 1+s) |_{[s < (\theta+r\theta)]}, \quad (10)$$

where

$$B(\vartheta_1, \vartheta_2) = \int_0^{\infty} z^{\vartheta_1-1} (1+z)^{-(\vartheta_1+\vartheta_2)} dz$$

and

$$B(\xi; \vartheta_1, \vartheta_2) = \int_0^{\xi} z^{\vartheta_1-1} (1+z)^{-(\vartheta_1+\vartheta_2)} dz$$

are the beta and the incomplete beta functions of the second type, respectively. The first incomplete moment of the EGLxP model,  $\mathbf{I}_1(t)$ , can be obtained by setting  $s = 1$  in (10). Another application of the first incomplete moment which is related to mean residual life and mean waiting time given by

$$m_1(t) = \frac{1 - \mathbf{I}_1(t)}{1 - F_{\alpha, \beta, \lambda, \theta}(t)} - t \text{ and } M_1(t) = t - \frac{\mathbf{I}_1(t)}{F_{\alpha, \beta, \lambda, \theta}(t)},$$

respectively. The amount of scatteredness in a population is evidently measured to some extent by the totality of deviations from the mean and median. The mean deviations about the mean is

$$[\delta_{\mu}(X) = \mathbf{E}(|X - \mu'_1|) = \int_0^{\infty} (|X - \mu'_1|) f_{\alpha, \beta, \lambda, \theta}(x) dx = 2\mu'_1 F(\mu'_1) - 2\mathbf{I}_1(\mu'_1)]$$

and about the median is

$$\begin{aligned}\delta_\mu(X) &= \mathbf{E} \left( \left| X - Q \left( \frac{1}{2} \right) \right| \right) \\ &= \int_0^\infty \left( \left| X - Q \left( \frac{1}{2} \right) \right| \right) f_{\alpha, \beta, \lambda, \theta}(x) dx \\ &= \mu'_1 - 2\mathbf{I}_1 \left( Q \left( \frac{1}{2} \right) \right),\end{aligned}$$

where  $\mu'_1 = \mathbf{E}(X)$  comes from (9),  $F(\mu'_1)$  is simply calculated,  $\mathbf{I}_1(\mu'_1)$  is the first incomplete moments and  $Q(\frac{1}{2})$  is the median of  $X$ ,  $Median(X) = Q(\frac{1}{2})$  is the median,  $F(\mu'_1)$  is easily calculated from (5) and  $\mathbf{I}_1(t)$  is the first incomplete moment given by (10) with  $s = 1$ . The main application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, demography, insurance, reliability and medicine. The answers to many important questions in economics require more than just knowing the mean of the distribution, but its shape as well. This is obvious not only in the study of econometrics but in other areas as well.

The  $n$ th central moment of  $X$ , say  $M_n$ , follows as

$$\mu_n = \mathbf{E}(X - \mu)^n = \sum_{h=0}^n (-1)^h (\mu'_1)^n \mu'_{n-h} \binom{n}{h}.$$

The cumulants  $(\kappa_n)$  of  $X$  follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1} \kappa_r \mu'_{n-r} \binom{n-1}{r-1},$$

where

$$\begin{aligned}\kappa_1 &= \mu'_1, \\ \kappa_2 &= \mu'_2 - (\mu'_1)^2,\end{aligned}$$

and

$$\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + (\mu'_1)^3.$$

### 3. NUMERICAL ANALYSIS OF THE $E(X)$ , $V(X)$ , $S(X)$ AND $K(X)$

Numerical analysis of the  $E(X)$ ,  $V(X)$ ,  $S(X)$  and  $K(X)$  are calculated in Table

1. Based on Table 1 we note that:

- 1-The mean of the EGLxP distribution increases as  $\beta$  increases
- 2-The mean of the EGLxP distribution decreases as  $\alpha$ ,  $\lambda$  and  $\theta$  increases.
- 3- The skewness of the EGLxP distribution is can positive and negative.
- 4- The kurtosis of the EGLxP distribution can be more than 3 and less than 3.

Table 1:  $E(X)$ ,  $V(X)$ ,  $S(X)$  and  $K(X)$  of the EGLxP distribution.

$\alpha$	$\beta$	$\lambda$	$\theta$	$E(X)$	$\text{Var}(X)$	$\text{Ske}(X)$	$\text{Ku}(X)$
5	1.25	-5	0.95	0.7283911	0.34054330	3.813461	63.8957
10				0.3003911	0.03737399	1.948094	11.40112
15				0.1887069	0.01298946	1.619420	8.403845
20				0.1374980	0.00648934	1.479558	7.376173
50				0.0522785	0.00084427	1.258053	6.013938
100				0.0257100	0.00019736	1.307164	4.947075
250				0.0101832	$3.03421 \times e^{-5}$	1.832651	-1.739131
450				0.0063489	$8.15736 \times e^{-6}$	8.486452	-10.24382
20	0.25	-3	1.5	0.02797689	0.001173043	2.389037	11.9058
	0.5			0.04315835	0.001611938	1.858937	8.680563
	1			0.06194953	0.002016842	1.518852	7.091069
	5			0.1149673	0.00267980	1.172759	5.892333
	10			0.1402637	0.002877628	1.122265	5.755609
	50			0.2024511	0.003270011	1.08043	5.650053
	100			0.2304725	0.003433577	1.075108	5.637130
	500			0.2982161	0.003830452	1.070836	5.626837
	1000			0.3285513	0.004012657	1.0703	5.625555
	2000			0.3596000	0.004202981	1.070033	5.624902
	5000			0.4017641	0.004468082	1.070149	5.619422
50	3	-100	0.5	0.2871793	0.004657163	1.393612	6.826510
		-75		0.2723970	0.004556759	1.391273	6.817779
		-50		0.2518173	0.004422224	1.386519	6.800162
		-25		0.2172592	0.004215819	1.371593	6.745989
		-5		0.1392291	0.003944029	1.253189	6.329854
		5		0.0342396	0.0005141585	2.792818	21.41780
		25		0.0152548	$4.640241 \times e^{-5}$	-3.279599	28.07781
		40		0.01253917	$2.887997 \times e^{-5}$	-2.395407	7.631431
		50		0.01145958	$2.338039 \times e^{-5}$	2.066845	-23.20995
		100		0.00874408	$1.262219 \times e^{-5}$	17.61696	-101.7821
5	10	5	0.5	0.2053520	0.007693594	1.79618	39.37493
			1	0.0971905	0.00152511	1.254992	10.31887
			3	0.0312584	0.0001459914	2.096629	-4.176897
			5	0.0186228	$5.103575 \times e^{-5}$	-4.252254	51.4869
			8	0.0115931	$1.961035 \times e^{-5}$	8.721583	-66.49622
			10	0.0092623	$1.248213 \times e^{-5}$	16.10603	-97.92557

3.1. **The moment generating function (MGF).** The MGF of  $X$ , say  $M_X(t) = \mathbf{E}(e^{tX})$ , is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{k,r=0}^{\infty} \frac{t^r}{r!} \zeta_k (1+r)\theta B((1+r)\theta - n, 1+n) |_{[n < (\theta+r\theta)]}.$$

**3.2. Probability weighted moments (PWMs).** The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The  $(s, r)$ th PWM of  $X$  following the EGLxP model, say  $\rho_{s,r}$ , is formally defined by

$$\rho_{s,r} = \mathbf{E} (X^s F(X)^r) = \int_{-\infty}^{\infty} x^s F(x)^r f(x) dx.$$

Using equations (5) and (6), we can write

$$f(x) F(x)^r = \sum_{h=0}^{\infty} v_h h_{\theta(1+h)}(x)$$

where

$$\begin{aligned} v_h &= \alpha\beta \sum_{k=0}^{\infty} \frac{(-1)^h \Gamma(k+1)}{h! \Gamma(k+1-h)} \sum_{w,i,j=0}^{\infty} \frac{(-1)^{w+i+j+k} (1+w)^i}{i! \lambda^{-i-1} \kappa_{(\lambda)}^{1+r}} \\ &\quad \times \binom{r}{w} \binom{\beta(1+i)-1}{j} \binom{\alpha(1+j)-1}{k}. \end{aligned}$$

Then, the  $(s, r)$ th PWM of  $X$  can be expressed as

$$\rho_{s,r} = \theta \sum_{h=0}^{\infty} v_h (1+h) \theta B((1+h)\theta - s, 1+s) |_{[s < (\theta+h\theta)]}.$$

**3.3. Entropies.** The Rényi entropy of a random variable  $X$  represents a measure of variation of the uncertainty. This entropy is defined by

$$I_{\delta}(X) = (1-\delta)^{-1} \log \int_{-\infty}^{\infty} f(x)^{\delta} dx |_{(\delta > 0 \text{ and } \delta \neq 1)}.$$

Using the power series expansion, the PDF in (5) can be expressed as

$$f_{\alpha,\beta,\lambda,\theta}(x)^{\delta} = \sum_{k=0}^{\infty} \Upsilon_k (\theta)^{\delta} (1+x)^{-\delta(1+\theta)} [1 - (1+x)^{-\theta}]^k,$$

where

$$\begin{aligned} \Upsilon_k &= \left( \frac{\alpha\beta}{\kappa_{(\lambda)}} \right)^{\delta} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}}{i! \lambda^{-\delta-i} \delta^{-i}} \\ &\quad \times \binom{\beta(i+\delta)-\delta}{j} \binom{\alpha(j+\delta)-\delta}{k} |_{(\delta > 0 \text{ and } \delta \neq 1)}. \end{aligned}$$

Therefore, the Rényi entropy of the EGLxP model is given by

$$I_{\delta}(X) = \frac{\log \left[ \sum_{k=0}^{\infty} \Upsilon_k \mathbf{I}_{(0,\infty)}^{(\delta)} \right]}{1-\delta},$$

where

$$\mathbf{I}_{(0,\infty)}^{(\delta)} = \int_0^{\infty} \theta^{\delta} (1+x)^{-\delta(1+\theta)} [1 - (1+x)^{-\theta}]^k dx.$$

The  $q$ -entropy, say  $H_q(X)$ , can be obtained as

$$H_q(X) = (q-1)^{-1} \log \left\{ 1 - \left[ \sum_{k=0}^{\infty} \Upsilon_k^{\star} \mathbf{I}_{(0,\infty)}^{(q)} \right] \right\},$$

where

$$\begin{aligned} \Upsilon_k^{\star} &= \left( \frac{\alpha\beta}{\kappa(\lambda)} \right)^q \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}}{i! \lambda^{-q-i} q^{-i}} \\ &\quad \times \binom{\beta(i+q)-q}{j} \binom{\alpha(j+q)-q}{k} \Big|_{(q>0 \text{ and } q \neq 1)}, \end{aligned}$$

and

$$\mathbf{I}_{(0,\infty)}^{(q)} = \int_0^{\infty} (\theta)^q (1+x)^{-q(1+\theta)} [1 - (1+x)^{-\theta}]^k dx$$

The Shannon entropy of a random variable  $X$ , say  $SH(X)$ , is defined by

$$SH(X) = \mathbf{E} \{ -[\log f_{\alpha,\beta,\lambda,\theta}(x)] \}.$$

It is the special case of the Rényi entropy when  $\delta \uparrow 1$ .

**3.4. Reversed residual life and mean inactivity time.** The  $n$ th moment of the reversed residual life, say

$$M_n(t) = \mathbf{E} [(t-X)^n] \Big|_{(X \leq t, t>0 \text{ and } n=1,2,\dots)}$$

uniquely determines  $F(x)$ . We obtain

$$M_n(t) = \frac{\int_0^t (t-x)^n dF_{\alpha,\beta,\lambda,\theta}(x)}{F_{\alpha,\beta,\lambda,\theta}(t)}.$$

Then, the  $n$ th moment of the reversed residual life of  $X$  becomes

$$M_n(t) = \frac{\theta}{F(t)} \sum_{h=0}^{\infty} \zeta_h^{\star} (1+h) B(t; (1+h)\theta - n, 1+n) \Big|_{[n < (\theta+h\theta)]}, \quad (11)$$

where

$$\zeta_h^{\star} = \zeta_h \sum_{r=0}^n (-1)^r \binom{n}{r} t^{n-r}.$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by

$$M_1(t) = \mathbf{E}(t-X) \Big|_{(X \leq t, t>0 \text{ and } n=1)},$$

and it represents the waiting time elapsed since the failure of an item on condition had occurred in  $(0, t)$ . The MIT of the EGLxP distribution can be obtained easily by setting  $n = 1$  in (11).

#### 4. STRESS-STRENGTH RELIABILITY MODEL

The stress-strength model is the most widely used approach for reliability estimation. This model is used in many applications of physics and engineering, such as strength failure and system collapse. In stress-strength modeling,

$$R(X_1, X_2 | X_2 < X_1) = \Pr(X_2 < X_1) = \int_0^{\infty} f(x_1) F(x_2) dx$$

is a measure of reliability of a system when it is subjected to random stress  $X_2$  and has strength  $X_1$ . The system fails if and only if the applied stress is greater



than its strength and the component will function satisfactorily whenever  $X_1 > X_2$ .  $R(X_1, X_2|_{X_2 < X_1})$  can be considered as a measure of system performance and naturally arises in electrical and electronic systems. Other interpretation can be that, the reliability, say  $R(X_1, X_2|_{X_2 < X_1})$ , is the probability that the system is strong enough to overcome the stress imposed on it. Let  $X_1$  and  $X_2$  be two independent rvs with EGLxP  $(x_1; \alpha_1, \beta_1, \lambda_1, \theta)$  and EGLxP  $(x_2; \alpha_2, \beta_2, \lambda_2, \theta)$  distributions. The reliability  $R(X_1, X_2|_{X_2 < X_1})$  is given by

$$R(X_1, X_2|_{X_2 < X_1}) = \int_0^\infty f_{\alpha_1, \beta_1, \lambda_1, \theta}^{(1)}(x) F_{\alpha_2, \beta_2, \lambda_2, \theta}^{(2)}(x) dx.$$

Then

$$R(X_1, X_2|_{X_2 < X_1}) = \sum_{k, w=0}^{\infty} \Phi_{k, w},$$

where

$$\begin{aligned} \Phi_{k, w} &= \alpha_1 \alpha_2 \beta_1 \beta_2^{-1} \kappa_{(\lambda_1)}^{-1} \kappa_{(\lambda_2)}^{-1} (-1)^{k+w} \sum_{i, j, m, h=0}^{\infty} \lambda_1^{i+1} \lambda_2^{m+1} \frac{(-1)^{i+j+m+h}}{i!m!} \\ &\quad \times \binom{(1+i)\beta_1-1}{j} \binom{(1+j)\alpha_1-1}{k} \binom{(m+1)\beta_2-1}{h} \binom{(1+h)\alpha_2-1}{w}, \end{aligned}$$

and  $\kappa_{(\lambda_i)}^{-1} = 1/[1 - \exp(-\lambda_i)] \quad \forall \quad i = 1, 2$ .

## 5. ORDER STATISTICS

Let  $X_1, \dots, X_n$  be a random sample (rs) from the EGLxP distribution and let  $X_{(1)}, \dots, X_{(n)}$  be the corresponding order statistics. The PDF of the  $i$ th order statistic, say  $X_{i:n}$ , can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \quad (12)$$

where  $B(\cdot, \cdot)$  is the beta function. Substituting (5) and (12) in equation (12), we get

$$f(x) F(x)^{j+i-1} = \sum_{p=0}^{\infty} \vartheta_p h_{\theta(1+p)}(x),$$

where

$$\begin{aligned} \vartheta_p &= \alpha \beta \sum_{k=0}^{\infty} \frac{(-1)^p \Gamma(k+1)}{p! \Gamma(k+1-p)} \sum_{w, m, h=0}^{\infty} \frac{(-1)^{w+m+h+k} (1+w)^m}{m! \lambda_{(\lambda)}^{-m-1} \kappa_{(\lambda)}^{j+i}} \\ &\quad \times \binom{(m+1)\beta-1}{h} \binom{(1+h)\alpha-1}{k} \binom{j+i-1}{w}. \end{aligned}$$

Moreover, the PDF of  $X_{i:n}$  can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \sum_{p=0}^{\infty} \vartheta_p h_{\theta(1+p)}(x),$$

therefore

$$\mathbf{E}(X_{i:n}^\tau) = \theta \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \sum_{p=0}^{\infty} \vartheta_p (1+p) B((1+p)\theta - \tau, 1+\tau) |_{[\tau < (\theta+p\theta)]}. \quad (13)$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (13), we can derive explicit expressions for the L-moments of  $X$ . They are linear functions of expected order statistics defined by

$$\lambda_{(r)} = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} \mathbf{E}(X_{r-d:r}), \quad r \geq 1.$$

The first four L-moments are given by:

$$\begin{aligned} \lambda_{(1)} &= \mathbf{E}(X_{1:1}), \quad \lambda_{(2)} = \frac{1}{2} \mathbf{E}(X_{2:2} - X_{1:2}), \\ \lambda_{(3)} &= \frac{1}{3} \mathbf{E}(X_{3:3} - 2X_{2:3} + X_{1:3}), \end{aligned}$$

and

$$\lambda_{(4)} = \frac{1}{4} \mathbf{E}(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}).$$

## 6. ESTIMATION

Let  $X_1, \dots, X_n$  be a random sample from the EGLxP distribution with parameters  $\lambda, \alpha, \beta$  and  $\theta$ . Let  $\underline{\Psi} = (\alpha, \beta, \lambda, \theta)^\top$  be a  $4 \times 1$  parameter vector. For determining the MLE of  $\underline{\Psi}$ , we have the log-likelihood function

$$\begin{aligned} \ell(\underline{\Psi}) &= n \log \alpha + n \log \beta + n \log \lambda + n \log \theta \\ &\quad - n \log [1 - \exp(-\lambda)] - (\alpha\theta + 1) \sum_{i=1}^n (1 + x_i) \\ &\quad + (\beta - 1) \sum_{i=1}^n \left[ 1 - (1 + x_i)^{-\alpha\theta} \right] - \lambda \sum_{i=1}^n \left[ 1 - (1 + x_i)^{-\alpha\theta} \right]^\beta. \end{aligned}$$

The components of the score vector

$$\begin{aligned} \mathbf{U}_{\underline{\Psi}}|_{[\underline{\Psi}=(\alpha, \beta, \lambda, \theta)^\top]} &= \frac{\partial}{\partial \underline{\Psi}} \ell(\underline{\Psi}) \\ &= \left( \mathbf{U}_\alpha = \frac{\partial \ell(\underline{\Psi})}{\partial \alpha}, \mathbf{U}_\beta = \frac{\partial \ell(\underline{\Psi})}{\partial \beta}, \mathbf{U}_\lambda = \frac{\partial \ell(\underline{\Psi})}{\partial \lambda}, \mathbf{U}_\theta = \frac{\partial \ell(\underline{\Psi})}{\partial \theta} \right)^\top, \end{aligned}$$

are easily to be derived. Setting the nonlinear system of equations  $\mathbf{U}_\alpha = \mathbf{U}_\beta = \mathbf{U}_\lambda = \mathbf{U}_\theta = 0$  and solving them simultaneously yields the MLE  $\hat{\underline{\Psi}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})^\top$ . To solve these equations, it is usually and more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize  $\ell$ . For interval estimation of the parameters, we obtain the  $4 \times 4$  observed information matrix  $\mathbf{J}(\underline{\Psi}) = \left\{ \frac{\partial^2 \ell}{\partial r \partial s} \right\}$  (for  $r, s = \alpha, \beta, \lambda, \theta$ ), whose elements can be computed numerically. Under standard regularity conditions when  $n \rightarrow \infty$ , the distribution of  $(\hat{\underline{\Psi}})$  can be approximated by a multivariate normal  $N_5(0, \mathbf{J}(\hat{\underline{\Psi}})^{-1})$  distribution

to construct approximate confidence intervals for the parameters. Here,  $\mathbf{J}(\hat{\underline{\Psi}})$  is the total observed information matrix evaluated at  $\mathbf{J}(\hat{\underline{\Psi}})$ . The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. The elements of  $\mathbf{J}(\underline{\Psi})$  are easily to be derived.

## 7. SIMULATION RESULTS

To assess the performance of the maximum likelihood estimation, the EGLxP model is simulated via taking  $n = 50, 100, 250, 500$  and  $1,000$ . For each sample size, the ML method is used to evaluate the parameters of the new distribution using the `optim` function of the R software. Then, we repeat this process  $N = 1000$  times and compute the averages of the estimates (AEs) and mean squared errors (MSEs). Table 2 gives all simulation results. The values in Table 2 indicate that the MSEs of estimators  $\hat{\alpha}$ ,  $\hat{\beta}$ ,  $\hat{\lambda}$  and  $\hat{\theta}$  decay toward zero when  $n$  increases for all settings of  $\alpha$ ,  $\beta$ ,  $\lambda$ , and  $\theta$  as expected. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 2 shows the AEs and MSEs based on  $N = 1000$  simulations where the true parameter values are **I** :  $\alpha = 5$ ,  $\beta = 2$ ,  $\lambda = 3$  and  $\theta = 2$ . **II** :  $\alpha = 3$ ,  $\beta = 4$ ,  $\lambda = -10$  and  $\theta = 4$ .

Table 2: AEs and MSE based on  $N = 1000$  simulations.

n	$\Psi$	AEs	MSEs
<b>I</b>			
50	5	5.84403323	0.69436011
	2	2.28163710	0.81909395
	3	3.56557123	0.49168712
	2	2.40355912	0.84609418
100	5	5.59411403	0.39804399
	2	2.23288039	0.52353585
	3	3.50863271	0.39988276
	2	2.35311438	0.51143248
250	5	5.21055377	0.12107023
	2	2.10986940	0.24312424
	3	3.40628724	0.18692885
	2	2.11621181	0.39823036
500	5	5.04934358	0.08617319
	2	2.04143102	0.10541143
	3	3.05185511	0.11233384
	2	2.06063187	0.18355114
1000	5	5.00433691	0.00063563
	2	2.00661126	0.00112410
	3	3.00822481	0.00210283
	2	2.00123235	0.00511023

Table 3: Bias and MSE based on  $N = 1000$  simulations.

n	$\Psi$	AEs	MSEs
<b>II</b>			
50	3	3.60554113	2.40659868
	4	4.57040399	0.99864321
	-10	-10.81995431	1.41688652
	4	4.71435066	0.63666524
100	3	3.51659323	1.67786037
	4	4.44318761	0.40081991
	-10	-10.61773233	0.81642610
	4	4.55435511	0.32710983
250	3	3.31655985	1.10306561
	4	4.14354240	0.21959917
	-10	-10.11808771	0.43987633
	4	4.18605676	0.17710983
500	3	3.10434442	0.38118071
	4	4.05040869	0.03633921
	-10	-10.01033678	0.10053533
	4	4.02113651	0.0077198
1000	3	3.00110565	0.01009398
	4	4.00102304	0.00062334
	-10	-10.00231168	0.00398648
	4	4.00058318	0.00008542

## 8. MODELING REAL DATA

Two real data applications are provided to illustrate the importance, potentiality and flexibility of the EGLxP model. According to these data, we compare the EGLxP distribution with BXII, Marshall-Olkin BXII (MOBXII) ([5]), Topp Leone BXII (TLBXII) ([31]), Zografos-Balakrishnan BXII (ZBBXII) ([4]), Beta BXII ([27]), Beta exponentiated BXII (BEBXII) ([22]), Kumaraswamy BXII (KwBXII) ([26]), BXIIBXII ([13]), Burr-Hatke BXII (BHBXII) ([30]), Burr-Hatke exponentiated BXII (BHEBXII) ([28]), Five parameters Beta BXII (FBBXII) (Paranaiba et al. (2011)), Five parameters (FKwBXII) distribution ([26]), Weibull Generalized BXII (WGBXII) ([2]), Weibull Generalized Lx (WGLx) ([14]) and Five parameters Weibull generalized-BXII (FWGBXII) ([15]) distributions.

Data set **I** (survival times in days of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by [7]): {0.1, 0.33, 1.08, 1.08, 1.08, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 2.54, 2.78, 2.93, 3.27, 3.42, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.09, 1.12, 1.13, 1.15, 1.36, 1.39, 1.44, 1.83, 1.95, 1.96, 1.97, 2.02, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 2.13, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55}. Data set **II** (called taxes revenue data or the actual taxes

revenue data (in 1000 million Egyptian pounds)): {5.9, 20.4, 13.3, 8.5, 21.6, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 18.5, 5.1, 6.7, 17, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.9, 7.1, 10.8}. This data set was used by [25] and [31].

The total time test plot (TTT) (see [1]) for the two real data sets is presented in Figure 3.

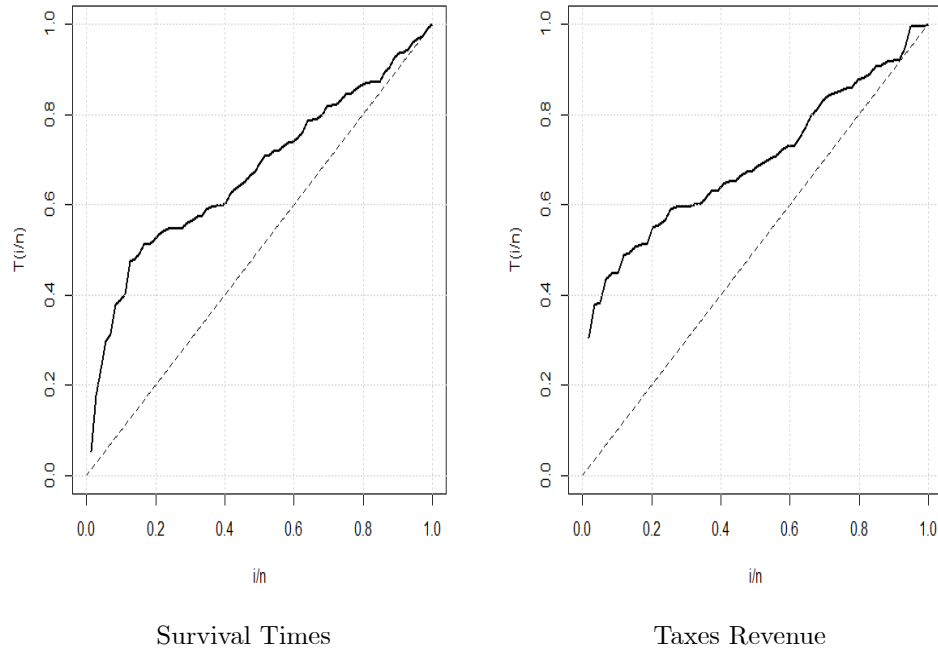


Figure 3: TTT plots for the survival times data set.

These plots indicates that the empirical HRF for the two data sets is increasing. We consider the goodness-of-fit statistics of the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and consistent Akaike information criterion (CAIC), where

$$\text{AIC} = 2 \left[ m - \ell \left( \hat{\Psi} \right) \right],$$

$$\text{BIC} = 2 \left[ \frac{1}{2} m \log(n) - \ell \left( \hat{\Psi} \right) \right],$$

$$\text{HQIC} = 2 \left\{ m \log[\log(n)] - \ell \left( \hat{\Psi} \right) \right\},$$

and

$$\text{CAIC} = 2 \left[ \frac{mn}{n-m-1} - \ell \left( \hat{\Psi} \right) \right],$$

where  $m$  is the number of parameters,  $n$  is the sample size and  $-2\ell\left(\hat{\underline{\Psi}}\right)$  is the maximized log-likelihood. Generally, the smaller these statistics indicate better fit. Based on the values in Tables 5 and 7 and Figures 4-8, the EGLxP model provides the best fit compared to other extensions of the Lx models with smallest values for BIC, AIC, CAIC and HQIC.

Table 4: MLEs, standard errors, confidence interval for survival times data set.

Model	Estimates
$BXII(\alpha, \beta)$	3.102, 0.465 (0.538), (0.077) (2.05, 4.16), (0.31, 0.62)
$MOBXII(\alpha, \beta, \gamma)$	2.259, 1.533, 6.760 (0.864), (0.907), (4.587) (0.57, 3.95), (0, 3.31), (0, 15.75)
$BHBXII(\theta, a, b)$	9.22, 1.96, 0.01 (6.82), (1.5), (0.01) (0, 22.8), (0, 4.96), (0, 0.03)
$WGLx(\alpha, \theta, b)$	2.027, 0.752, 0.82, 0.18, 3.88, 4.25 (1.7, 2.4), (0, 8.4), (0, 9.4)
$BHEBXII(\theta, \alpha, a, b)$	7.95, 7.945, 0.085, 72.75 (4.51), (0.00), (0.00), (34.4) (0, 16.9), -, -, (3.95, 141.55)
$TLBXII(\alpha, \beta, \gamma)$	2.393, 0.458, 1.796 (0.907), (0.244), (0.915) (0.62, 4.17), (0, 0.94), (0.002, 3.59)
$KwBXII(\lambda, \theta, \alpha, \beta)$	14.105, 7.424, 0.525, 2.274 (10.805), (11.850), (0.279), (0.990) (0, 35.28), (0.30.65), (0, 1.07), (0.33, 4.21)
$BBXII(\lambda, \theta, \alpha, \beta)$	2.555, 6.058, 1.800, 0.294, (1.859), (10.391), (0.955), (0.466) (0, 6.28), (0, 26.42), (0, 3.67), (0, 1.21)
$WGBXII(\gamma, \theta, a, b)$	12.91, 1.798, 2.61, 0.052 (19.4), (1.05), (1.1), (0.08) (0, 51.7), (0, 3.8), (0.4, 4.8), (0, 21)
$BEBXII(\lambda, \theta, \alpha, \beta, \gamma)$	1.876, 2.991, 1.780, 1.341, 0.572 (0.094), (1.731), (0.702), (0.816), (0.325) (1.7, 2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)
$BXIIBXII(\alpha, \beta, a, b, c)$	99.99, 84.82, 0.016, 1.023, 257.75 ( $9.89 \times 10^1$ ), ( $4.089 \times 10^2$ ), ( $1.17 \times 10^{-3}$ ), ( $3.55 \times 10^{-2}$ ), (0.00) (46.2, 153.8), (24.2, 145.4), (5, 0.135), (0, 1.9), -
$FWGBXII(\beta, \theta, a, b, c)$	6.98, 0.07, 0.22, 20.196, 12.26 (0.00), (0.00), (0.00), (0.00), (0.00),
$EGBXIIP(\alpha, \beta, \lambda, a, b)$	1.45, 3.07, -4.265, 1.166, 2.047 (0.000), (0.000), (1.6), (0.000), (0.000), -, -, (-7.5, -1.1), -, -
$EGLxP(\alpha, \beta, \lambda, \theta)$	16.72, 3.36, -5.44, 0.21 (0.00), (1.6), (1.88), (0.00) -, (0.16, 6.56), (-9.2, -1.68), -



Table 5: AIC, BIC, CAIC and HQIC values for the survival times data.

Model	AIC, BIC, CAIC, HQIC
BHBXII	1103.2, 1110, 1103.5, 1105.8
BHEBXII	872.7, 881.85, 873.34, 876.37
BXII	209.60, 214.15, 209.77, 211.40
MOBXII	209.74, 216.56, 210.09, 212.44
TLBXII	211.80, 218.63, 212.15, 214.52
KwBXII	208.76, 217.86, 209.36, 212.38
BBXII	210.44, 219.54, 211.03, 214.06
BEBXII	212.10, 223.50, 213.00, 216.60
BXIIBXII	228.12, 239.50, 229.03, 232.65
WGBXII	213.88, 222.98, 214.5, 217.5
FWGBXII	215.76, 216.7, 227.14, 220.29
EGBXIIP	207.23, 18.61, 208.14, 211.76
WGL <sub>x</sub>	208.80, 215.70, 209.21, 211.6
<b>EGL<sub>x</sub>P</b>	<b>205.41, 214.52, 206.01, 209.04</b>

Table 6: MLEs, standard errors, confidence interval for taxes revenue data set.

Model	Estimates
$BXII(\alpha, \beta)$	5.615, 0.072 (15.048), (0.194) (0, 35.11), (0, 0.45)
$MOBXII(\alpha, \beta, \gamma)$	8.017, 0.419, 70.359 (22.083), (0.312), (63.831) (0, 51.29), (0, 1.03), (0, 195.47)
$TLBXII(\alpha, \beta, \gamma)$	91.320, 0.012, 141.073 (15.071), (0.002), (70.028) (61.78,120.86), (0.008, 0.02), (3.82,278.33)
$WGLx(\alpha, \theta, b)$	3.97, 1.927, 0.1299 (0.386), (0.00), (0.00) (3.2, 4.7), -, -
$BHBXII(\theta, a, b)$	33.93, 5.3, 0.0022 (4.197 $\times e^1$ ), (6.4), 2.6 $\times e^{-4}$ (11.1, 56.7), (0, 18.1), (0, 09)
$KwBXII(\lambda, \theta, \alpha, \beta)$	18.130, 6.857, 10.694, 0.081 (3.689), (1.035), (1.166), (0.012) (10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10)
$BBXII(\lambda, \theta, \alpha, \beta)$	26.725, 9.756, 27.364, 0.020 (9.465), (2.781), (12.351), (0.007) (8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03)
$BEBXII(\lambda, \theta, \alpha, \beta, \gamma)$	2.924, 2.911, 3.270, 12.486, 0.371 (0.564), (0.549), (1.251), (6.938), (0.788) (1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)
$FBBXII(\lambda, \theta, \alpha, \beta, \gamma)$	30.441, 0.584, 1.089, 5.166, 7.862 (91.745), (1.064), (1.021), (8.268), (15.036) (0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)
$FKwBXII(\lambda, \theta, \alpha, \beta, \gamma)$	12.878, 1.225, 1.665, 1.411, 3.732 (3.442), (0.131), (0.034), (0.088), (1.172) (6.13,19.62), (0.97,1.48), (1.56,1.73), (1.24,1.58), (1.43,6.03)
<b>EGLxP</b> ( $\alpha, \beta, \lambda, \theta$ )	7.86, 30.16, -15.05, 0.332 (0.00), (89.61), (33.002), (0.00) , (0, 209.4), (-81.05, 50, 95), -

Table 7: AIC, BIC, CAIC and HQIC values for the taxes revenue data set.

Model	AIC, BIC, CAIC, HQIC
BXII	518.46, 522.62, 518.67, 520.08
BHBXII	520.51, 526.74, 520.94, 522.94
WGLx	395.08, 401.32, 395.52, 397.52
MOBXII	387.22, 389.38, 387.66, 389.68
TLBXII	385.94, 392.18, 386.38, 388.40
KwBXII	385.58, 393.90, 386.32, 388.86
BBXII	385.56, 394.10, 386.30, 389.10
BE BXII	387.04, 397.42, 388.17, 391.09
FB BXII	386.74, 397.14, 387.87, 390.84
FKwBXII	386.96, 397.36, 388.09, 391.06
<b>EGLxP</b>	<b>385.1, 393.41, 385.84, 388.34</b>

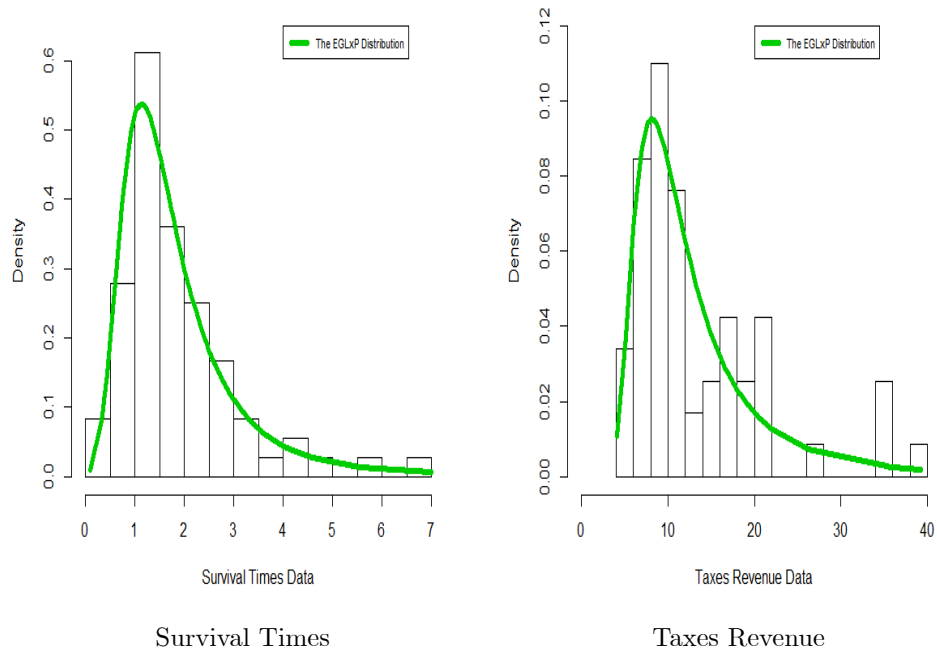


Figure 4: Estimated PDFs.

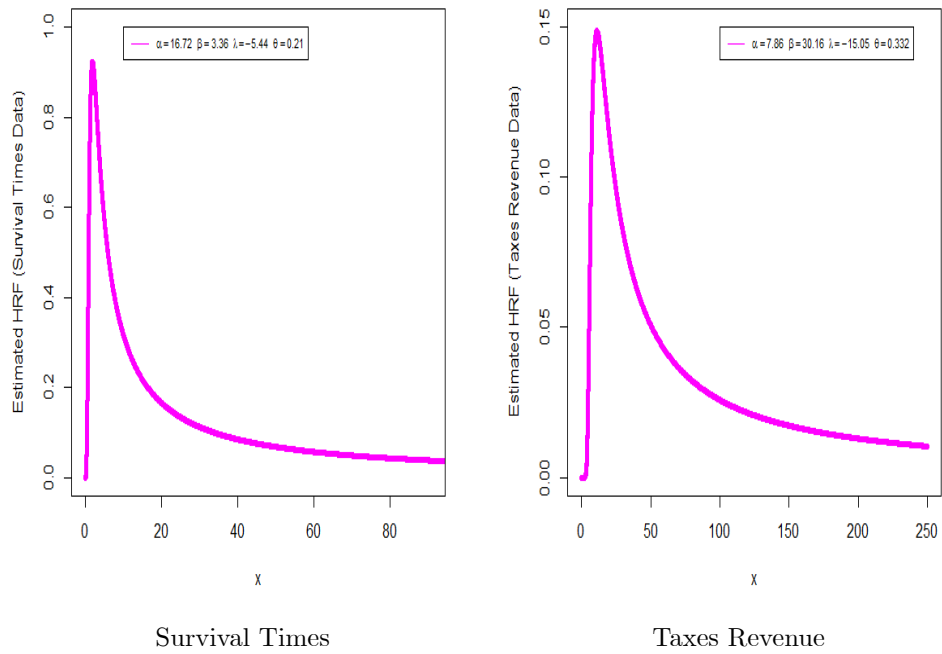
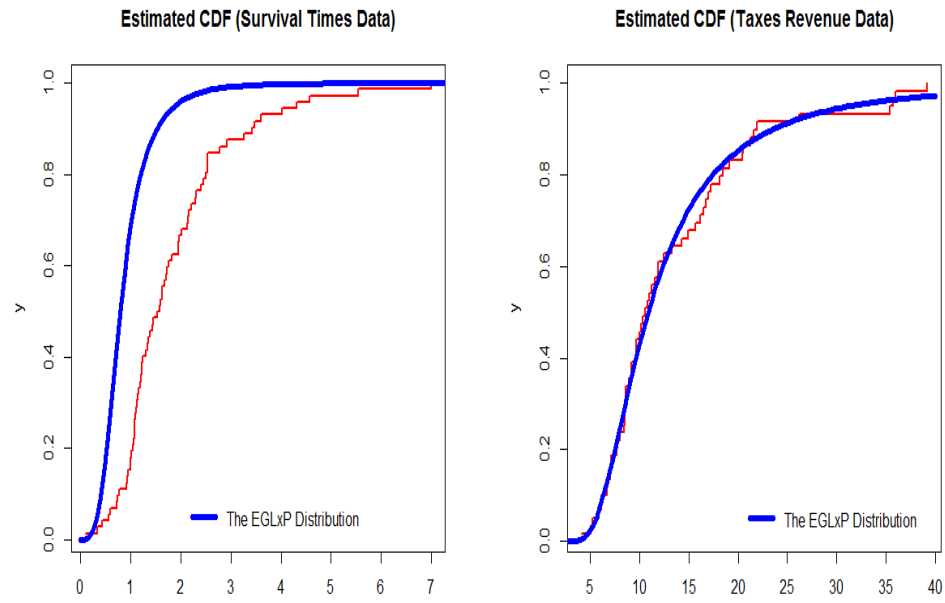


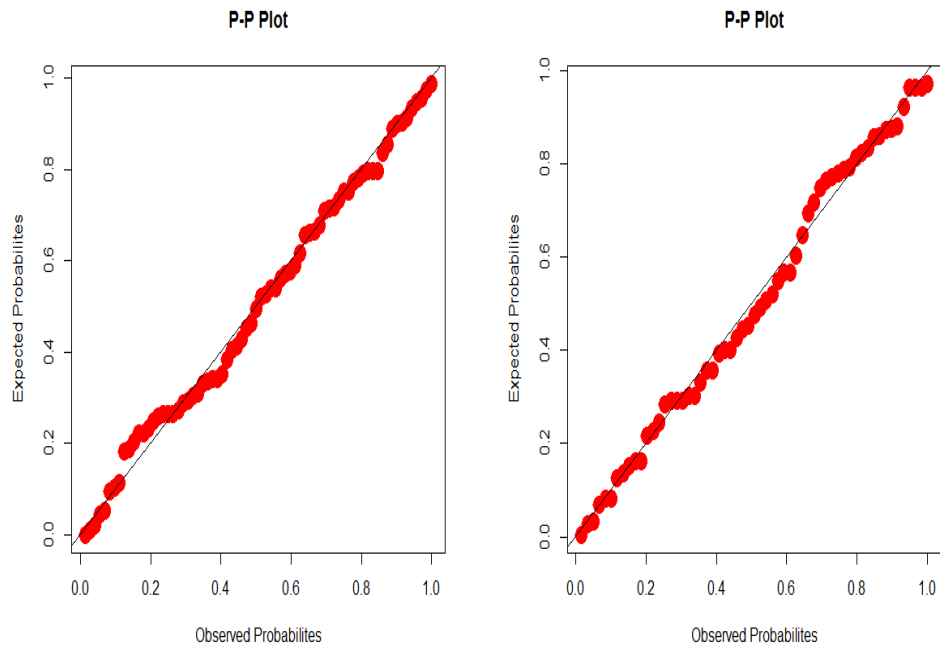
Figure 5: Estimated HRFs.



Survival Times

Taxes Revenue

Figure 6: Estimated CDFs.



Survival Times

Taxes Revenue

Figure 7: P-P plots.

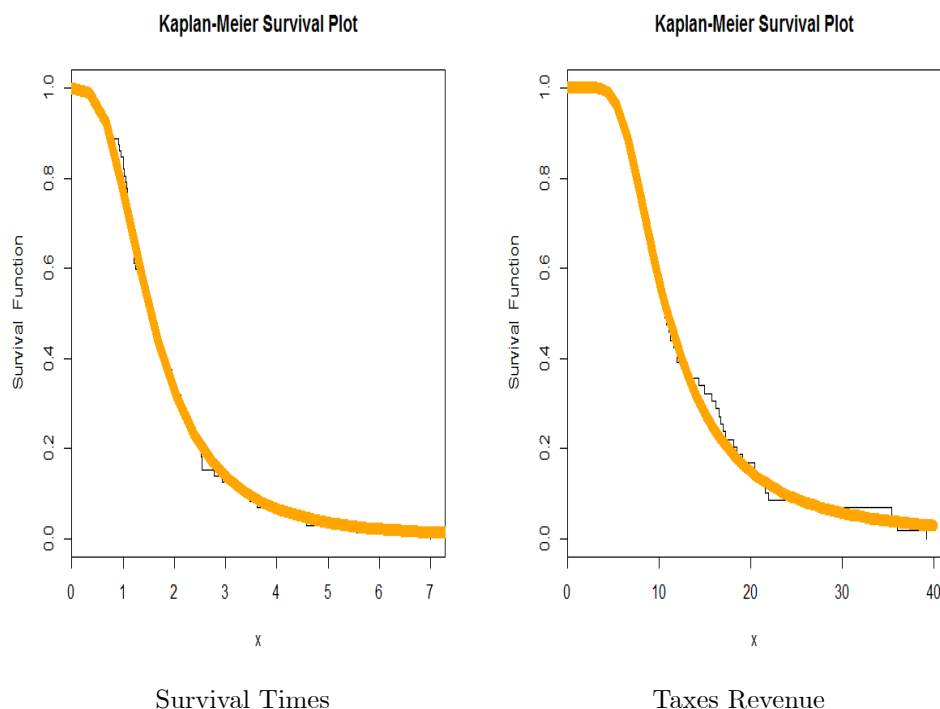


Figure 8: Kaplan-Meier survival plots.

## 9. CONCLUSIONS

In this work, a new compound extension of the Lomax distribution is introduced. Some of its properties are derived. The skewness of the new extension can be right-skewed, left-skewed, unimodal and symmetric. The kurtosis of the new distribution can be more than 3 or less than 3. The hazard rate function of the new distribution can be upside down, increasing or decreasing. The new density is expressed as a linear combination of the Lomax densities. The method of maximum likelihood is used to estimate the unknown parameters as well as a Monte Carlo simulation study is conducted. Two applications is provided along with some important plots to illustrate the importance and the flexibility of the new Lomax distribution. The new model relatively provided an adequate fit compared to other related models with the smallest values for AIC, BIC, CAIC and HQIC. The new model is much better than many other useful well-known Lomax extensions.

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