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THE WEIBULL GENERALIZED EXPONENTIATED WEIBULL DISTRIBUTION: THEORY AND APPLICATIONS

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Abstract. In this work, a new extension of the exponentiated Weibull model is introduced with its mathematical properties and applications to failure times and medical data using the maximum likelihood method. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study. We prove empirically the importance and ‡exibility of the new model in modeling two types of lifetime data. We conclude that, the new model is much better than the Weibull , exponetiated Weibull, beta Weibull, Kumaraswamy Weibull, Transmuted Weibull, Weibull generalized Weibull and McDonald Weibul models in modeling failure times and breast cancer data.

1. Introduction and motivation

A random variable (r.v.) Z is said to have the Exponentiated Weibull (EW) distribution if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$g_{(EW)}^{(\alpha,\beta)}(z) = \alpha\beta z^{\beta-1}\exp\left(-z^{\beta}\right)\left[1-\exp\left(-z^{\beta}\right)\right]^{\alpha-1},$$

and

$$G_{(EW)}^{(\alpha,\beta)}(z) = \left[1-\exp\left(-z^{\beta}\right)\right]^{\alpha},$$

respectively, for $z>0$, $\alpha>0$ and $\beta>0$, when $\alpha=1$ we get the the one parameter Weibull distribution.

Let $g_{(EW)}^{(\boldsymbol{\xi})}(x)$ and $G_{(EW)}^{(\boldsymbol{\xi})}(x)$ denote the pdf and the cdf of the EW with parameter vector $\boldsymbol{\xi}=(\alpha,\beta)$. Then the cdf of the Weibull Generalized-EW (WGEW) based on Yousof et al. (2018) is defined by

$$F(x) = F_{(WGEW)}^{(\gamma,\theta,\alpha,\beta)}(x) = 1-\exp\left[-\left(\frac{1-\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}\right)^{\gamma}\right], \quad (1)$$

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the corresponding pdf to (1) is given by

$$\begin{aligned} f(x) &= f^{(\gamma,\theta,\alpha,\beta)}_{(WGEW)}(x) = \gamma\,\theta\alpha\beta x^{\beta-1} \\ &\times \exp\left(-x^{\beta}\right) \frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\gamma\alpha-2\alpha-1}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta\gamma+1}} \\ &\times \exp\left[-\left(\frac{1-\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}\right)^{\gamma}\right], \end{aligned} \quad (2)$$

where $\gamma > 0$ and $\theta > 0$ are two additional shape parameters. For $\alpha = 1$ we have the WGW distribution (see [13]). The additional parameters induced by the new Weibull generator are sought as a manner to furnish a more flexible distribution. In this paper, we study the WGEW model and give a comprehensive description of its mathematical properties. The new model is motivated by its important flexibility in applications. By means of two applications, it is noted that the WGEW model provides better fits than seven other models each having the same (or more) number of parameters. The cdf of the WGEW model can be expressed as

$$F(x) = 1 - \sum_{k=0}^{\infty} d_k\,\Pi_{(1+k)\alpha}(x), \quad (3)$$

where

$$d_k = \sum_{i,j=0}^{\infty} \left[(-1)^{i+j+k}/i!\right] \binom{i\gamma}{j} \binom{\theta(j-i\gamma)}{k},$$

and

$$\Pi_{\delta}(x) = \left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\delta},$$

is the cdf of the EW with power parameter δ. Upon differentiating (3), we obtain the same mixture representation for the pdf

$$f(x) = \sum_{k=0}^{\infty} a_k \pi_{(1+k)\alpha}(x), \quad (4)$$

where $a_k = -d_k$ and

$$\pi_{\delta}(x) = \delta\alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha-1}\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\delta-1}.$$

Equations (3) and (4) are the main results of this section.

According to [2], a physical interpretation of the WGEW distribution can be shown as follows: soppose that we have a lifetime r.v., Z, having a certain continuous EW distribution. The generalized ratio $\left\{1-\left[\overline{G}^{(\alpha,\beta)}_{(EW)}(x)\right]^{\theta}\right\} / \left[\overline{G}^{(\alpha,\beta)}_{(EW)}(x)\right]^{\theta}$,that an individual (or component) following the lifetime Z will die (fail) at time t is $\frac{1-\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}$. Consider that the variability of this ratio of death is represented by the r.v. X and assume that it follows the Weibull model with shape

γ. We can write $\Pr(Z \leq x) = \Pr\left(X \leq \frac{1-\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\theta}}\right) = F(x)$,which is given by (1).

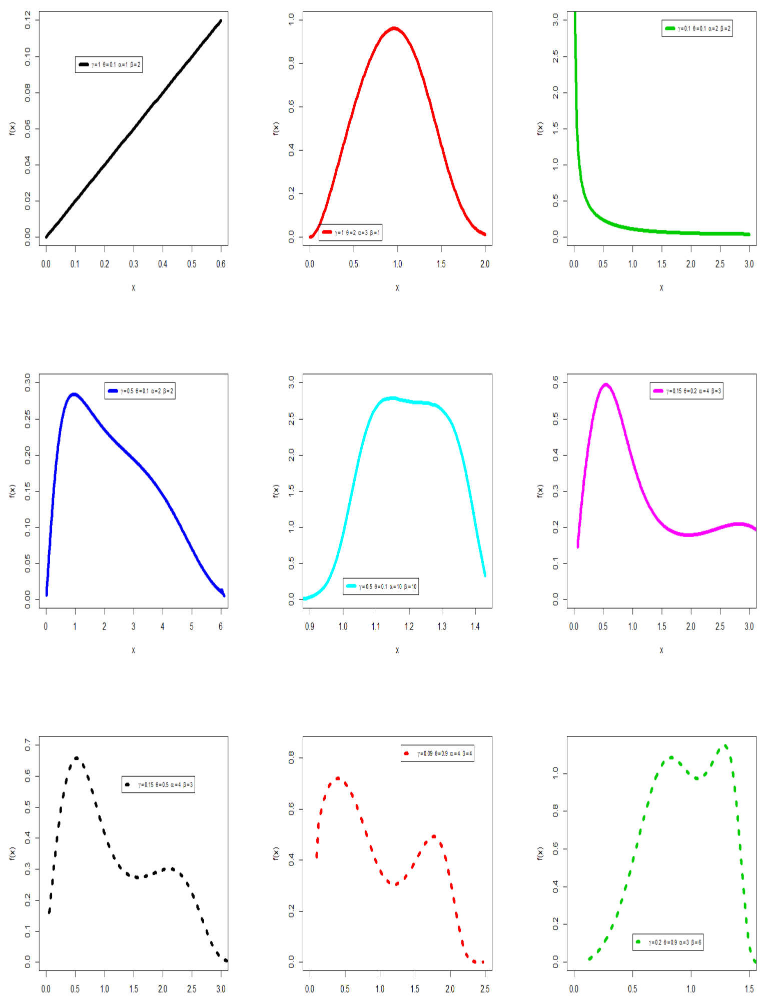

Figure 1: Plots of the WGEW pdf .

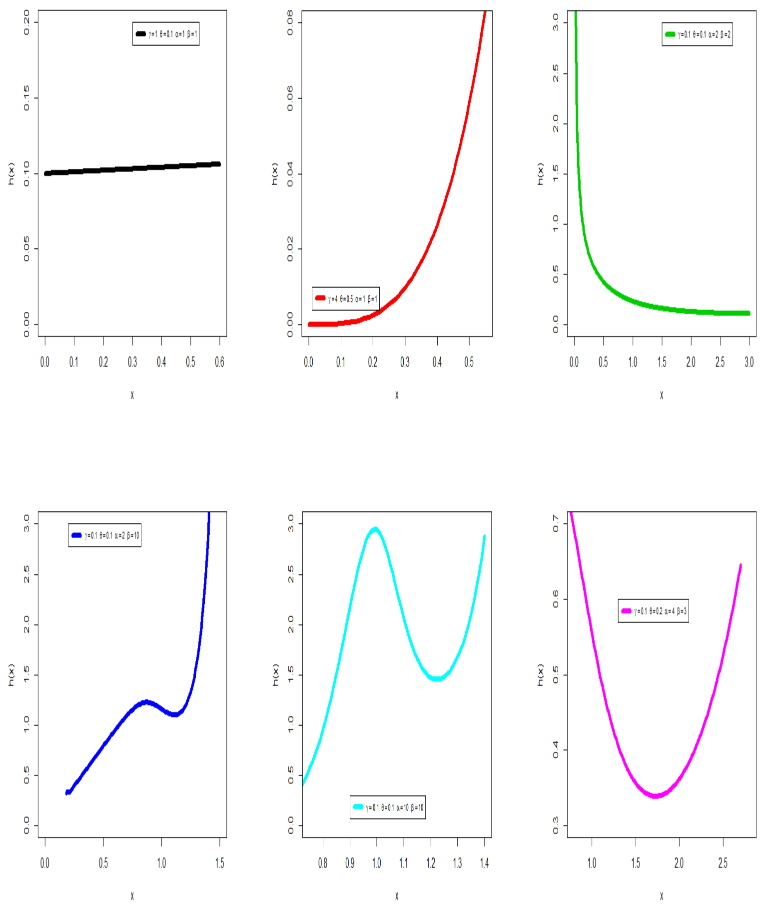

Figure 2: Plots of the WGEW hrf .

From Figure 1 we conclude that the pdf of the WGEW distribution exhibits various important shapes like left skewed, symmetric, right skewed and bimodal, from Figure 2 we conclude that the hrf WGEW distribution exhibits constant, increasing, decreasing, unimodal then increasing, unimodal then bathtub and bathtub hazard rates.

This paper is organized as follows. In Section 2, we derive some of mathematical properties for the new model. Maximum likelihood estimation for the model parameters is addressed in Section 3. Section 4 introduces the simulation studies. In Section 5, the potentiality of the proposed model is illustrated by means of two real data sets. Finally, Section 4 ends with some conclusions.

2. Properties

2.1. Main statistical properties. The r^{th} ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r\, f(x)\, dx,$$

then, we obtain (for any $r > -\beta$)

$$E(X^r) = \mu_r' = \Gamma\left(1 + \frac{r}{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,r)}, \tag{5}$$

where

$$a_{k,m}^{((1+k)\alpha,r)} = a_k a_m^{((1+k)\alpha,r)},$$

$$a_m^{((1+k)\alpha,r)} = \left[(1+k)\,\alpha\,(-1)^m \ \ (m+1)^{-(r+\beta)/\beta}\right]\binom{(1+k)\,\alpha - 1}{m},$$

$$\Gamma(1+a)\,|_{(a\in\mathbb{R}^+)} = \prod_{m=0}^{a-1}(a-m) = a \times (a-1) \times (a-2) \times ... \times 1 = a!,$$

and $\Gamma(a) = \int_0^\infty x^{a-1}\, e^{-t} dx$ is the complete gamma function. Setting $r = 1, 2, 3$ and 4 in (5) we get (for any $r > -\beta$)

$$\begin{aligned} E(X) &= \mu' = \Gamma\left(1 + \frac{1}{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,1)}, \\ E\left(X^2\right) &= \mu_2' = \Gamma\left(1 + \frac{2}{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,2)}, \\ E\left(X^3\right) &= \mu_3' = \Gamma\left(1 + \frac{3}{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,3)}, \end{aligned}$$

and

$$E\left(X^4\right) = \mu_4' = \Gamma\left(1 + \frac{4}{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,4)},$$

The r^{th} incomplete moment, say $I_r(t)$, of X can be expressed from (4) as (for any $r > -\beta$)

$$\begin{aligned} I_r(t) &= \int_{-\infty}^{t} x^r f(x)\, dx = \sum_{k=0}^{\infty} t_k \int_0^t x^r\, \pi_{1+k}(x)\, dx \\ &= \gamma\left(1 + \frac{r}{\beta}, \left(\frac{1}{t}\right)^{\beta}\right) \sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,r)}, \end{aligned} \tag{6}$$

where $\gamma(\zeta, q)$ is the incomplete gamma function.

$$\begin{aligned} \gamma(a,q)\,|_{(a\neq 0,-1,-2,\ldots)} &= \int_0^q t^{a-1} \exp(-t)\, dt \\ &= \frac{q^a}{a}\left\{{}_1\mathbf{F}_1\left[a; a+1; -q\right]\right\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(a+k)} q^{a+k}, \end{aligned}$$

where ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. Setting $r = 1, 2, 3$ and 4 in (6) we get (for any $r > -\beta$)

$$\begin{aligned} I_r(t) &= \Gamma\left(1+\frac{1}{\beta},\left(\frac{1}{t}\right)^{\beta}\right)\sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,1)}, \\ I_r(t) &= \Gamma\left(1+\frac{2}{\beta},\left(\frac{1}{t}\right)^{\beta}\right)\sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,2)}, \\ I_r(t) &= \Gamma\left(1+\frac{3}{\beta},\left(\frac{1}{t}\right)^{\beta}\right)\sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,3)}, \end{aligned}$$

and

$$I_r(t) = \Gamma\left(1+\frac{4}{\beta},\left(\frac{1}{t}\right)^{\beta}\right)\sum_{k.m=0}^{\infty} a_{k,m}^{((1+k)\alpha,4)}.$$

2.2. **Probability weighted moments.** The $(s,r)^{th}$ probability weighted moments (PWMs) of X following the WGEW model, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\left\{X^s\, F(X)^r\right\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\, dx,$$

the $(s,r)^{th}$ PWM of X can be expressed as

$$\rho_{s,r} = \sum_{k=0}^{\infty} v_k \int_{-\infty}^{\infty} x^s \pi_{(1+k)\alpha}(x)dx,$$

where

$$\begin{aligned} v_k &= \left\{\gamma\,\theta\,(-1)^k / [(1+k)\,\alpha]\right\}\sum_{m,i,j=0}^{\infty}\left\{(-1)^{m+i+j}\,(r)_k\,(m+1)^i / [m!i!]\right\} \\ &\times\binom{\gamma\,(i+1)-1}{j}\binom{\theta\,[-\gamma\,(i+1)+j]-1}{k}, \end{aligned}$$

and

$$(r)_k = r\times(r-1)\times\ldots\times(r-1+k)$$

is the descending factorial, k is a positive integer, then

$$\rho_{s,r} = \Gamma\left(1+\frac{s}{\beta}\right)\sum_{k,m=0}^{\infty} c_{k,m}^{((1+k)\alpha,s)},\ \forall\ s > -\beta,$$

where

$$c_{k,m}^{((1+k)\alpha,s)} = v_k a_m^{((1+k)\alpha,s)}.$$

2.3. **Order statistics.** Let $X_{(1)},\ldots,X_{(n)}$ be a random sample (r.s.) from the WGEW model of distributions and let $X_{1:n},\ldots,X_{n:n}$be the corresponding order statistics. The pdf of i^{th} order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \mathrm{B}^{-1}(i, n-i+1)\, f(x)\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F^{j+i-1}(x), \tag{7}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (1) and (6) in equation (7) and using a power series expansion, we get

$$f(x)\ F(x)^{j+i-1} = \sum_{k=0}^{\infty} u_k \pi_{(1+k)\alpha}(x),$$

where

$$\begin{aligned} u_k &= \frac{\gamma\,\theta\,(-1)^k}{(1+k)\,\alpha} \sum_{m,l,w=0}^{\infty} \left\{(-1)^{m+l+w}(j+i-1)_k\,(m+1)^l / [m!l!]\right\} \\ &\quad \times \binom{\gamma\,(i+1)-1}{w} \binom{\theta\,[-\gamma\,(l+1)+w]-1}{k}, \end{aligned}$$

and the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} (-1)^j\ \mathrm{B}^{-1}(i, n-i+1) \binom{n-i}{j} \sum_{k=0}^{\infty} u_k \pi_{(1+k)\alpha}(x),$$

i.e, the density function of the WGEW order statistic is a mixture of EW density. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those of Y_{1+k}. For example, the moments of $X_{i:n}$ can be expressed as. Then we have

$$E\left(X_{i:n}^q\right) = \Gamma\left(1+\frac{q}{\beta}\right) \sum_{k,m=0}^{\infty} \sum_{j=0}^{n-i} a_{k,m,j}^{((1+k)\alpha,q)},\ \forall\ q > -\beta,$$

where

$$a_{k,m,j}^{((1+k)\alpha,q)} = (-1)^j\ \mathrm{B}^{-1}(i, n-i+1)\, u_k a_m^{((1+k)\alpha,q)} \binom{n-i}{j}.$$

2.4. **Moments of residual life and reversed residual life.** The n^{th} moment of the residual life, say

$$\tau_n(t) = E\left\{(X-t)^n \mid \left[\begin{smallmatrix}(n=1,2,\ldots)\\(X>t)\end{smallmatrix}\right]\right\},$$

The n^{th} moment of the residual life of X is given by

$$\tau_n(t) = [1-F(t)]^{-1} \int_t^{\infty} (x-t)^n dF(x).$$

Therefore

$$\tau_n(t) = [1-F(t)]^{-1} \sum_{k=0}^{\infty} a_k^{(\tau)} \int_t^{\infty} x^r \pi_{(1+k)\alpha}(x),$$

where

$$a_k^{(\tau)} = a_k \sum_{r=0}^{n} (1-t)^n.$$

The n^{th} moment of the reversed residual life, say

$$\omega_n(t) = E\left\{(t-X)^n \mid \left[\begin{smallmatrix}(n=1,2,\ldots)\\(X\le t,t>0)\end{smallmatrix}\right]\right\},$$

We obtain

$$\omega_n(t) = [F(t)]^{-1} \int_0^t (t-x)^n dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$\omega_n(t) = [F(t)]^{-1} \sum_{k=0}^{\infty} a_k^{(\omega)} \int_0^t x^r \pi_{(1+k)\alpha}(x),$$

where

$$a_k^{(\omega)} = a_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

For the WGEW model we have

$$\tau_n(t) = [1-F(t)]^{-1}\, \Gamma\left(1+\frac{n}{\beta}, \left(\frac{1}{t}\right)^{\beta}\right) \sum_{k,m=0}^{\infty} a_{k,m}^{(1+k,n)^{(\tau)}}\ ,\ \forall\ n > -\beta,$$

and

$$\omega_n(t) = [F(t)]^{-1}\, \Gamma\left(1+\frac{n}{\beta}, \left(\frac{1}{t}\right)^{\beta}\right) \sum_{k,m=0}^{\infty} a_{k,m}^{(1+k,n)^{(\omega)}},\ \forall\ n > -\beta,$$

where

$$a_{k,m}^{(1+k,n)^{(\tau)}} = a_k^{(\tau)} a_m^{(1+k,n)} \text{ and } a_{k,m}^{(1+k,n)^{(\omega)}} = a_k^{(\omega)} a_m^{(1+k,n)}$$

3. Maximum likelihood estimation

Let $x_1, \ldots, x_n$ be a r.s. from the WGEW distribution with parameters θ,γ,α and β. For determining the MLE of Θ, we have the log-likelihood function

$$\begin{aligned}
\ell \quad = \quad & \ell(\Theta) = n\log\gamma + n\log\theta + n\log\alpha + n\log\beta \\
& -\beta\sum_{i=1}^{n}\log x_i + (\gamma\alpha - 2\alpha - 1)\sum_{i=1}^{n}\log\left[1-\exp\left(-x_i^{\beta}\right)\right] \\
& -(\theta\gamma+1)\sum_{i=1}^{n}\log\left\{1-\left[1-\exp\left(-x_i^{\beta}\right)\right]^{\alpha}\right\} - \sum_{i=1}^{n} z_i^{\gamma},
\end{aligned}$$

where

$$\begin{aligned}
a_i &= 1-\left\{1-\left[1-\exp\left(-x_i^{\beta}\right)\right]^{\alpha}\right\}^{\theta}, \\
b_i &= \left\{1-\left[1-\exp\left(-x_i^{\beta}\right)\right]^{\alpha}\right\}^{\theta},
\end{aligned}$$

and

$$z_i = \frac{a_i}{b_i}.$$

The components of the score vector

$$\mathbf{U}(\Theta) = \frac{\partial\ell}{\partial\Theta} = \left(\frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\gamma}, \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}\right)^{\intercal},$$

are easily to be obtained.

4. Simulation studies

We used computer software R Core for the simulation study. For each combination of specific parameter values, we simulated the sample data. MLEs are computed based on this data using `R` function `optimx` (see [5]). To maximize (9), we used [10] method as it provides more robust results than other methods. Finally, standard errors (SE) of the estimates are obtained from the Hessian matrix provided by `optimx`. We use the inversion method to simulate the WGEW ($\alpha = 0.5, 2, \beta = 2, 0.5, \gamma = 1, \theta = 1.5$) model by taking n=50, 150 and 300. For each sample size, we evaluate the MLEs of the parameters using the optim function of the R software. Then, we repeat this process 1,000 times and compute the averages of the estimates (AEs), biases and mean squared errors (MSEs). The simulation results are reported in Table 1. The figures in Table1 indicate that the MSEs and the biases of $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ and $\widehat{\theta}$ decay toward zero when the n increases for all settings of α and β, as expected under first-under asymptotic theory. The AEs of the parameters tend to be closer to the true parameter values when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. Table 1 gives the AEs, biases and MSEs based on 1000 simulations of the WGEW distribution for some values of α and β when $\gamma = 1$ and $\theta = 1.5$ by taking $n = 50, 150$ and 300.

Table 1: The AEs, biases and MSEs based on 1000 simulations.

		$\alpha=0.5$, $\beta=2$				$\alpha=2.0$, $\beta=0.5$		
n	Θ	AE	Bias	MSE	Θ	AE	Bias	MSE
50	α	0.4688	0.0303	0.3314	α	2.0233	0.0281	0.4944
	β	2.1332	0.0733	0.2832	β	0.5041	0.0161	0.0853
	γ	1.0232	0.4343	0.2911	γ	1.0444	0.1973	0.2443
	θ	1.7236	0.3345	0.2578	θ	1.6762	0.2940	0.2974
150	α	0.4711	0.0194	0.2309	α	2.0212	0.0292	0.2248
	β	2.0685	0.0602	0.1733	β	0.5045	0.0070	0.0271
	γ	1.0353	0.2467	0.0448	γ	1.0215	0.1209	0.1202
	θ	1.5055	0.1266	0.1132	θ	1.5033	0.1634	0.0943
300	α	0.4912	0.0128	0.0044	α	2.0051	0.0240	0.0810
	β	2.0025	0.0382	0.0591	β	0.5022	0.0029	0.0221
	γ	0.9934	0.0149	0.0411	γ	1.0035	0.0135	0.0145
	θ	1.5033	0.0768	0.0534	θ	1.5069	0.0811	0.0139

5. Real data applications

In this section, we provide two applications to real data to illustrate the importance of the WGEW model presented in Section 1. The MLEs of the parameters for these models are calculated and two goodness-of-fit statistics are used to compare the new model with other models. We compared the fits of the WGEW distribution with some of its special cases and other models such as Weibull (W) (see [12]), exponetiated Weibull (EW) (see [7]) and [8]), beta Weibull (BW) (see [6]), Kumaraswamy Weibull (KwW) (see [4]), Transmuted Weibull (TW) (see [1]), WGW (see [13]) and McDonald Weibull (McW) (see [3]) distributions given by:

- WG-W :

$$f(x;\gamma,\theta,\beta)=\gamma\theta\beta x^{\beta-1}\mathrm{e}^{-(\theta\gamma+2)x^{\beta}}\left[1-\mathrm{e}^{-\theta x^{\beta}}\right]^{\gamma-1}\exp\left\{-\left[\frac{1-\mathrm{e}^{-\theta x^{\beta}}}{\mathrm{e}^{-\theta x^{\beta}}}\right]^{\gamma}\right\},$$

- W :

$$f(x)=\beta x^{\beta-1}e^{-x^{\beta}},$$

- BW :

$$f(x)=\beta x^{\beta-1}e^{-\gamma x^{\beta}}\left(1-e^{-x^{\beta}}\right)^{\alpha-1}/B(\alpha,\gamma),$$

- KwW :

$$f(x) = \alpha\gamma\beta x^{\beta-1} e^{-x^\beta} \left(1 - e^{-x^\beta}\right)^{\alpha-1} \left(1 - \left(1 - e^{-x^\beta}\right)^{\alpha}\right)^{\gamma-1},$$

- TW :

$$f(x) = \beta x^{\beta-1} e^{-\alpha x^\beta} \left[1 + \lambda - 2\lambda \left(1 - e^{-x^\beta}\right)\right],$$

- McW :

$$f(x) = \gamma\beta x^{\beta-1} e^{-ax^\beta} \left(1 - e^{-x^\beta}\right)^{\alpha\gamma-1} \left(1 - \left(1 - e^{-x^\beta}\right)^{\gamma}\right)^{a-1} / B(\alpha, a).$$

The first data set consists of failure times for a particular windshield model including 88 observations that are classified as failed times of windshields. These data were previously studied by [9]. The second real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (see [6]). The data was examined by [11]. In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion (AIC) and Bayesian information criterion (BIC) as

$$AIC = -2\hat{\ell} + 2p \text{ and } BIC = -2\hat{\ell} + p \log(n),$$

where p is the number of parameters, n is the sample size and $\hat{\ell}$ is the log-likelihood function evaluated at the MLEs. The smaller are values of these statistics, the better are the fits.Tables 2 and 3 list the MLEs of the models parameters and the numerical values of the model selection statistics AIC and BIC and K-S. We note from the figures in Table 2 that the WGEW model has the lowest values of the AIC and BIC (for the first data set) as compared to other models. The fitted PDF, CDF, HRF and P-P plots for the 1^{st} data of the WGEW model is displayed in Figure 3. Similarly, it is also evident from Table 3 that the WGEW gives the lowest values the AIC, BIC (for the second data set) as compared to other models, The fitted PDF, CDF, HRF and P-P plot for the 2^{nd} data of the WGEW distribution is displayed in Figure 4.

Table 2: The MLEs and the goodness-of-fit statistics for the first data set.

Distribution	Parameter Estimates	AIC	BIC
$W_{(\beta)}$	2.562	331.9	334.4
$EW_{(\alpha,\beta)}$	3.595,1.316	286.7	291.6
$BW_{(\beta,\alpha,\gamma)}$	1464.1, 3.52, 2014.8	282.7	290.1
$KwW_{(\beta,\alpha,\gamma)}$	80.66, 2.41, 3351.1	268.9	276.2
$TW_{(\beta,\lambda)}$	1.749, -0.996	297.6	302.5
$McW_{(\gamma,a,\alpha,\beta)}$	27.80, 8.68, 0.256, 3.73	269.2	279.0
$WGW_{(\theta,\gamma,\beta)}$	1.68, 1.79, 7.12	264.1	271.4
$WGEW_{(\theta,\gamma,\alpha,\beta)}$	6.32, 0.13, 0.7, 0.033	**261.76**	**271.3**

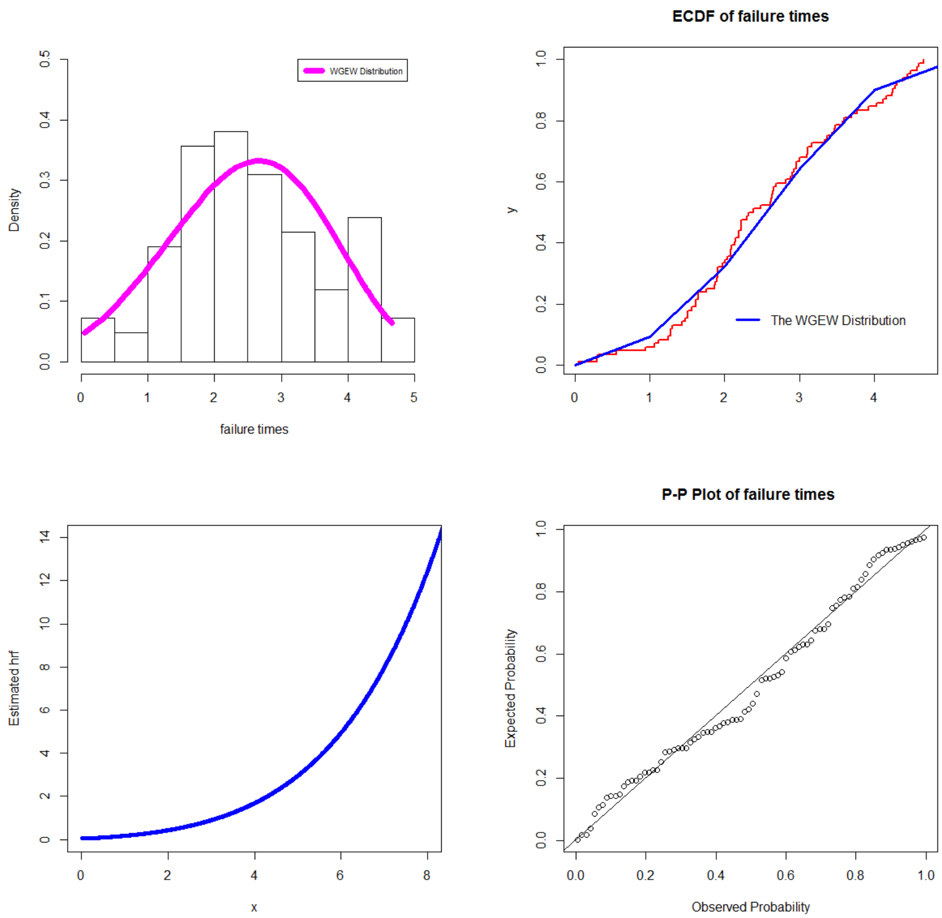

Figure 3: The fitted PDF, CDF, HRF and P-P plot for the first data set.

Table 3: The MLEs and the goodness-of-fit statistics for the second data set.

Distribution	Parameter Estimates	AIC	BIC
$W_{(\beta)}$	46.35	1172.2	1175.1
$EW_{(\beta,\alpha)}$	36.03, 1.515	1166.1	1173.7
$BW_{(\beta,\alpha,\gamma)}$	12635.4, 1.492, 406.1	1165.6	1173.9
$McW_{(\gamma,a,\alpha,\beta)}$	9.05, 2.28, 0.508, 169.4	1166.0	1177.2
$WGW_{(\theta,\gamma,\beta)}$	0.126, 0.957,10.06	1165.5	1172.8
$WGEW_{(\theta,\gamma,\alpha,\beta)}$	5.4, 1.015, 0.09, 2.049	**838.254**	**849.662**

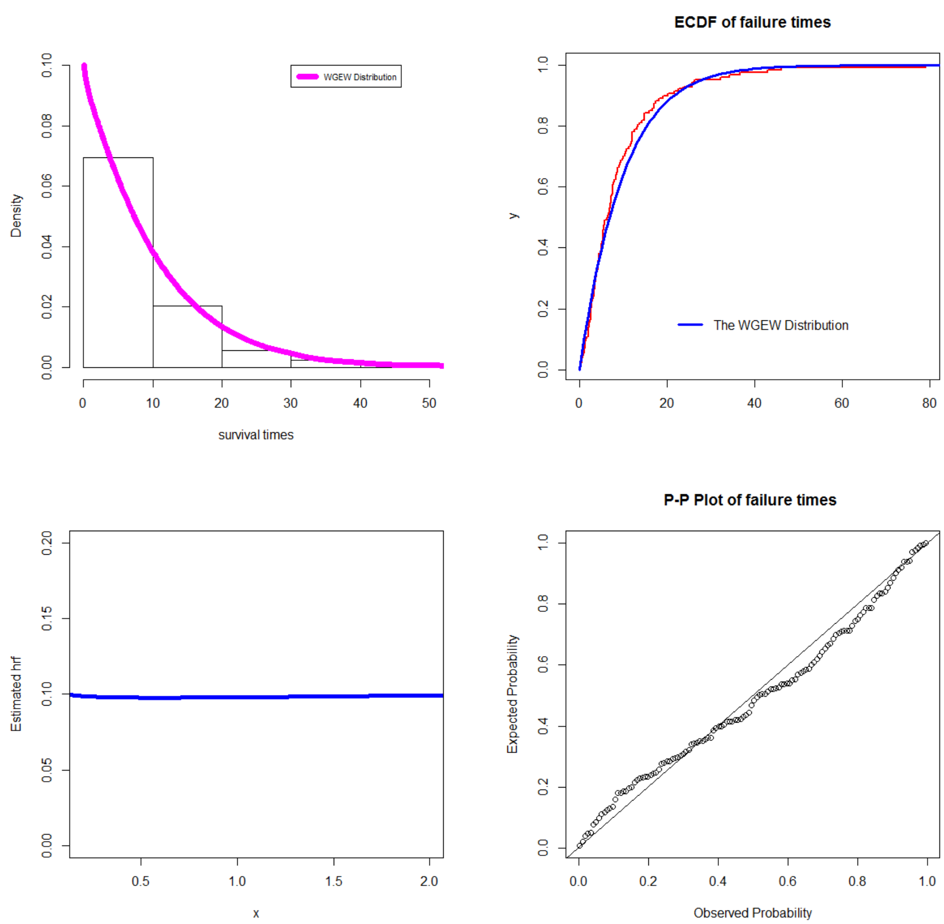

Figure 4: The fitted PDF, CDF, HRF and P-P plot for the second data set.

6. Conclusions

In this paper, a new extension of the exponentiated Weibull model is introduced with its mathematical properties and applications to failure times and medical data using the maximum likelihood method. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of a simulation study. We prove empirically the importance and flexibility of the new model in modeling two types of lifetime data. We note from the figures in Tables 1 and 2 that the new model has the lowest values of the AIC and BIC (for the first data set). Similarly, it is also evident from Table 2 that the WGEW gives the lowest values the AIC, BIC (for the second data set) as compared to other models. The new model is much better than the Weibull , exponetiated Weibull, beta Weibull, Kumaraswamy Weibull, Transmuted Weibull, Weibull generalized Weibull and McDonald Weibul models in modeling failure times and breast cancer data.

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