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A NEW EXTENSION OF THE LOMAX DISTRIBUTION AND ITS APPLICATIONS

MONA MUSTAFA EL BIELY[1] AND HAITHAM M. YOUSOF[2]

ABSTRACT. In this work, we shall introduce a new extension of the Lomax distribution with a simple physical motivation. Some of its mathematical and statistical properties such as ordinary moment, moment generating function, incomplete moment, weighted moments, order statistics and their moments and moment of the reversed residual life are derived. The method of maximum likelihood is used to estimate the unknown parameters. Three applications are provided with its related plots to illustrate the importance of the new Lomax model. The new Lomax model is better than other nine competitive models.

1. INTRODUCTION AND PHYSICAL MOTIVATION

The Lomax (Lx) or Pareto II (Pa II) (or the shifted Pa II) model was originally pioneered for modeling failure data in business by [9]. This model has found wide application in fields such as the size of cities, income and inequality of wealth, the actuarial science, engineering, medical and biological sciences, lifetime and reliability modeling. The cumulative distribution function (CDF) of the one parameter Lomax (Lx) or Pareto type II (PaII) model is given as

$$H_{[b]}(x)|_{(b>0)}^{(x>0)} = 1 - (1 + x)^{-b},$$

where b is a shape parameter. Scale parameter can easily be added for getting other extensions of the Lx models as follows

$$H_{(b,\lambda)}(x)|_{(b>0,\lambda>0)}^{(x>0)} = 1 - \left(1 + \frac{x}{\lambda}\right)^{-b},$$

and

$$H_{(b,\lambda)}(x)|_{(b>0,\lambda>0)}^{(x>0)} = 1 - (1 + \lambda x)^{-b}.$$

The characterization of the Lx model is studied in a number of several studies. The Lx distribution suggested as an alternative model wuth heavy tailed to the exponential, Weibull and gamma (E, W and Ga) distributions. It is known as a special case of Pearson type VI (Per VI) distribution and considered as a mixture of E and Ga models. In the lifetime field, the Lx model belongs to the decreasing failure rates family. Further, the above the one parameter Lx model is a special case from the well known Burr type XII (BXII) model, so several properties of the

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Lx model can easily be obtained from the BXII model (for more details about the BXII model and its relations with other models like the Lx model see [5], [6], [8], [7], [11] and [12]. The corresponding probability density function (PDF) of $H_{[b]}(x)$ is given by

$$h_{[b]}(x)|_{(b>0)}^{(x>0)} = b\,(1+x)^{-b-1}.$$

Let $h_{[b]}(x)$ and $H_{[b]}(x)$ denote the PDF and the CDF of the Lx model. Then the CDF of the Weibull Generalized Lx (WGLx) due to [15] is defined by

$$F_{[\alpha,\theta,b]}\,(x)\,|_{(\alpha>0,\theta>0,b>0)}^{(x>0)} = 1 - \mathrm{e}^{-\left[-1+(1+x)^{\theta b}\right]^{\alpha}}, \tag{1}$$

and its corresponding PDF in (1) is given by

$$f_{[\alpha,\theta,b]}\,(x)\,|_{(\alpha>0,\theta>0,b>0)}^{(x>0)} = \alpha\,\theta b\,(1+x)^{b\theta-1}\left[-1+(1+x)^{\theta b}\right]^{-1+\alpha}\,\mathrm{e}^{-\left[-1+(1+x)^{\theta b}\right]^{\alpha}}, \tag{2}$$

where both α and θ are shape parameters. The effect of the new two parameters (α and θ) on the skewness and kurtosis of the WGLx distribution is shown in Fig.1 (b) and table 1.The additional parameters α and θ are sought as a manner to furnish a more flexible Lx distribution (see Fig.1). The hazard rate function (HRF) can be written as

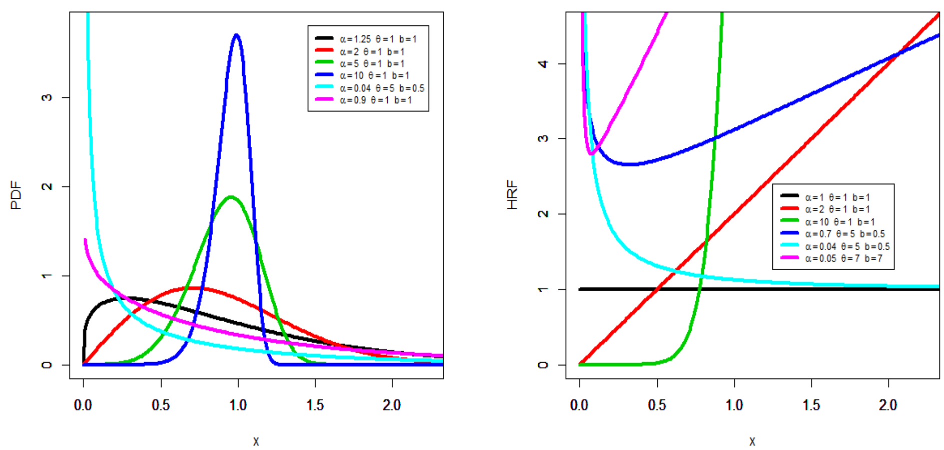
$$\tau_{[\alpha,\theta,b]}\,(x)\,|_{(\alpha>0,\theta>0,b>0)}^{(x>0)} = \alpha\,\theta b\frac{\left[-1+(1+x)^{\theta b}\right]^{-1+\alpha}}{(1+x)^{-(b\theta-1)}}$$

In this work, we study the WGLx model and give a sufficient description of its mathematical properties. The new model is motivated by its flexibility in applications (see section. 4), by means of three applications, it is noted that the WGLx model provides better fits than nine models.

Table 1: Sub-models of the WGLx model.

α	θ	b	Reduced model	CDF	Author
2	θ	b	Rayleigh GLx	$1-\mathrm{e}^{-\left[-1+(1+x)^{\theta b}\right]^2}$	[2]
1	θ	b	Exponential GLx	$1-\mathrm{e}^{1-(1+x)^{\theta b}}$	[2]
α	1	b	WLx (WPaII)	$1-\mathrm{e}^{-\left[(1+x)^{b}-1\right]^{\alpha}}$	[13]
2	θ	b	Rayleigh Lx (Rayleigh PaII)	$1-\mathrm{e}^{-\left[(1+x)^{b}-1\right]^{2}}$	[13]
1	1	b	Exponential Lx (EPaII)	$1-\mathrm{e}^{1-(1+x)^{b}}$	[2]
α	θ	1	Quasi W type I	$1-\mathrm{e}^{-\left[(1+x)^{\theta}-1\right]^{\alpha}}$	New
2	θ	1	Quasi Rayleigh type I	$1-\mathrm{e}^{-\left[(1+x)^{\theta}-1\right]^{2}}$	New
1	θ	1	Quasi W type II	$1-\mathrm{e}^{1-(1+x)^{\theta}}$	New
α	1	1	W	$1-\mathrm{e}^{-x^{\alpha}}$	[14]

Figure (Fig.) 1 gives the plots of PDF and HRF for the WGLx model. From Fig. 1 we conclude that the the PDF of the WGLx model can be unimodal or symmetric and right skewed, the HRF can be constant, bathtub, decreasing and increasing shaped.



(a) (b)

Fig.1: Plots of PDF and HRF for the WGLx model.

Due to [15], the PDF of the WGLx model can be simplified as

$$f\,(x) = \sum_{r=0}^{\infty} \zeta_r h_{(b+rb)}(x), \tag{3}$$

where $h_{(b+rb)}(x)$ is the Lx PDF with parameter $(b+rb)$, $\zeta_r = -v_r$ and

$$v_r \;=\; \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+r+j+k}}{i!r!} \binom{i\alpha}{j} \binom{\theta\,(j-i\alpha)}{k} \binom{1+k}{r}, \tag{4}$$

similarly, the CDF presented in (1) of X can be expressed as

$$F(x) = \sum_{r=0}^{\infty} \zeta_r H_{(b+rb)}(x),$$

where $H_{(b+rb)}(x)$ is the Lx CDF with parameter $(b+rb)$. A physical interpretation of the WGLx distribution can be shown as follows:

2. Mathematical properties

2.1. **Moment.** The nth ordinary moment of X is given by

$$\mathbf{E}(X^n) = \mu_n' = \int_{-\infty}^{\infty} f\,(x)\; x^n dx.$$

Then, we obtain

$$\mu_n'|_{(n<(b+rb))} = \sum_{r=0}^{\infty} \zeta_r\,(b+rb)\; B\,(1+n, (b+rb)-n)\,, \tag{5}$$

where

$$B(a,c) = \int_0^{\infty} (1+t)^{-(a+c)} t^{a-1} dt$$

is the beta function of the second type. By setting $n = 1$ in (5), we get the mean of X. The skewness (Ske) and kurtosis (Kur) measures can be calculated from the ordinary moments using well-known relationships (see Table 2).The moment generating function $M_X(t) = \mathbf{E}\left(e^{t\,X}\right)$ of X can be derived from (3) as

$$M_X(t)\,|_{(n<(b+rb))} = \sum_{r,n=0}^{\infty} \frac{t^n}{n!}\zeta_r\,(b+rb)\;B\left(1+n,(b+rb)-n\right),$$

The nth incomplete moment $(\boldsymbol{\phi}_n(t))$ of X can be gotten from (3) as

$$\boldsymbol{\phi}_n(t)\,|_{(n<(b+rb))} = \int_{-\infty}^{t} x^n f(x)\,dx = \sum_{r=0}^{\infty}\zeta_r\,(b+rb)\;B\left(t^a;1+n,(b+rb)-n\right), \tag{6}$$

where

$$B(q;a,c) = \int_0^q (1+t)^{-(a+c)} t^{a-1}\,dt$$

is the incomplete beta function of the second type, when $n = 1$ we obtain the 1^{st} incomplete moment which is necessary to get the mean deviations and the Bonferroni (Bon) and Lorenz (Loz) curves which are useful in demography and econometrics. The skewness of the WGLx distribution can range in the interval $(-4.5, 11)$, whereas the kurtosis of the WGLx distribution varies in the interval $(-46.5, 98.7)$ also the mean of X decreases as b increases, the Ske is always positive (see Table 2).

Table 2: Mean, variance, Ske and Kur of the WGLx distribution.

α	θ	b	Mean	Variance	Ske	Kur
1	1	1	1	1	2	9
		2	0.37893610	0.09853526	1.253915	4.772774
		3	0.2323270	0.03305351	1.057228	4.002385
		6	0.1072515	0.006321044	0.8797764	3.420981
		10	0.06239426	0.002051677	0.8134962	3.230566
		25	0.01203411	7.266611e^{-5}	0.4640694	6.717835
		50	0.01203411	$7.266611\times\text{e}^{-5}$	0.4640694	6.717835
		100	0.00599017	$1.381345\times\text{e}^{-5}$	8.915002	-17.92345
		110	0.00544339	$2.750172\times\text{e}^{-5}$	0.663236	1.565649
1.5	2.5	1	0.2786300	0.02521419	0.5906033	3.013179
		2	0.12863710	0.004808392	0.4475122	2.748751
		3	0.08356701	0.00195659	0.4013848	2.677017
		6	0.0407299	0.000448291	0.3559833	2.740309
		10	0.02419184	0.000155894	0.2631833	-1.126099
		15	0.01778916	0.0001127403	2.322934	-2.675555
		17	0.01567427	$8.735643\times\text{e}^{-5}$	1.733498	2.613976
		20	0.01200487	$3.79788\times\text{e}^{-5}$	-4.562879	17.40263
		22	0.01208298	$5.205916\times\text{e}^{-5}$	-3.103545	14.78127
		25	0.01062267	$3.993474\times\text{e}^{-5}$	-2.626373	8.525641
		27	0.009830597	$3.417619\times\text{e}^{-5}$	-1.079657	1.713844
		29	0.009148447	2.957897e-05	0.7977443	-5.239335
		30	0.00798317	$1.665469\times\text{e}^{-5}$	10.91303	-46.47535
		31	0.008554823	$2.58553\times\text{e}^{-5}$	2.704588	-11.53195
0.95	0.75	0.85	2.367222	10.4909	3.656463	26.84072
		2	0.5599422	0.2656862	1.57895	6.288152
		4	0.2342739	0.03651031	1.129819	4.217071
		10	0.08491655	0.00419917	0.9059771	3.449846
3.661	0.619	0.820	2.618983	1.021084	0.3466342	2.906629
2.027	0.752	0.821	1.880109	1.321671	1.013182	4.313426
1.800	1.07	0.188	45.69999	7130.462	6.655114	98.66605

2.2. **Probability weighted moments (PWMs).** The (s-r)th PWM of X following the WGLx model, say $P_{s,r}$, is formally defined by

$$P_{s,r} = \int_{-\infty}^{\infty} F^r(x)\, x^s f(x)\; dx = \mathbf{E}\left[\, F^r(x) X^s \right].$$

The (s-r)th PWM will be

$$P_{s,r}|_{(s<(b+rb))} = \sum_{r=0}^{\infty} a_r \left(b+rb\right) \, B\left(1+s,(b+rb)-s\right),$$

where

$$\begin{aligned} a_r &= \alpha\,\theta \sum_{m,i,j,k=0}^{\infty} \frac{(-1)^{k+i+m+j}\,(r)_k\,(m+1)^i}{i!m!}\binom{1+k}{r} \\ &\times \binom{(1+i)\,\alpha-1}{j}\binom{\theta\left[-(1+i)\,\alpha+j\right]-1}{k}, \end{aligned}$$

and

$$(\nu_1)_{\nu_2}|_{(\nu_2\in\mathbf{I}^+)} = \nu_1\left(\nu_1-1\right)...\left(1+\nu_1-\nu_2\right)$$

is the descending factorial.

2.3. **Order statistics.** The PDF of ith order statistic of a andom sample from the WGLx model $(X_1, X_2, \ldots, X_n)$, say $X_{i:n}$, can be written as

$$g_{i:n}(x) = \sum_{j=0}^{n-i} (-1)^j \, B^{-1}\left(i,1+n-i\right) f\left(x\right) \;\; F^{-1+j+i}\left(x\right)\binom{n-i}{j}, \qquad (7)$$

where $B(\zeta_1, \zeta_2)$ is the beta function. Inserting (1) and (2) in (7) and using a power series expansion, we have

$$F^{-1+j+i}\left(x\right) f\left(x\right) = \sum_{r=0}^{\infty} \tau_r h_{(b+rb)}(x),$$

where

$$\begin{aligned} \tau_r &= \alpha\,\theta \sum_{m,l,w,k=0}^{\infty}\sum_{j=0}^{n-i} \frac{(-1)^{m+l+w+k+j}\,(m+1)^l\,(-1+j+i)_k}{m!l!B\left(i,n-i+1\right)}\binom{n-i}{j} \\ &\times \binom{1+k}{-1+j+i}\binom{(1+i)\,\alpha-1}{w}\binom{\theta\left[-\alpha\left(l+1\right)+w\right]-1}{k}, \end{aligned}$$

and the PDF of $X_{i:n}$ can be expressed as

$$g_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{r=0}^{\infty} \frac{\binom{n-i}{j}(-1)^j\,\tau_r}{B\left(i,1+n-i\right)} h_{(b+rb)}(x),$$

and the qth moments will be

$$\mathbf{E}\left(X_{i:n}^q\right)|_{(q<(b+rb))} = \sum_{r=0}^{\infty} \tau_r \left(b+rb\right) \, B\left(1+q,(b+rb)-q\right),$$

3. Moments of the reversed residual life (MRL)

The nth MRL, say

$$\Upsilon_n(t)|_{(X\le t,\;\; t>0\;)} = \mathbf{E}\left[(t-X)^n\right] \;\forall\; n=1,2,\ldots.$$

Then, we obtain

$$\Upsilon_n(t) = F^{-1}(t)\int_0^t (t-x)^n dF(x).$$

Then, the nth MRL of X becomes

$$\Upsilon_n(t) = F^{-1}(t) \sum_{r=0}^{\infty} \zeta_r^{[\omega]} (b + rb) \; B\left(t^a; (b + rb) - n, 1 + n\right),$$

where

$$\sum_{r=0}^{n} t^{n-r} (-1)^r \binom{n}{r} \zeta_r = \zeta_r^{[\omega]}.$$

4. Parameter estimation

In this article, we will consider the maximum likelihood (ML) method for estimating the unknown parameters (α, θ, b) of the WGLx. Let $x_1, x_2, \cdots, x_n$ be a RS from the WGLx model, where $\mathbf{\Theta} = (\alpha, \theta, b)^{\intercal}$. Then the log-likelihood (LogL) function $(\ell_n(\mathbf{\Theta}))$ for $\mathbf{\Theta}$ is given by

$$\begin{aligned} \ell_n(\mathbf{\Theta}) &= n \log\theta + n \log b + n \log\alpha + (b\theta - 1) \sum_{i=1}^{n} \log\left(1 + x_i\right) \\ &\quad + (-1 + \alpha) \sum_{i=1}^{n} \log\left[(1 + x_i)^{\theta b} - 1\right] - \sum_{i=1}^{n} \left[(1 + x_i)^{\theta b} - 1\right]^{\alpha}, \end{aligned}$$

the above LogL can be numerically maximized via many software programs like `R` (`optim`). The components of the score vector,

$$\mathbf{U}\left(\mathbf{\Theta}\right) = \frac{\partial \ell}{\partial \mathbf{\Theta}} = \left(\frac{\partial}{\partial \alpha}\ell_n(\mathbf{\Theta}), \frac{\partial}{\partial \theta}\ell_n(\mathbf{\Theta}), \frac{\partial}{\partial b}\ell_n(\mathbf{\Theta})\right)^{\intercal}$$

are easily to be derived.

5. Simulation studies

We simulate the WGLx model by taking $n = 20, 50, 150, 500$ and 1000. For each sample size (n), we evaluate the ML estimations (MLEs) of the parameters using the optim function of the R software. Then, we repeat the process 1000 times $(N = 1000)$ and compute the averages of the estimates (AEs) and mean squared errors (MSEs). Table 3 gives all simulation results. The values in Table 3 indicate that the MSEs and the bias of $\widehat{\alpha}, \widehat{\theta}$ and $\widehat{b}$ decay towards the zero when n increases for all settings of α, θ and b as expected under first-under asymptotic theory. The AEs of the parameters tend to be closer to the true parameter values ($\alpha = 1.5$, $\theta = 1.5$, $b = 2$) when n increases. These results supports that the asymptotic normal model provides good approximation to the finite sample model of the MLEs. Table 3 gives the AEs and MSEs based on $N = 1000$ simulations.

Table 3: AEs and MSE for $N = 1000$.

n	Θ	AE	MSE
20	α	1.8841401	0.2148603
	θ	1.2503191	0.0862224
	b	1.7648493	0.1551691
50	α	1.6754943	0.1685464
	θ	1.4472617	0.1316134
	b	1.8613436	0.1316134
150	α	1.5804481	0.1020621
	θ	1.4901349	0.0700229
	b	1.9519941	0.1256115
500	α	1.5193465	0.0460722
	θ	1.5109881	0.0683902
	b	1.9917632	0.0225906
1000	α	1.5003614	0.0007538
	θ	1.5009841	0.0679572
	b	2.0008719	0.0023129

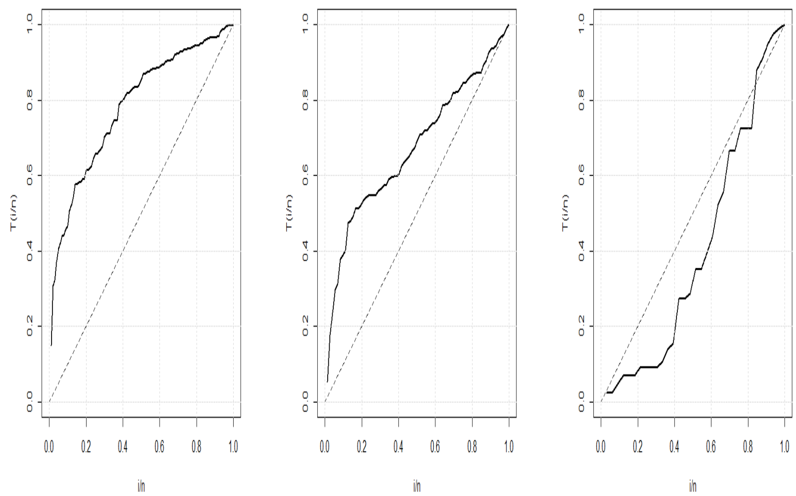
6. Applications

We provide three applications to show and illustrate the importance, potentiality and flexibility of the WGLx model. For these data, we compare the WGLx distribution, with BXII, Marshall-Olkin BXII (MOBXII), Topp Leone BXII (TL-BXII), Zografos-Balakrishnan BXII (ZBBXII), Five Parameters beta BXII (FB-BXII), BBXII, B exponentiated BXII (BEBXII), Five Parameters Kumaraswamy BXII (FKwBXII) and KwBXII distributions given in [3]. Data Set I called breaking stress data. This data consists of 100 observations of breaking stress of carbon fibres (in Gba) given by [10]. Data Set II called survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by [4]. Data set III called leukaemia data. This real data set gives the survival times (in weeks) of 33 patients suffering from acute Myelogeneous Leukaemia (see the data sets in Appendix A).

The total time test (TTT) plot is an important graphical approach to verify whether the real data can be applied to a specific model or not. According to [1], the empirical version of the TTT plot is given by plotting

$$T(q/m) = \frac{1}{\sum_{j=1}^{m} y_{j:m}} \left[\sum_{j=1}^{q} y_{j:m} + y_{q:m}(-q+m) \right]$$

against q/m, where $q = 1, 2, \ldots, m$ and $y_{j:m}(j = 1, 2, \ldots, m)$ are the order statistics of the sample. The HRF is constant if the TTT plot is graphically presented as a straight diagonal, the HRF is increasing (or decreasing) if the TTT plot is concave (or convex). The HRF is U-shaped (bathtub) if the TTT plot is firstly convex and then concave, if not, the HRF is unimodal. The TTT plots the three real data sets is presented in Fig.2. There three plot indicates that the empirical HRFs of the three data sets are increasing, increasing and U-shaped.



Data set **I** Data set **II** Data set **III**

Fig.2: TTT plots.

We shall consider the following goodness-of-fit statistics: the Akaike information criterion (AIc), Bayesian Ic (BIc), Hannan-Quinn Ic (HQIc), consistent Akaike Ic (CAIc), where

$$\mathrm{AIc} = 2\left[-\ell\left(\widehat{\Theta}\right) + k\right],$$

$$\mathrm{BIc} = 2\left[-\ell\left(\widehat{\Theta}\right) + \frac{1}{2}k\log\left(n\right)\right],$$

$$\mathrm{HQIc} = 2\left\{k\log\left[\log\left(n\right)\right] - \ell\left(\widehat{\Theta}\right)\right\}$$

and

$$\mathrm{CAIc} = 2\left[-\ell\left(\widehat{\Theta}\right) + \frac{kn}{(n-k-1)}\right],$$

where k refer to the number of parameters, n is the sample size, $-2\ell\left(\widehat{\Theta}\right)$ is the maximized log-likelihood. Generally, the smaller these criterion are, the better the fits. Based on the values in Tables 4-6 and Fig.2-5 the WGLx model provides the best fits as compared to other Lx extensions in the three applications with small values for BIc, AIc, CAIc and HQIc.

Table 4: MLEs and standard errors (SEs), confidence interval (CI) (in parentheses) with AIc, BIc, CAIc and HQIc values for the data set **I.**

Model	$\widehat{\alpha}$,$\widehat{\theta}$,$\widehat{a}$,$\widehat{b}$,$\widehat{\gamma}$	AIc, BIc, CAIc, HQIc
BXII	—,—, 5.941, 0.187,— —,—, (1.279) ,(0.044),— —,—, (3.43,8.45),(0.10,0.27),—	382.94, 388.15, 383.06, 385.05
MOBXII	—,—, 1.192,4.834,838.73 —,—, (0.952),(4.896),(229.34) —,—, 0, 3.06),(0, 14.43),(389.22,1288.24)	305.78, 313.61, 306.03, 308.96
TLBXII	—,—, 1.350,1.061,13.728 —,—, 0.378) ,(0.384) ,(8.400) —,—, (0.61, 2.09), (0.31,1.81) ,(0, 30.19)	323.52, 331.35, 323.77, 326.70
KwBXII	48.103 ,79.516 ,0.351 ,2.730,— (19.348) ,(58.186) ,(0.098) ,(1.077) ,— (10.18,86.03) ,(0,193.56) ,(0.16,0.54), (0.62,4.84),—	303.76, 314.20, 304.18, 308.00
BBXII	359.683 ,260.097 ,0.175 ,1.123 ,— (57.941) ,(132.213),(0.013),(0.243),— (246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6),—	305.64, 316.06, 306.06, 309.85
BEBXII	0.381, 11.949, 0.937, 33.402, 1.705 (0.078), (4.635), (0.267), (6.287),(0.478) (0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)	305.82, 318.84, 306.46, 311.09
FBBXII	0.421, 0.834, 6.111, 1.674, 3.450 (0.011), (0.943), (2.314), (0.226), (1.957) (0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)	304.26, 317.31, 304.89, 309.56
FKwBXII	0.542,4.223, 5.313, 0.411, 4.152 (0.137), (1.882), (2.318), (0.497), (1.995) (0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)	305.50, 318.55, 306.14, 310.80
ZBBXII	123.101,—,0.368, 139.247,— (243.011), —, (0.343), (318.546),— (0, 599.40), —,(0, 1.04), (0, 763.59),—	302.96, 310.78, 303.21, 306.13
WGLx	3.661, 0.619, —, 0.82,— 0.29, 0.00, —, 0.00,— (3.1,4.3), ,—,—,—,—	**288.7, 296.5, 288.9, 291.8**

Table 5: MLEs and SEs, CI with
AIc, BIc, CAIc and HQIc values for the data set **II.**

Model	$\widehat{\alpha}$,$\widehat{\theta}$,$\widehat{a}$,$\widehat{b}$,$\widehat{\gamma}$	AIc, BIc, CAIc, HQIc
BXII	—,—, 3.102, 0.465, — —,—, (0.538), (0.077),— —,—, (2.05,4.16), (0.31,0.62),—	209.60, 214.15, 209.77, 211.40
MOBXII	—,—, 2.259,1.533, 6.760 —,—, (0.864), (0.907), (4.587) —,—, (0.57,3.95), (0,3.31), (0, 15.75)	209.74, 216.56, 210.09, 212.44
TLBXII	—,—, 2.393,0.458,1.796 —,—, (0.907), (0.244),(0.915) —,—, (0.62,4.17),(0, 0.94),(0.002,3.59)	211.80, 218.63, 212.15, 214.52
KwBXII	14.105,7.424, 0.525, 2.274,— (10.805), (11.850), (0.279),(0.990),— (0, 35.28), (0.30.65), (0, 1.07),(0.33, 4.21),—	208.76, 217.86, 209.36, 212.38
BBXII	2.555, 6.058,1.800,0.294, — (1.859), (10.391), (0.955),(0.466),— (0, 6.28), (0, 26.42), (0, 3.67),(0, 1.21), —	210.44, 219.54, 211.03, 214.06
BEBXII	1.876,2.991, 1.780, 1.341, 0.572 (0.094), (1.731), (0.702), (0.816), (0.325) (1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)	212.10, 223.50, 213.00, 216.60
WGLx	2.027, 0.752, —, 0.82, — 0.18, 3.88, —, 4.25, — (1.7,2.4), (0,8.4), —, (0,9.4),—	**208.8, 215.7, 209.2, 211.6**

Table 6: MLEs and SEs, CI with
AIc, BIc, CAIc and HQIc values for the data set **III.**

Model	$\widehat{\alpha}$,$\widehat{\theta}$,$\widehat{a}$,$\widehat{b}$,$\widehat{\gamma}$	AIc, BIc, CAIc, HQIc
BXII	—,—, 58.711,0.006,— —,—, (42.382), (0.004),— —,—, (0, 141.78), (0, 0.01),—	328.20, 331.19, 328.60, 329.19
MOBXII	—,—, 11.838, 0.078, 12.251 —,—, (4.368), (0.013), (7.770) —,—, (0, 141.78), (0, 0.01), (0, 27.48)	315.54, 320.01, 316.37, 317.04
TLBXII	—,—,0.281, 1.882 ,50.215 —,—, (0.288), (2.402), (176.50) —,—, (0, 0.85), (0, 6.59), (0, 396.16)	316.26, 320.73, 317.09, 317.76
KwBXII	9.201, 36.428, 0.242,0.941,— (10.060), (35.650), (0.167), (1.045),— (0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99),—	317.36, 323.30, 318.79, 319.34
BBXII	96.104, 52.121, 0.104, 1.227,— (41.201), (33.490), (0.023), (0.326),— (15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9),—	316.46, 322.45, 317.89, 318.47
BEBXII	0.087, 5.007, 1.561, 31.270, 0.318 (0.077), (3.851), (0.012), (12.940), (0.034) (0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)	317.58, 325.06, 319.80, 320.09
FBBXII	15.194, 32.048, 0.233, 0.581, 21.855 (11.58), (9.867), (0.091), (0.067), (35.548) (0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)	317.86, 325.34, 320.08, 320.36
FKwBXII	14.732, 15.285, 0.293, 0.839, 0.034 (12.390), (18.868), (0.215), (0.854), (0.075) (0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)	317.76, 325.21, 319.98, 320.26
ZBBXII	41.973,—,0.157, 44.263,— (38.787),—,(0.082), (47.648),— (0, 117.99),—,(0, 0.32,) (0, 137.65),—	313.86, 318.35, 314.39, 315.36
WLx	1.79, ,—, —, 0.2, — (0.26), ,—, —,(0.015), — (1.25,2.33), ,—, —,(0.17,0.23), —	310.59, 313.58, 310.99, 311.59
WGLx	1.8, 1.07, —, 0.188,— 0.27, 0.00, —,0.00,— (1.2, 2.4), , —, —, —, —	**312.59, 317.07, 313.4, 314.10**

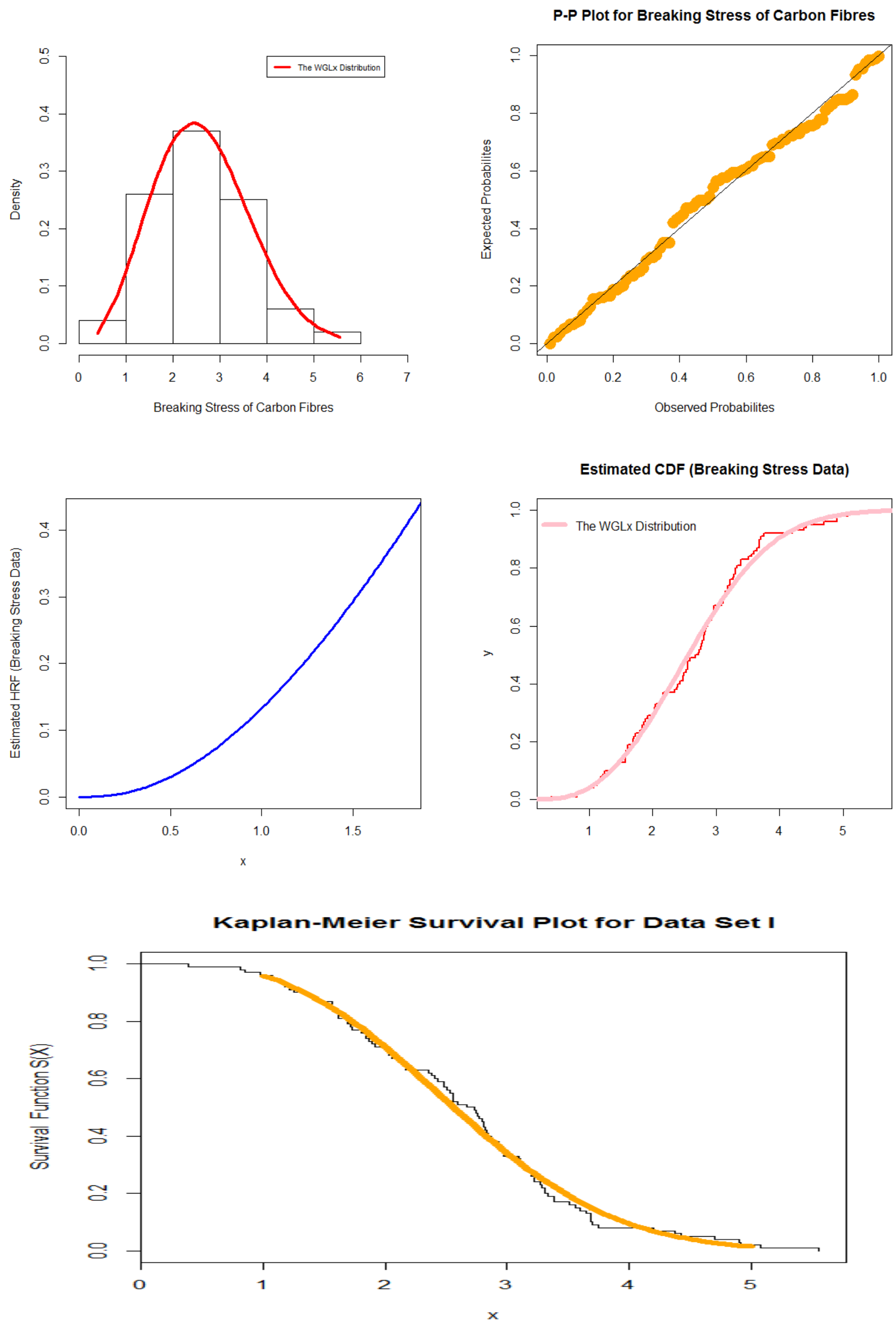


Fig.3: Estimated PDF, P-P plot, estimated HRF, estimated CDF and Kaplan-Meier survival plot for data set **I.**

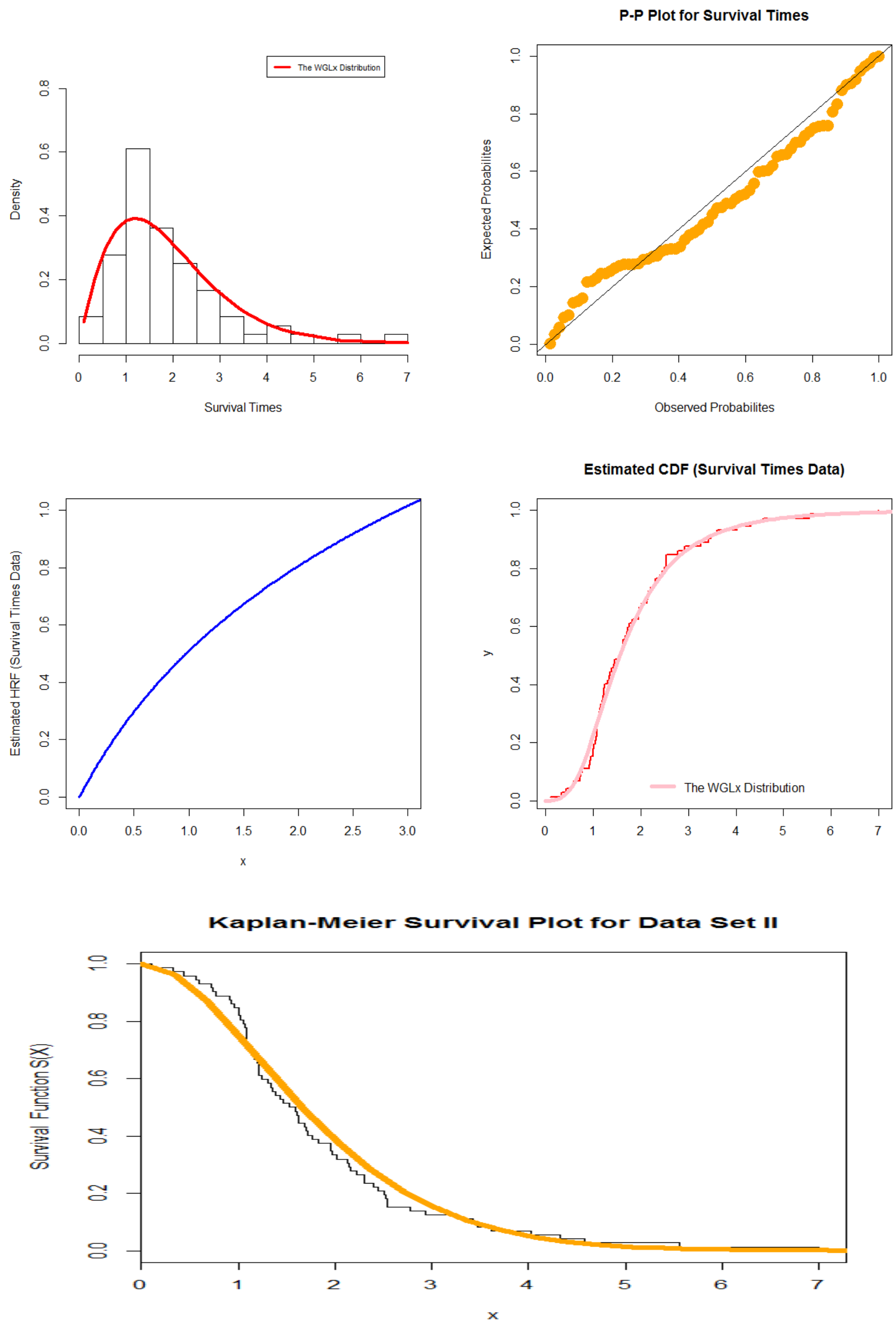


Fig.4: Estimated PDF, P-P plot, estimated HRF, estimated CDF and Kaplan-Meier survival plot for data set **II.**

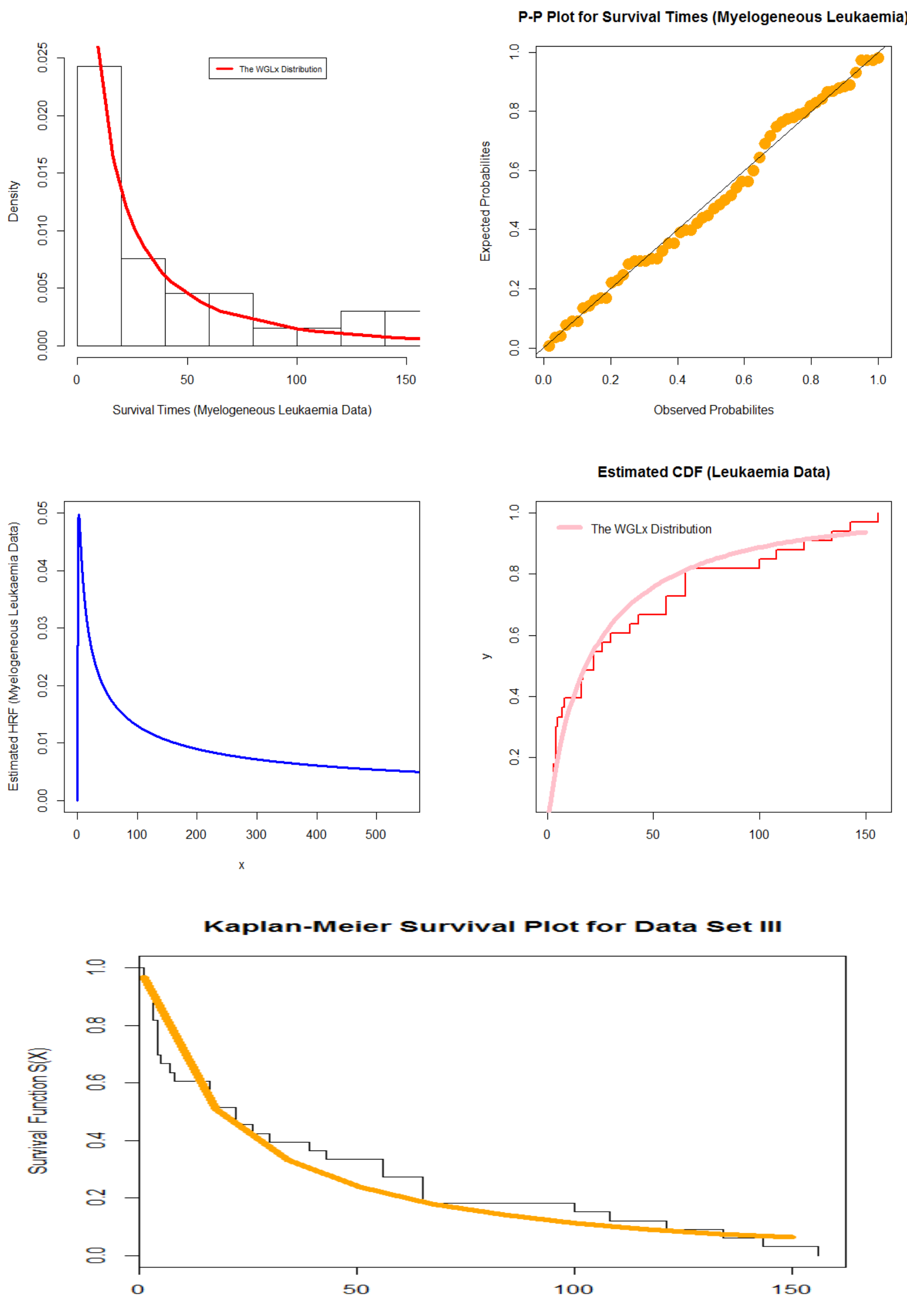


Fig.5: Estimated PDF, P-P plot, estimated HRF, estimated CDF and Kaplan-Meier survival plot for data set **III.**

7. Conclusions

In this work, we shall introduce a new extension of the Lomax distribution with a simple physical motivation. Some of its mathematical and statistical properties such as ordinary moment, moment generating function, incomplete moment, weighted moments, order statistics and their moments and moment of the reversed residual life are derived. The method of maximum likelihood is used to estimate the unknown parameters. Three applications are provided with its related plots to illustrate the importance of the new Lomax model. The new Lomax model is better than other nine competitive models. The skewness of the new distribution can range in the interval $(-4.5, 11)$, whereas the kurtosis of the WGLx distribution varies in the interval $(-46.5, 98.7)$ also the mean of X decreases as b increases, the Ske is always positive

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Appendix A

Data Set I:

{0.98, 5.56, 5.08, 0.39, 1.57, 3.19, 4.90, 2.93, 2.85, 2.77, 2.76, 1.73, 2.48, 3.68, 1.08, 3.22, 3.75, 3.22, 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.40, 3.15, 2.67,3.31, 2.81, 2.56, 2.17, 4.91, 1.59, 1.18, 2.48, 2.03, 1.69, 2.43, 3.39, 3.56, 2.83, 3.68, 2.00, 3.51, 0.85, 1.61, 3.28, 2.95, 2.81, 3.15, 1.92, 1.84, 1.22, 2.17, 1.61, 2.12, 3.09, 2.97, 4.20, 2.35, 1.41, 1.59, 1.12, 1.69, 2.79, 1.89, 1.87, 3.39, 3.33, 2.55, 3.68, 3.19, 1.71, 1.25, 4.70, 2.88, 2.96, 2.55, 2.59, 2.97, 1.57, 2.17, 4.38, 2.03, 2.82, 2.53, 3.31, 2.38, 1.36, 0.81, 1.17, 1.84, 1.80, 2.05, 3.65}.

Data Set II:

{0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55}.

Data Set III:

{65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43}

[1]Department of Statistics, Mathematics and Insurance, Damanhur University, Egypt.,
[2]Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt.
E-mail address: elbielym@yahoo.com & haitham.yousof@fcom.bu.edu.eg