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WEIGHTED NORLUND-EULER λ-STATISTICAL CONVERGENCE AND APPLICATION

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Abstract. In this paper we introduce the concepts of weighted Norlund-Euler λ-statistical convergence and statistical summability $(N_\lambda, p, q)(E_\lambda, q)$. We also establish inclusion relation and some related for these new summability methods. Further, we determine a Korovkin type approximation theorem through statistical sumability $(N_\lambda, p, q)(E_\lambda, q)$.

1. Background and preliminaries

1.1. **Definition of statistical convergence.** The concept of convergence of a sequence of real numbers had been extended to statistical convergence by Fast [11]. Let $K \subseteq \mathbb{N}$, the set of positive integers and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = lim_{n\to\infty}\frac{1}{n}|K_n|$ if the limit exist, where $|K_n|$ denotes the cardinality of K_n.A sequence $x = (x_k)$ is said to be statisticaly convergent to the number L if for each $\varepsilon > 0, \delta(K_\varepsilon) = 0$, where $K_\varepsilon = \{k \leq n : |x_k - L| \geq \varepsilon\}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0 \tag{1.1}$$

In these case we write $st - \lim x = L$. It is known that every convergent sequence is statisticaly convergent, but not conversely.

1.2. **Definition of generalized weighted Norlund-Euler statistical convergence.** The generalized weighted Norlund-Euler statistical convergence is defined by E. A. Aljimi, V. Loku [1] , as follow: Let $\sum_{k=0}^{n} x_k$ be a given infinite series with sequence of its n^{th} partial sum $\{S_n\}$. If (E, q) transform is defined as

$$E_n^{(E,q)} = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) q^{n-k} S_k \tag{1.2}$$

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and we say that this summability method is convergent if $E_n^{(E,q)} \to S$ as $n \to \infty$. In this case we say the series $\sum_{k=0}^{n} x_k$ is $(E, q)-$ summable to a definite number S. And we will write $S_n \to S(E, q)$ as $n \to \infty$. Let (p_n) and (q_n) be the two sequences of non-zero real constants such that

$$P_n = p_0 + p_1 + \cdots + p_n, P_{-1} = p_{-1} = 0$$
$$Q_n = q_0 + q_1 + \cdots + q_n, Q_{-1} = q_{-1} = 0$$

For the given sequences (p_n) and (q_n), convolution $p * q$ and is defined by:

$$R_n = p * q = \sum_{k=0}^{n} p_k q_{n-k} \tag{1.3}$$

The series $\sum_{k=0}^{n} x_k$ or the sequence $\{S_n\}$ is summable to S by generalized Norlund method and it is denoted by $S_n \to S(N, p, q)$ if

$$t_n^{(N,p,q)} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-\nu} q_\nu S_\nu \tag{1.4}$$

tends to S as $n \to \infty$.

Let us use in consideration the following method of summability:

$$t_n^{(N,p,q)(E,q)} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k E_k^q = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu=0}^{n} \begin{pmatrix} k \\ \nu \end{pmatrix} q^{k-\nu} S_\nu \tag{1.5}$$

If $t_n^{(N,p,q)(E,q)} \to L$ as $n \to \infty$, then we say that the series $\sum_{k=0}^{n} x_k$ or the sequence $\{S_n\}$ is summable to S by Norlund-Euler method and it is denoted by $S_n \to S(N, p, q)(E, q)$.

Remark. *If $p_k = 1$,$q_k = 1$, then we get Euler summability method.*

Now we are able to give the definition of the generalized weighted statistical convergence related to the $(N, p, q)(E, q)$ summability method. We say that E have weighted density, denoted by $\delta_{NE}(E)$, if

$$\delta_{NE}^q(E) = \lim_{n\to\infty} \frac{1}{R_n} |\{k \le R_n : k \in E\}| \tag{1.6}$$

A sequence $x = (x_k)$ is said to be generalized weighted Norlund-Euler statistical convergent (S_{NE}^q −convergent) if for every $\varepsilon > 0$:

$$\lim_{n\to\infty} \frac{1}{R_n} \left| \left\{ k \le R_n : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu=0}^{n} \begin{pmatrix} \mathrm{k} \\ \nu \end{pmatrix} q^{k-\nu} |x_\nu - L| \ge \varepsilon \right\} \right| \tag{1.7}$$

In these case we write $L = S_{NE}^q(st) - \lim x$.

2. Main resultes

2.1. **Weighted Norlund -Euler λ-statistical convergence.** In these section we get second generalization of weighted Norlund-Euler statistical convergence following the line of Mursaleen [17]; V. Karakaya, T. Chishti [13]; M. Mursaleen, V. Karakaya [18]; C. Belen, S.A. Mohiuddine [3], Sengul, Hacer; Et, Mikail [19], Et, Mikail; Sengul, Hacer [20-21] Basarir, M.; Konca, S.[22], Cinar, Muhammed; Et,

Mikail.[23] and E. A. Aljimi, V. Loku [1] and we call these new generalization, weighted Norlund -Euler λ-statistical convergence.
Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1 \tag{2.1}$$

The collection of such a sequence will be denoted by Δ.
Let (p_n) and (q_n) be the two sequences of non-zero real constants such that

$$P_{\lambda n} = \sum_{k\in I_n} p_k, P_{-1} = p_{-1} = 0$$

$$Q_{\lambda n} = \sum_{k\in I_n} p_k, Q_{-1} = q_{-1} = 0$$

For the given sequences (p_n) and (q_n), convolution $p * q$ and is defined by:

$$R_{\lambda n} = p * q = \sum_{k\in I_n} p_k q_{n-k} \tag{2.2}$$

and

$$t_n^{(N_\lambda,p,q)(E_\lambda,q)} = \frac{1}{R_{\lambda n}} \sum_{k\in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu\in I_k} \binom{k}{\nu} q^{k-\nu} S_\nu \tag{2.3}$$

where $I_n = [n - \lambda_n + 1, n]$.
If $t_n^{(N_\lambda,p,q)(E_\lambda,q)} \to S$ as $n \to \infty$, then we say that the series $\sum x_k$ is summable to S by new generalization of weighted Norlund-Euler method and it is denoted by $S_n \to S(N_\lambda, p, q)(E_\lambda, q)$. Now we give the following definitions.

2.1.1. *Definition.* A sequence $x = (x_k)$ is said to be $(N_\lambda, p, q)(E_\lambda, q)$ summable to L if

$$\lim_n t_n^{(N_\lambda,p,q)(E_\lambda,q)} = L \tag{2.4}$$

2.1.2. *Definition.* A sequence $x = (x_k)$ is said to be strongly $(N_\lambda, p, q)(E_\lambda, q)$ summable to L if

$$\frac{1}{R_{\lambda n}} \sum_{k\in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu\in I_k} \binom{k}{\nu} q^{k-\nu} |x_\nu - L| = 0 \tag{2.5}$$

In these case we write $x_k \to L[(N_\lambda, p, q)(E_\lambda, q)]$ and $[(N_\lambda, p, q)(E, q)]$ denotes the set of all strongly $(N_\lambda, p, q)(E_\lambda, q)$ $-$ summable sequences.

2.1.3. *Definition.* A sequence $x = (x_k)$ is said to be summable $(N_\lambda, p, q)(E_\lambda, q)$ to L if

$$st - \lim t_n^{(N_\lambda,p,q)(E_\lambda,q)} = L \tag{2.6}$$

In these case we write

$$(NE)_\lambda(st) - \lim x = L \tag{2.7}$$

Now, we get weighted Norlund-Euler λ-statistical convergence by using the notion of $(N_\lambda, p, q)(E_\lambda, q)$ $-$summability.
Let $K \subseteq \mathbb{N}$. The number

$$\delta_{(NE)_\lambda}(K) = \lim_n \frac{1}{R_{\lambda n}} \left|\{k \leq R_{\lambda n} : k \in K|\}\right. \tag{2.8}$$

is said to be weighted Norlund-Euler λ-density of K. In case $\lambda_n = n$, the weighted Norlund-Euler λ-density reduced to generalized weighted Norlund-Euler density.

2.1.4. *Definition.* A sequence $x = (x_k)$ is said to be weighted Norlund-Euler λ-statistically convergent (or $S_{(NE)_\lambda}$-convergent) to L if for each $\varepsilon > 0$

$$\lim_n \frac{1}{R_{\lambda n}} \left| \left\{ k \leq R_{\lambda n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} |x_\nu - L| \geq \varepsilon \right| \right\} = 0 \tag{2.9}$$

In these case we write $L = S_{(NE)_\lambda} - \lim x$ and $S_{(NE)_\lambda}$ denotes the set of the all weighted Norlund-Euler λ-statistically convergent sequences.

Theorem 2.1. *Let* $p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} |x_\nu - L| \leq M$*for all* k*. If a sequence* $x = (x_k)$ *is* $S_{(NE)_\lambda}$*-convergent to* L *then it is statistically summable* $(N_\lambda, p, q)(E_\lambda, q)$ *to* L *but not conversely.*

Proof. Let

$$K_{R_{\lambda_n}}(\varepsilon) = \left\{ k \leq R_{\lambda n} : p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} |x_\nu - L| \geq \varepsilon \right\} \tag{2.10}$$

Since $L = S_{(NE)_\lambda} - \lim x$, $\lim_n \frac{1}{R_{\lambda_n}} \left|K_{R_{\lambda_n}}(\varepsilon)\right| = 0$. Also, we have

$$\begin{aligned}
\left| t_n^{(N_\lambda,p,q)(E_\lambda,q)} - L \right| &= \left| \frac{1}{R_{\lambda_n}} \sum_{k \in I_n} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} (x_\nu - L) \right| \leq \\
&\leq \frac{1}{R_{\lambda_n}} \sum_{k \in I_n, k \notin R_{\lambda_n}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} (x_\nu - L) + \\
&+ \left| \frac{1}{R_{\lambda_n}} \sum_{k \in I_n, k \in R_{\lambda_n}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} (x_\nu - L) \right| \leq \\
&\leq \frac{1}{R_{\lambda_n}} \sum_{k \in I_n, k \notin R_{\lambda_n}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} (x_\nu - L) + \\
&+ \frac{1}{R_{\lambda_n}} \sum_{k \in I_n, k \in R_{\lambda_n}} p_{n-k} q_k \frac{1}{(1+q)^k} \sum_{\nu \in I_k} \binom{k}{\nu} q^{k-\nu} (x_\nu - L) \\
&\leq \frac{M}{R_{\lambda_n}} \left|K_{R_{\lambda_n}}(\varepsilon)\right| + \varepsilon \to 0 + \varepsilon
\end{aligned} \tag{2.11}$$

as $n \to \infty$, which implies that $t_n^{(N_\lambda,p,q)(E_\lambda,q)} \to L$, i.e. $x = (x_k)$ is $(N_\lambda, p, q)(E_\lambda, q)$ $-$summable to L. Hence $x = (x_k)$ is statistically $(N_\lambda, p, q)(E_\lambda, q)$ $-$summable to L. For converse, take $\lambda_n = n$ and then see the example in Theorem 2.1 [1].
It can be easily seen that $S_{(NE)_\lambda} \subseteq S_{NE}$. Since $\frac{\lambda_n}{n}$ is bounded by 1 and so $\frac{R_{\lambda n}}{R_n}$ is bounded by 1. We prove following for the reverse inclusion. □

Theorem 2.2. *If* $\lim_{n\to\infty} \inf \left(\frac{R_{\lambda n}}{R_n}\right) > 0$ *then* $S_{NE} \subseteq S_{(NE)_\lambda}$.

Proof. Let $S_{NE} - \lim x = L$. We may write

$$\frac{1}{R_n}\left|\left\{k \le R_n : p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| \ge \varepsilon\right\}\right| \ge$$

$$\ge \frac{1}{R_n}\left|\left\{k \le R_{\lambda n} : p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| \ge \varepsilon\right\}\right| =$$

$$= \frac{R_{\lambda n}}{R_n}\frac{1}{R_{\lambda n}}\left|\left\{k \le R_{\lambda n} : p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| \ge \varepsilon\right\}\right| \quad (2.12)$$

Taking the limit as $n \to \infty$ and using the condition of $\lim_{n\to\infty} \inf \left(\frac{R_{\lambda n}}{R_n}\right) > 0$, we have

$$S_{NE} - \lim x = L \Leftrightarrow S_{(NE)_\lambda} - \lim x = L$$

□

Theorem 2.3.

a) If $x_k \to L[(N_\lambda, p, q)(E_\lambda, q)]$*, then* $S_{(NE)_\lambda} - \lim x = L$ *and the inclusion* $[(N_\lambda, p, q)(E_\lambda, q)] \subset S_{(NE)_\lambda}$ *is strict.*

b) If $p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| \le M$ *for all* k *and* $S_{(NE)_\lambda} - \lim x = L$*, then* $x_k \to L[(N_\lambda, p, q)(E_\lambda, q)]$*, hence* $x_k \to L[(N, p, q)(E, q)]$.

Proof.
a) Let $x_k \to L[(N_\lambda, p, q)(E_\lambda, q)]$ and

$$K_{R\lambda n}(\varepsilon)\left\{k \le R_{\lambda n} : p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| \ge \varepsilon\right\}$$

for a given $\varepsilon > 0$. Then, we have

$$\frac{1}{R_{\lambda n}}\sum_{k\in I_n} p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| =$$

$$= \frac{1}{R_{\lambda n}}\sum_{k\in I_n, k\in K_{R\lambda}(\varepsilon)} p_{n-k}q_k \frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu} q^{k-\nu}|x_\nu - L| +$$

$$+\frac{1}{R_{\lambda n}}\sum_{k\in I_n, k\notin K_{R\lambda}(\varepsilon)} p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}|x_\nu-L|\geq$$

$$\geq\frac{\varepsilon}{R_{\lambda n}}\left|\left\{k\leq R_{\lambda n}:p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}|x_\nu-L|\geq\varepsilon\right\}\right| \tag{2.13}$$

and these imply that $S_{(NE)_\lambda}-\lim x=L$. To show that the inclusion is strict let $\lambda_n=n$, and define $x=x_k$ as follows :

$$x_k=\begin{cases}\sqrt{k},\text{ if } k=m^2\\ 0,\text{ otherwice}\end{cases} \tag{2.14}$$

Then for $p_k=q_k=1$ then

$$R_n=\sum_{k=0}^{n}1\cdot1=1+1+\cdots+1=n+1 \tag{2.15}$$

and

$$\frac{1}{R_n}\left|\left\{k\leq n+1:p_{n-k}q_k\frac{1}{2^k}\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}|x_\nu-0|\geq\varepsilon\right\}\right|\leq\frac{\sqrt{n+1}}{n}\to0\ \ as\ \ n\to\infty \tag{2.16}$$

but

$$\frac{1}{R_n}\sum_{k=1}^{n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu=1}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}|x_\nu-0|\to\infty\ \ as\ \ n\to\infty \tag{2.17}$$

Hence the inclusion is strict.

b) The first implication is obvious from the proof of theorem 2.1. Further, we have

$$\frac{1}{R_n}\sum_{k=1}^{n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu=1}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L)=$$

$$=\frac{1}{R_n}\sum_{k=1}^{n-\lambda_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu=1}^{k-\lambda_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L)+$$

$$+\frac{1}{R_n}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L)\leq$$

$$\leq\frac{1}{R_{\lambda n}}\sum_{k=1}^{n-\lambda_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu=1}^{k-\lambda_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L)+$$

$$+\frac{1}{R_{\lambda n}}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L)\leq$$

$$\leq\frac{2}{R_{\lambda n}}\sum_{k\in I_n}p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\left(\begin{array}{c}k\\ \nu\end{array}\right)q^{k-\nu}(x_\nu-L) \tag{2.18}$$

which implies that $x_k \to L[(N,p,q)(E,q)]$. □

Theorem 2.4. *A sequence $x=(x_k)$ is statistically summable $(N_\lambda,p,q)(E_\lambda,q)$ to L if and only if there exist a set of $K=\{k_1<k_2<\cdots<k_n<\cdots\}\subseteq\mathbb{N}$ such that $\delta(K)=1$ and (x_{k_n}) is $(N_\lambda,p,q)(E_\lambda,q)$ summable to L.*

Proof. Suppose that there exists a set $K\subset\mathbb{N}$ such that $\delta(K)=1$ and (x_{k_n}) is $(N_\lambda,p,q)(E_\lambda,q)$ summable to L. Then there is positive integer n_0 such that for every $n>n_0$ we have $\left|t_n^{(N_\lambda,p,q)(E_\lambda,q)}-L\right|<\varepsilon$.Put
$K_t(\varepsilon)=\left\{n\in\mathbb{N}:|t_{k_n}^{(N_\lambda,p,q)(E_\lambda,q)}-L|\geq\varepsilon\right\}$ and $K'=k_{n_0+1},k_{n_0+2},\cdots$. Then $\delta(K')=1$ and $K_t(\varepsilon)\subset\mathbb{N}-K'$, which implies that $\delta(K_t(\varepsilon))=0$. Hence $x=(x_k)$ is statistically summable $(N_\lambda,p,q)(E_\lambda,q)$ to L.

Conversely let $x=(x_k)$ be statistically summable $(N_\lambda,p,q)(E_\lambda,q)$ to L. For $r=1,2,3,\cdots$ put $K_t(r):=\left\{j\in\mathbb{N}:\left|t_{k_j}^{(N_\lambda,p,q)(E_\lambda,q)}-L\right|\geq\frac{1}{r}\right\}$ and
$M_t(r):=\left\{j\in\mathbb{N}:\left|t_{k_j}^{(N_\lambda,p,q)(E_\lambda,q)}-L\right|<\frac{1}{r}\right\}$. Then $\delta(K_t(r))=0$ and

$$M_t(1)\supset M_t(2)\supset\cdots\supset M_t(i)\supset M_t(i+1)\supset\cdots \tag{2.19}$$

and

$$\delta(M_t(r))=1 \tag{2.20}$$

Now we have to show that for $j\in M_t(r)$,(x_{k_j}) is $(N_\lambda,p,q)(E_\lambda,q)$ summable to L. Suppose that (x_{k_j}) is not $(N_\lambda,p,q)(E_\lambda,q)$ summable to L. Therefore there is $\varepsilon>0$ such that $\left|t_{k_j}^{(N_\lambda,p,q)(E_\lambda,q)}-L\right|\geq\varepsilon$ for infinitely many terms. Let $M_t(\varepsilon):=\left\{j\in\mathbb{N}\left|t_{k_j}^{(N_\lambda,p,q)(E_\lambda,q)}-L\right|<\varepsilon\right\}$ and $\varepsilon>\frac{1}{r},(r=1,2,3,\cdots)$. Then from (2.19) we have $M_t(r)\subset M_t(\varepsilon)$. Hence $\delta(M_t(r))=0$, which contradicts (2.20) and therefore (x_{k_j}) is $(N_\lambda,p,q)(E_\lambda,q)$ summable to L. □

3. Application

In this section, we prove a Korovkin approximation theorem through statistical summability $(N_\lambda,p,q)(E_\lambda,q)$.
Let $C[a,b]$. be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with the norm $\parallel f\parallel_\infty=sup_{a\leq x\leq b}|f|, f\in C[a,b]$. Suppose that L be a linear operator from $C[a,b]$ into $C[a,b]$. Then as usual, we say that L is positive linear operator provided that $f\geq 0$ implies $Lf\geq 0$. Also, we denote the value of Lf at a point x by $L(f,x)$. The classical Korovkin theorem states as follows [14]:
Suppose that (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n\parallel T_n(f;x)-f(x)\parallel_\infty=0$, for all $f\in C[a,b]$., if and only if $\lim_n\parallel T_n(f_i;x)-f_i(x)\parallel_\infty=0$, for $i=0,1,2$, where $f_i(x)=x^i$. The statistical case of this theorem has been given by Gadjiev and Orhan [10] and later many authors proved Korovkin type approximation theorems by using different summability methods, see for instance [2-10,12,15,16].

Theorem 3.1. *Suppose that (T_k) is a sequence of positive linear operators from $C[a,b]$ into itself.*
Then

$$(NE)_\lambda(st) - \lim_k \parallel T_k(f;x) - f(x) \parallel_\infty = 0, for\ all\ f \in C[a,b] \tag{3.1}$$

if and only if

$$(NE)_\lambda(st) - \lim_k \parallel T_k(f;x) - f(x) \parallel_\infty = 0, for\ i = 0,1,2 \tag{3.2}$$

where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

Proof. Since each $1, x, x^2$ belong to $C[a,b]$, (3.1) follows immediately from (3.2). Let $f \in C[a,b]$. Then there exists a constant $M > 0$ such that $|f(x)| \leq M for\ all -\infty < x < +\infty$. Therefore

$$|f(t) - f(x)| \leq 2M, -\infty < x < +\infty \tag{3.3}$$

Let $\varepsilon > 0$. By hypothesis there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|f(t) - f(x)| < \varepsilon, \forall |t - x| < \delta. \tag{3.4}$$

Using (3.3) and (3.4) and putting $\psi(t) = (t-x)^2$, we get

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}\psi, \forall |t - x| < \delta. \tag{3.5}$$

These means

$$-\varepsilon - \frac{2M}{\delta^2}\psi < f(t) - f(x) < \varepsilon + \frac{2M}{\delta^2}\psi. \tag{3.6}$$

Now, we operating $T_k(1,x)$ to this inequality since $T_k(f;x)$ is monotone and linear. Hence

$$T_k(1,x)(-\varepsilon - \frac{2M}{\delta^2}\psi) < T_k(1,x)(f(t) - f(x)) < T_k(1,x)(\varepsilon + \frac{2M}{\delta^2}\psi). \tag{3.7}$$

Note that x is fixed and so $f(x)$ is constant number. Therefore

$$-\varepsilon T_k(1,x) - \frac{2M}{\delta^2}T_k(\psi,x) < T_k(f,x) - f(x)T_k(1,x) < \varepsilon T_k(1,x) + \frac{2M}{\delta^2}T_k(\psi,x). \tag{3.8}$$

But

$$\begin{aligned} T_k(f,x) - f(x) &= T_k(f,x) - f(x)T_k(1,x) + f(x)T_k(1,x) - f(x) = \\ &= [T_k(f,x) - f(x)T_k(1,x)] + f(x)[T_k(1,x) - 1]. \end{aligned} \tag{3.9}$$

Using (3.8) in (3.9), we obtain

$$T_k(f,x) - f(x) < \varepsilon T_k(1,x) + \frac{2M}{\delta^2}T_k(\psi,x) + f(x)(T_k(1,x) - 1). \tag{3.10}$$

Let us estimate $T_k(\psi,x)$

$$T_k(\psi,x) = T_k((t-x)^2,x) = T_k(t^2-2tx+x^2,x) = T_k(t^2,x) - 2xT_k(t,x) + x^2T_k(1,x) = \\ = [T_k(t^2,x)-x^2] + 2x[T_k(t,x)-x] + x^2[T_k(1,x)-1]. \tag{3.11}$$

Substituting the value in (3.10), we get

$$T_k(f,x) - f(x) < \varepsilon T_k(1,x) + \frac{2M}{\delta^2}\left\{[T_k(t^2,x)-x^2] + 2x[T_k(t,x)-x] + x^2[T_k(1,x)-1]\right\} + \\ +f(x)(T_k(1,x)-1) = \\ = \varepsilon(T_k(1,x)-1) + \varepsilon + \frac{2M}{\delta^2}\left\{[T_k(t^2,x)-x^2] + 2x[T_k(t,x)-x] + x^2[T_k(1,x)-1]\right\} + \\ +f(x)(T_k(1,x)-1). \tag{3.12}$$

Therefore

$$|T_k(f,x)-f(x)| \le \varepsilon + \left(\varepsilon + \frac{2Mb^2}{\delta^2} + M\right)|T_k(1,x)-1| + \frac{2M}{\delta^2}|T_k(t^2,x)-x^2| + \\ + \frac{4Mb}{\delta^2}|T_k(t,x)-x|. \tag{3.13}$$

where $b = max|x|$. Now taking supremum over $x \in [a,b]$, we get

$$\| T_k(f;x) - f(x) \|_\infty \le \varepsilon + K\sum_{i=0}^{2} \| T_k(f_i;x) - f_i(x) \|_\infty . \tag{3.14}$$

where $K = max\left\{\varepsilon + \frac{2Mb^2}{\delta^2} + M, \frac{2M}{\delta^2}, \frac{4Mb}{\delta^2}\right\}$. Hence

$$\| T_k(f;x)p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu} - f(x) \|_\infty \le \\ \le \varepsilon + K\sum_{i=0}^{2} \| T_k(f_i;x)p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu} - f_i(x) \|_\infty \tag{3.15}$$

Now replace $T_k(.;x)p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-v}$ by

$$L_m(.;x) = \frac{1}{R_{\lambda m}}\sum_{k\in I_m} T_k(.;x)p_{n-k}q_k\frac{1}{(1+q)^k}\sum_{\nu\in I_k}\binom{k}{\nu}q^{k-\nu} \tag{3.16}$$

in (3.15). For given $r > 0$ choose a $\eta > 0$ such that $\eta < r$ and define the following sets:

$$D = \{m \in N : \| L_m(f;x) - f(x) \|_\infty \ge r\}$$

$$D_i = \left\{m \in N : \| L_m(f_i;x) - f_i(x) \|_\infty \ge \frac{r-\eta}{3K}\right\}, i = 0,1,2.$$

Then $D \subset {}_{i=0}^{2}D_i$ and hence using conditions (3.2), we get

$$(NE)_\lambda(st) - lim_k \parallel T_k(f;x) - f(x) \parallel_\infty = 0, for\, all f \in C[a,b] \tag{3.17}$$

These completes the proof of the theorem. □

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