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# COMMON FIXED POINT FOR KANNAN TYPE CONTRACTIONS VIA INTERPOLATION

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Abstract. In this paper, we use interpolation to obtain a common fixed point result for a new type of Kannan contraction mappings.

## 1. Introduction and Preliminaries

In 1968 Kannan introduced an interesting type of contraction mapping which is not continuous and it posses a fixed point [1]. Kannan's theorem asserts that if $\mathcal{M}$ is a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ is a mapping satisfying the following condition

$$d\,(Tp, Tq) \leq \lambda\,[d(p, Tp) + d(q, Tq)]\,.$$

for all $p, q \in \mathcal{M}$, where $\lambda \in [0, \frac{1}{2})$. Then $T$ has a unique fixed point. Kannan's theorem has been generalized in different ways by many authors; one of the latest generalizations was given by Karapınar in [2]. Karapınar introduced a Kannan type contraction mapping called *interpolative Kannan type contraction* and proved a fixed point result on it.

**Difinition 1.1.** [2] *Let $(\mathcal{M}, d)$ be a metric space. A self mapping $T : \mathcal{M} \to \mathcal{M}$ is said to be an* interpolative Kannan type contraction *if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Tp, Tq) \leq \lambda\,[d(p, Tp)]^{\alpha}\,[d(q, Tq)]^{1-\alpha}\,.$$

**Theorem 1.2.** [2] *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ be an interpolative Kannan type contraction mapping. Then, $T$ has a unique fixed point.*

## 2. Main result

In this section we are following Karapınar's result in [2] to obtain a common fixed point result.

**Theorem 2.1.** *Let $\mathcal{M}$ be a complete metric space, $S, T : \mathcal{M} \to \mathcal{M}$ be self mappings. Assume that there are some $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ such that the condition*

$$d\,(Tp, Sq) \leq \lambda\,[d\,(p, Tp)]^{\alpha}\,[d\,(q, Sq)]^{1-\alpha}\,. \tag{2.1}$$

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*is satisfied for all $p, q \in \mathcal{M}$ such that $Tp \neq p$ whenever $Sq \neq q$. Then, $S$ and $T$ have a unique common fixed point.*

*Proof.* Let $p_0 \in \mathcal{M}$, define the sequence $\{p_n\}$ by

$$p_{2n+1} = Tp_{2n}, \;\; p_{2n+2} = Sp_{2n+1} \quad \forall n \in \{0, 1, 2, \cdots\}.$$

If there exists $n \in \{0, 1, 2, \cdots\}$ such that $p_n = p_{n+1} = p_{n+2}$, then $p_n$ is a common fixed point of $S$ and $T$; so suppose that there does not exist three consecutive identical terms in the sequence $\{p_n\}$ and that $p_0 \neq p_1$.

Now, using (2.1) we deduce that

$$\begin{aligned} d(p_{2n+1}, p_{2n+2}) &= d(Tp_{2n}, Sp_{2n+1}) \\ &\leq \lambda \left[d(p_{2n}, p_{2n+1})\right]^{\alpha} \left[d(p_{2n+1}, p_{2n+2})\right]^{1-\alpha}. \end{aligned}$$

Thus,

$$\left[d(p_{2n+1}, p_{2n+2})\right]^{\alpha} \leq \lambda \left[d(p_{2n}, p_{2n+1})\right]^{\alpha};$$

or,

$$\begin{aligned} d(p_{2n+1}, p_{2n+2}) &\leq \lambda^{\frac{1}{\alpha}} d(p_{2n}, p_{2n+1}) \\ &\leq \lambda d(p_{2n}, p_{2n+1}). \end{aligned}$$

Hence,

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda d(p_{2n}, p_{2n+1}) \leq \lambda^2 d(p_{2n-1}, p_{2n}) \leq \lambda^3 d(p_{2n-2}, p_{2n-1}) \cdots \leq \lambda^{2n+1} d(p_0, p_1);$$

or,

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda^{2n+1} d(p_0, p_1). \tag{2.2}$$

Similarly,

$$\begin{aligned} d(p_{2n+1}, p_{2n}) &= d(Tp_{2n}, Sp_{2n-1}) \\ &\leq \lambda \left[d(p_{2n}, p_{2n+1})\right]^{\alpha} \left[d(p_{2n-1}, p_{2n})\right]^{1-\alpha}. \end{aligned}$$

Thus,

$$\left[d(p_{2n+1}, p_{2n})\right]^{1-\alpha} \leq \lambda \left[d(p_{2n-1}, p_{2n})\right]^{1-\alpha};$$

or,

$$\begin{aligned} d(p_{2n+1}, p_{2n}) &\leq \lambda^{\frac{1}{1-\alpha}} d(p_{2n-1}, p_{2n}) \\ &\leq \lambda d(p_{2n-1}, p_{2n}). \end{aligned}$$

Hence,

$$d(p_{2n+1}, p_{2n}) \leq \lambda d(p_{2n-1}, p_{2n}) \leq \lambda^2 d(p_{2n-2}, p_{2n-1}) \leq \lambda^3 d(p_{2n-3}, p_{2n-2}) \cdots \leq \lambda^{2n} d(p_0, p_1).$$

Thus,

$$d(p_{2n+1}, p_{2n}) \leq \lambda^{2n} d(p_0, p_1). \tag{2.3}$$

From (2.2) and (2.3) we can deduce that

$$d(p_n, p_{n+1}) \leq \lambda^{n} d(p_0, p_1). \tag{2.4}$$

Now, using (2.4) we prove that the sequence $\{p_n\}$ is a Cauchy sequence. Let $m, r \in \{0, 1, 2, \cdots\}$

$$\begin{aligned} d(p_m, p_{m+r}) &\leq d(p_m, p_{m+1}) + d(p_{m+1}, p_{m+2}) + \cdots + d(p_{m+r-1}, p_{m+r}) \\ &\leq \left[\lambda^m + \lambda^{m+1} + \cdots \lambda^{m+r-1}\right] d(p_0, p_1) \\ &\leq \left[\lambda^m + \lambda^{m+1} + \cdots \lambda^{m+r-1} + \cdots\right] d(p_0, p_1) \\ &= \frac{\lambda^m}{1-\lambda} d(p_0, p_1). \end{aligned}$$

Letting $m \longrightarrow \infty$, we deduce that $\{p_n\}$ is a Cauchy sequence.

As $\mathcal{M}$ is complete, so there exists $u \in \mathcal{M}$ such that $\lim_{n\to\infty} p_n = u$. Using the continuity of the metric in its both variables we can prove that $u$ ia a fixed point of $T$ as follows

$$\begin{aligned} d(Tu, p_{2n+2}) &= d(Tu, Sp_{2n+1}) \\ &\leq \lambda [d(u, Tu)]^{\alpha} [d(p_{2n+1}, p_{2n+2})]^{1-\alpha}. \end{aligned}$$

Letting $n \longrightarrow \infty$ we get $d(Tu, u) = 0$, so $Tu = u$. Similarly,

$$\begin{aligned} d(p_{2n+1}, Su) &= d(Tp_{2n}, Su) \\ &\leq \lambda [d(p_{2n}, p_{2n+1})]^{\alpha} [d(u, Su)]^{1-\alpha}. \end{aligned}$$

Letting $n \longrightarrow \infty$ we get $u = Su$.

To prove that $u$ is the unique common fixed point of $S$ and $T$, suppose that $v$ is another common fixed point of $S$ and $T$, then

$$d(u, v) = d(Tu, Sv) \leq \lambda [d(u, Tu)]^{\alpha} [d(v, Sv)]^{1-\alpha} = 0.$$

Hence, $u = v$. $\square$

Now, we give an example of the previous result using a metric defined in [2].

**Example 2.2.** *Let $\mathcal{M} = \{p, q, z, w\}$ , define a metric $d$ on $\mathcal{M}$ as follows*

$$\begin{aligned} d(p,p) = d(q,q) = d(z,z) = d(w,w) &= 0 \\ d(p,q) = d(q,p) &= 3 \\ d(z,p) = d(p,z) &= 4 \\ d(q,z) = d(z,q) &= \tfrac{3}{2} \\ d(w,p) = d(p,w) &= \tfrac{5}{2} \\ d(w,q) = d(q,w) &= 2 \\ d(w,z) = d(z,w) &= \tfrac{3}{2} \end{aligned}$$

*Define self maps $T, S$ as follows*

$$T : \begin{pmatrix} p & q & z & w \\ p & w & z & w \end{pmatrix}, \; S : \begin{pmatrix} p & q & z & w \\ p & q & w & z \end{pmatrix}$$

*It is clear that $S, T$ satisfies (2.1) with $\lambda = \frac{9}{10}$ and $\alpha = \frac{1}{2}$, and $S$ and $T$ has a unique common fixed point $p$.*

## Conclusion

We can obtain more common fixed point results in similar ways and use them in more applications.

## References


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