

**NEW CONFORMABLE FRACTIONAL HERMITE HADAMARD  
 TYPE INEQUALITIES FOR HARMONICALLY CONVEX  
 FUNCTIONS**

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**ABSTRACT.** In this paper, we prove three new conformable fractional Hermite-Hadamard type inequalities for harmonically convex functions by using the left and the right fractional integrals independently. Also, we give two new conformable fractional identities for differentiable functions. By using these identities, we obtained some new trapezoidal type inequalities for harmonically convex functions. Our results generalizes the results given by İşcan in [5] and Şanlı et al. in [18].

1. INTRODUCTION

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

is well known in the literature as Hermite-Hadamard's inequality. There are so many generalizations and extensions of inequalities (1.1) for various classes of functions. One of this classes of functions is harmonically convex functions defined by İşcan.

In [5], İşcan gave the definition of harmonically convex functions as follows:

**Definition 1.1.** [5] Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (1.2)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1.2) is reversed, then  $f$  is said to be harmonically concave.

For some similar studies with this work about harmonically convex functions, readers can see [2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 15, 18] and references therein.

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In [5], İşcan gave Hermite-Hadamard type inequalities for harmonically convex functions as follows:

**Theorem 1.2.** [5] Let  $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  then the following inequalities hold:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \quad (1.3)$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

**Definition 1.3.** Let  $a, b \in \mathbb{R}$  with  $a < b$  and  $f \in L[a, b]$ . The left and right Riemann-Liouville fractional integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where  $\Gamma(\alpha)$  is the Gamma function defined by  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$  (see [8, page 69]).

In [7], İşcan and Wu presented Hermite-Hadamard type inequalities for harmonically convex functions in fractional integral forms as follows.

**Theorem 1.4.** Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a function such that  $f \in L[a, b]$ , where  $a, b \in I$  with  $a < b$ . If  $f$  is a harmonically convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left[ J_{1/a-}^\alpha (f \circ h)(1/b) + J_{1/b+}^\alpha (f \circ h)(1/a) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (1.4)$$

with  $\alpha > 0$  and  $h(x) = 1/x$ .

In [18], Şanlı et al. proved the following three Riemann-Liouville fractional Hermite-Hadamard type inequalities for harmonically convex functions by using the left and the right fractional integrals separately as follows:

**Theorem 1.5.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the left Riemann-Liouville fractional integral holds:

$$f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right) \leq \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^\alpha J_{\frac{1}{a}+}^\alpha (f \circ h)\left(\frac{1}{b}\right) \leq \frac{f(a) + \alpha f(b)}{\alpha+1} \quad (1.5)$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

**Theorem 1.6.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the right Riemann-Liouville fractional integral holds:

$$f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right) \leq \Gamma(\alpha+1) \left(\frac{ab}{b-a}\right)^\alpha J_{\frac{1}{b}-}^\alpha (f \circ h)\left(\frac{1}{a}\right) \leq \frac{\alpha f(a) + f(b)}{\alpha+1} \quad (1.6)$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

**Theorem 1.7.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the left Riemann-Liouville fractional integral holds:

$$\begin{aligned} \frac{f\left(\frac{(\alpha+1)ab}{a+\alpha b}\right) + f\left(\frac{(\alpha+1)ab}{\alpha a+b}\right)}{2} &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left[ J_{\frac{1}{b}-}^\alpha (f \circ h)(\frac{1}{a}) + J_{\frac{1}{a}+}^\alpha (f \circ h)(\frac{1}{b}) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (1.7)$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

The following definition of conformable fractional integrals could be found in [1, 2, 17].

**Definition 1.8.** Let  $\alpha \in (n, n+1]$ ,  $n = 0, 1, 2, \dots$ ,  $\beta = \alpha - n$ ,  $a, b \in \mathbb{R}$  with  $a < b$  and  $f \in L[a, b]$ . The left and right conformable fractional integrals  $I_\alpha^a f$  and  $I_\alpha^b f$  of order  $\alpha > 0$  are defined by

$$I_\alpha^a f(x) = \frac{1}{n!} \int_a^x (x-t)^n (t-a)^{\beta-1} f(t) dt, \quad x > a$$

and

$$I_\alpha^b f(x) = \frac{1}{n!} \int_x^b (t-x)^n (b-t)^{\beta-1} f(t) dt, \quad x < b$$

respectively.

It is easily seen that if one takes  $\alpha = n+1$  in the Definition 1.8 (for the left and right conformable foractional integrals), one has the Definition 1.3 (the left and right Riemann-Liouville fractional integrals) for  $\alpha \in \mathbb{N}$ .

In [2, Theorem 2.1], Awan et al. presented Hermite-Hadamard type inequalities for harmonically convex functions in conformable fractional integral forms as follows.

**Theorem 1.9.** Let  $I = [a, b] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonic convex function such that  $f \in L_1[a, b]$ , then

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{2\Gamma(\alpha-n)}{\Gamma(\alpha+1)} \left(\frac{ab}{b-a}\right)^\alpha \left[ I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) + I_\alpha^{\frac{1}{b}}(f \circ h)(\frac{1}{a}) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (1.8)$$

where  $h(x) = 1/x$ .

We noticed that the correct inequality should be as follows:

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2\Gamma(\alpha-n)} \left(\frac{ab}{b-a}\right)^\alpha \left[ I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) + I_\alpha^{\frac{1}{b}}(f \circ h)(\frac{1}{a}) \right] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (1.9)$$

and  $I = [a, b] \subset (0, \infty)$ .

In (1.9), if one takes  $\alpha = 1$ , one obtaines the inequality (1.4) in the Theorem 1.4.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals. The papers [2, 6, 7, 9, 10, 11, 12, 13, 15, 17, 18] are based on Hermite-Hadamard type inequalities involving several fractional integrals.

We recall the following inequality and special functions which are known as Beta, incomplete Beta and hypergeometric functions respectively

$$\begin{aligned}\beta(x, y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \\ \beta_w(x, y) &= \int_0^w t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad 0 < w \leq 1, \\ {}_2F_1(a, b; c; z) &= \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \\ c > b > 0, |z| < 1. &\quad (\text{see [8]})\end{aligned}$$

The following properties of convex functions are used for forward results.

**Definition 1.10.** [16, page 12] A function  $f$  defined on  $I$  has a support at  $x_0 \in I$  if there exists an affine functions  $A(x) = f(x_0) + m(x - x_0)$  such that  $A(x) \leq f(x)$  for all  $x \in I$ . The graph of the support function  $A$  is called a line of support for  $f$  at  $x_0$ .

**Theorem 1.11.** [16, page 12]  $f : (a, b) \rightarrow \mathbb{R}$  is a convex function if and only if there is at least one line of support for  $f$  at each  $x_0 \in (a, b)$ .

**Remark.** [4] Let  $[a, b] \subset I \subseteq (0, \infty)$ , if the function  $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$  defined  $g(x) = f(\frac{1}{x})$ , then  $f$  is harmonically convex on  $[a, b]$  if and only if  $g$  is convex on  $[\frac{1}{b}, \frac{1}{a}]$ .

In literature, there are so many studies for Hermite-Hadamard type inequalities by using the left and right fractional integrals (such as Riemann-Liouville fractional integrals, Hadamard fractional integrals, Conformable fractional integrals etc.). In all of them, the left and right fractional integrals are used together. As much as we know, the studies [12, 13] are the first two works by using only the right fractional integrals or the left fractional integrals.

In this paper, our aim is to obtain new conformable fractional Hermite-Hadamard type inequalities by using only the right or the left fractional integrals separately for harmonically convex functions.

## 2. CONFORMABLE FRACTIONAL HERMITE HADAMARD TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS

**Theorem 2.1.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the left conformable fractional integral holds:

$$\begin{aligned}f\left(\frac{(\alpha+1)ab}{(n+1)a+(\alpha-n)b}\right) &\leq \left(\frac{ab}{b-a}\right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} I_{\alpha}^{\frac{1}{b}}(f \circ h)(\frac{1}{b}) \\ &\leq \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}\end{aligned}\tag{2.1}$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

*Proof.* Let  $\alpha > 0$ . Since  $f$  is harmonically convex on  $[a, b]$  by using Remark 1,  $g(x) = f(\frac{1}{x})$  is convex on  $[\frac{1}{b}, \frac{1}{a}]$ . Hence, using Theorem 1.11, there is at least one line of support

$$A(x) = g\left(\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}\right) + m\left(x - \frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}\right) \leq g(x) \quad (2.2)$$

for all  $x \in [\frac{1}{b}, \frac{1}{a}]$  and  $m \in \left[g'_- \left(\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}\right), g'_+ \left(\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}\right)\right]$ . From (2.2) and harmonically convexity of  $f$ , we have

$$\begin{aligned} A\left(\frac{ta + (1-t)b}{ab}\right) &= f\left(\frac{(\alpha+1)ab}{(n+1)a + (\alpha-n)b}\right) + m\left(-\frac{\frac{ta+(1-t)b}{ab}}{-\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}}\right) \\ &\leq f\left(\frac{ab}{ta + (1-t)b}\right) \leq tf(b) + (1-t)f(a) \end{aligned} \quad (2.3)$$

for all  $t \in [0, 1]$ . Multiplying all sides of (2.3) with  $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$  and integrating over  $[0, 1]$  respect to  $t$ , we have

$$\begin{aligned} &\frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} \left[ f\left(\frac{(\alpha+1)ab}{(n+1)a + (\alpha-n)b}\right) + m\left(-\frac{\frac{ta+(1-t)b}{ab}}{-\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}}\right) \right] dt \\ &= f\left(\frac{(\alpha+1)ab}{(n+1)a + (\alpha-n)b}\right) \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} dt \\ &\quad + \frac{m}{n!} \left[ -\frac{\int_0^1 t^n (1-t)^{\alpha-n-1} \frac{ta+(1-t)b}{ab} dt}{-\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab}} \int_0^1 t^n (1-t)^{\alpha-n-1} dt \right] \\ &= f\left(\frac{(\alpha+1)ab}{(n+1)a + (\alpha-n)b}\right) \frac{B(n+1, \alpha-n)}{n!} \\ &\quad + \frac{m}{n!} \left[ \frac{\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab} B(n+1, \alpha-n)}{-\frac{(n+1)a + (\alpha-n)b}{(\alpha+1)ab} B(n+1, \alpha-n)} \right] \\ &= f\left(\frac{(\alpha+1)ab}{(n+1)a + (\alpha-n)b}\right) \frac{B(n+1, \alpha-n)}{n!} \\ &\leq \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} f\left(\frac{ab}{ta + (1-t)b}\right) dt \\ &= \left(\frac{ab}{b-a}\right)^\alpha \frac{1}{n!} \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - t\right)^n \left(t - \frac{1}{b}\right)^{\alpha-n-1} f\left(\frac{1}{t}\right) dt \\ &= \left(\frac{ab}{b-a}\right)^\alpha I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) \\ &\leq f(b) \frac{1}{n!} \int_0^1 t^{n+1} (1-t)^{\alpha-n-1} dt + f(a) \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n} dt \\ &= \frac{(\alpha-n)f(a) + (n+1)f(b)}{\alpha+1} \frac{B(n+1, \alpha-n)}{n!} \end{aligned}$$

It means that

$$\begin{aligned} f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) &\leq \frac{n!}{B(n+1,\alpha-n)}\left(\frac{ab}{b-a}\right)^\alpha I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) \\ &\leq \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} \end{aligned} \quad (2.4)$$

Since  $\frac{n!}{B(n+1,\alpha-n)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}$ , by using (2.4), we have (2.1). This completes the proof.  $\square$

**Remark.** In Theorem 2.1,

- (1) If one takes  $\alpha = n+1$ , one has the inequality (1.5).
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality (1.3).

**Theorem 2.2.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the right conformable fractional integral holds:

$$\begin{aligned} f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) &\leq \left(\frac{ab}{b-a}\right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \frac{1}{b} I_\alpha(f \circ h)(\frac{1}{a}) \\ &\leq \frac{(n+1)f(a)+(\alpha-n)f(b)}{\alpha+1} \end{aligned} \quad (2.5)$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

*Proof.* Let  $\alpha > 0$ . Since  $f$  is harmonically convex on  $[a, b]$  by using Remark 1,  $g(x) = f(\frac{1}{x})$  is convex on  $[\frac{1}{b}, \frac{1}{a}]$ . Hence, using Theorem 1.11, there is at least one line of support

$$A(x) = g\left(\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}\right) + m\left(x - \frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}\right) \leq g(x) \quad (2.6)$$

for all  $x \in [\frac{1}{b}, \frac{1}{a}]$  and  $m \in [g'_-(\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}), g'_+(\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab})]$ . From (2.6) and harmonically convexity of  $f$ , we have

$$\begin{aligned} A\left(\frac{tb+(1-t)a}{ab}\right) &= f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) + m\left(-\frac{\frac{tb+(1-t)a}{ab}}{-\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}}\right) \\ &\leq f\left(\frac{ab}{tb+(1-t)a}\right) \leq tf(a) + (1-t)f(b) \end{aligned} \quad (2.7)$$

for all  $t \in [0, 1]$ . Multiplying all sides of (2.7) with  $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$  and integrating over  $[0, 1]$  respect to  $t$ , we have

$$\begin{aligned} \frac{1}{n!} \int_0^1 t^n(1-t)^{\alpha-n-1} &\left[ f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) + m\left(-\frac{\frac{tb+(1-t)a}{ab}}{-\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}}\right) \right] dt \\ &= f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) \frac{1}{n!} \int_0^1 t^n(1-t)^{\alpha-n-1} dt \\ &\quad + \frac{m}{n!} \left[ -\frac{\int_0^1 t^n(1-t)^{\alpha-n-1} \frac{tb+(1-t)a}{ab} dt}{-\frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab}} \int_0^1 t^n(1-t)^{\alpha-n-1} dt \right] \end{aligned}$$

$$\begin{aligned}
&= f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) \frac{B(n+1, \alpha-n)}{n!} \\
&\quad + \frac{m}{n!} \left[ \begin{array}{l} \frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab} B(n+1, \alpha-n) \\ - \frac{(\alpha-n)a+(n+1)b}{(\alpha+1)ab} B(n+1, \alpha-n) \end{array} \right] \\
&= f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) \frac{B(n+1, \alpha-n)}{n!} \\
&\leq \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n-1} f\left(\frac{ab}{tb+(1-t)a}\right) dt \\
&= \left(\frac{ab}{b-a}\right)^\alpha \frac{1}{n!} \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t - \frac{1}{b}\right)^n \left(\frac{1}{a} - t\right)^{\alpha-n-1} f\left(\frac{1}{t}\right) dt \\
&= \left(\frac{ab}{b-a}\right)^\alpha I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) . \\
&\leq f(b) \frac{1}{n!} \int_0^1 t^{n+1} (1-t)^{\alpha-n-1} dt + f(a) \frac{1}{n!} \int_0^1 t^n (1-t)^{\alpha-n} dt \\
&= \frac{1}{n!} \frac{(\alpha-n)f(a) + (n+1)f(b)}{\alpha+1} B(n+1, \alpha-n)
\end{aligned}$$

It means that

$$\begin{aligned}
f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right) &\leq \frac{n!}{B(n+1, \alpha-n)} \left(\frac{ab}{b-a}\right)^{\alpha-\frac{1}{b}} I_\alpha(f \circ h)(\frac{1}{a}) \\
&= \frac{(\alpha-n)f(a) + (n+1)f(b)}{\alpha+1}
\end{aligned} \tag{2.8}$$

Since  $\frac{n!}{B(n+1, \alpha-n)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)}$ , by using (2.8), we have (2.5). This completes the proof.  $\square$

**Remark.** In Theorem 2.2,

- (1) If one takes  $\alpha = n+1$ , one has the inequality (1.6).
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality (1.3).

**Theorem 2.3.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$ , then the following inequality for the Riemann-Liouville fractional integrals holds:

$$\begin{aligned}
\frac{f\left(\frac{(\alpha+1)ab}{(n+1)a+(\alpha-n)b}\right) + f\left(\frac{(\alpha+1)ab}{(\alpha-n)a+(n+1)b}\right)}{2} &\leq \frac{1}{2} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \left(\frac{ab}{b-a}\right)^\alpha \\
&\quad \times \left[ I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{b}) + \frac{1}{b} I_\alpha^{\frac{1}{a}}(f \circ h)(\frac{1}{a}) \right] \\
&\leq \frac{f(a) + f(b)}{2}
\end{aligned} \tag{2.9}$$

where  $h(x) = \frac{1}{x}$  and  $\alpha > 0$ .

*Proof.* Adding the inequalities (2.1) and (2.7) side by side, then multiplying the resulting inequalities by  $\frac{1}{2}$ , we have the inequalities (2.9).  $\square$

**Remark.** In Theorem 2.3;

- (1) If one takes  $\alpha = n + 1$ , one has the inequality (1.7).
- (2) If one takes  $\alpha = n + 1$ , after that if one takes  $\alpha = 1$ , one has the inequality (1.3).

**Corollary 2.4.** The left hand side of (2.9) is better than the left hand side of (1.9).

*Proof.* Since  $f$  is harmonically convex on  $[a, b]$ , it is clear from

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &= f\left(\frac{1}{\frac{a+b}{2ab}}\right) = f\left(\frac{1}{\frac{(\alpha+1)(a+b)}{2ab(\alpha+1)}}\right) = f\left(\frac{\frac{1}{(\alpha+1)a+(\alpha-n)b}}{\frac{(n+1)a+(\alpha-n)b}{2ab(\alpha+1)} + \frac{(\alpha-n)a+(n+1)b}{2ab(\alpha+1)}}\right) \\ &= f\left(\frac{\frac{ab(\alpha+1)}{(n+1)a+(\alpha-n)b} \frac{ab(\alpha+1)}{(\alpha-n)a+(n+1)b}}{\frac{ab(\alpha+1)}{(n+1)a+(\alpha-n)b} \frac{1}{2} + \frac{ab(\alpha+1)}{(\alpha-n)a+(n+1)b} \frac{1}{2}}\right) \leq \frac{f\left(\frac{ab(\alpha+1)}{(\alpha-n)a+(n+1)b}\right) + f\left(\frac{ab(\alpha+1)}{(n+1)a+(\alpha-n)b}\right)}{2}. \end{aligned}$$

□

### 3. LEMMAS

In this section we will prove two new identities used in forward results.

**Lemma 3.1.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$ , then the following equality for the left conformable fractional integrals holds:

$$\begin{aligned} &\frac{(\alpha - n)f(a) + (n + 1)f(b)}{\alpha + 1} - \left(\frac{ab}{b-a}\right)^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n)} I_{\alpha}^{\frac{1}{\alpha}}(f \circ h)(\frac{1}{b}) \\ &= \frac{ab(b-a)}{(\alpha + 1)B(n+1, \alpha - n)} \int_0^1 \left[ \begin{array}{c} (\alpha + 1)B_t(n+1, \alpha - n) \\ -(\alpha - n)B(n+1, \alpha - n) \end{array} \right] \\ &\quad \times \frac{1}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \end{aligned} \tag{3.1}$$

with  $n = 0, 1, 2, \dots$  and  $\alpha \in (n, n + 1]$ .

*Proof.* It could be prove directly by applying the partial integration to the right hand side of the equation (3.1) as follows:

$$\begin{aligned} &\frac{ab(b-a)}{(\alpha + 1)B(n+1, \alpha - n)} \int_0^1 \left[ \begin{array}{c} (\alpha + 1)B_t(n+1, \alpha - n) \\ -(\alpha - n)B(n+1, \alpha - n) \end{array} \right] \\ &\quad \times \frac{1}{(ta + (1-t)a)^2} f'\left(\frac{ab}{ta + (1-t)a}\right) dt \\ &= \frac{ab(b-a)}{(\alpha + 1)} \int_0^1 \left[ (\alpha + 1) \frac{B_t(n+1, \alpha - n)}{B(n+1, \alpha - n)} - (\alpha - n) \right] \\ &\quad \times \frac{1}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \\ &= \left[ \begin{array}{c} \frac{1}{B(n+1, \alpha - n)} \int_0^1 B_t(n+1, \alpha - n) \frac{ab(b-a)}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \\ -\frac{\alpha - n}{\alpha + 1} \int_0^1 \frac{ab(b-a)}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \end{array} \right] \\ &= \left[ \begin{array}{c} \frac{1}{B(n+1, \alpha - n)} \int_0^1 \left( \int_0^t x^n (1-x)^{\alpha - n - 1} dx \right) \frac{ab(b-a)}{(ta + (1-t)b)^2} f'\left(\frac{ab}{ta + (1-t)b}\right) dt \\ -\frac{\alpha - n}{\alpha + 1} f\left(\frac{ab}{ta + (1-t)b}\right) \Big|_0^1 \end{array} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{1}{B(n+1, \alpha-n)} \begin{pmatrix} \left( \int_0^t x^n (1-x)^{\alpha-n-1} dx \right) f \left( \frac{ab}{ta+(1-t)b} \right) \Big|_0^1 \\ - \int_0^1 t^n (1-t)^{\alpha-n-1} f \left( \frac{ab}{ta+(1-t)b} \right) dt \\ - \frac{\alpha-n}{\alpha+1} (f(b) - f(a)) \end{pmatrix} \right] \\
&= \left[ \frac{1}{B(n+1, \alpha-n)} \left( (B(n+1, \alpha-n)) f(b) - \int_0^1 t^n (1-t)^{\alpha-n-1} f \left( \frac{ab}{ta+(1-t)b} \right) dt \right) + \frac{\alpha-n}{\alpha+1} (f(a) - f(b)) \right] \\
&= \left[ - \frac{\frac{\alpha-n}{\alpha+1} (f(a) - f(b)) + f(b)}{\frac{1}{B(n+1, \alpha-n)}} \int_0^1 t^n (1-t)^{\alpha-n-1} f \left( \frac{ab}{ta+(1-t)b} \right) dt \right] \\
&= \left[ - \frac{\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}}{\frac{1}{B(n+1, \alpha-n)}} \int_0^1 t^n (1-t)^{\alpha-n-1} f \left( \frac{ab}{ta+(1-t)b} \right) dt \right] \\
&= \frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} I_\alpha^{\frac{1}{b}} f \circ h(\frac{1}{b}) .
\end{aligned}$$

This completes the proof.  $\square$

**Remark.** In Lemma 3.1,

- (1) If one takes  $\alpha = n+1$ , one has the inequality proved in [18, Lemma 3],
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, 2.5. Lemma].

**Lemma 3.2.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$ , then the following equality for the right conformable fractional integrals holds:

$$\begin{aligned}
&\frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} \frac{1}{b} I_\alpha^{\frac{1}{b}} (f \circ h)(\frac{1}{\alpha}) \quad (3.2) \\
&= \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} \int_0^1 \left[ \frac{(\alpha-n)B(n+1, \alpha-n)}{(\alpha+1)B_{1-t}(n+1, \alpha-n)} \right. \\
&\quad \times \left. \frac{1}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt \right]
\end{aligned}$$

with  $n = 0, 1, 2, \dots$  and  $\alpha \in (n, n+1]$ .

*Proof.* It could be prove directly by applying the partial integration to the right hand side of the equation (3.2) as follows:

$$\begin{aligned}
&\frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} \int_0^1 \left[ \frac{(\alpha-n)B(n+1, \alpha-n)}{(\alpha+1)B_{1-t}(n+1, \alpha-n)} \right. \\
&\quad \times \left. \frac{1}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt \right] \\
&= \frac{ab(b-a)}{(\alpha+1)} \int_0^1 \left[ (\alpha-n) - (\alpha+1) \frac{B_{1-t}(n+1, \alpha-n)}{B(n+1, \alpha-n)} \right] \\
&\quad \times \frac{1}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt
\end{aligned}$$

$$\begin{aligned}
&= \left[ \begin{array}{l} \frac{\alpha-n}{\alpha+1} \int_0^1 \frac{ab(b-a)}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt \\ -\frac{1}{B(n+1,\alpha-n)} \int_0^1 B_{1-t}(n+1,\alpha-n) \frac{ab(b-a)}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt \end{array} \right] \\
&= \left[ \begin{array}{l} \frac{\alpha-n}{\alpha+1} f \left( \frac{ab}{ta+(1-t)b} \right) \Big|_0^1 \\ -\frac{1}{B(n+1,\alpha-n)} \int_0^1 \left( \int_0^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{1}{(ta+(1-t)b)^2} f' \left( \frac{ab}{ta+(1-t)b} \right) dt \end{array} \right] \\
&= \left[ \begin{array}{l} \frac{\alpha-n}{\alpha+1} (f(b) - f(a)) \\ -\frac{1}{B(n+1,\alpha-n)} \left( \left( \int_0^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) f \left( \frac{ab}{ta+(1-t)b} \right) \Big|_0^1 + \int_0^1 t^n (1-t)^{\alpha-n-1} f \left( \frac{ab}{ta+(1-t)b} \right) dt \right) \end{array} \right] \\
&= \left[ \begin{array}{l} \frac{\alpha-n}{\alpha+1} (f(b) - f(a)) \\ -\frac{1}{B(n+1,\alpha-n)} \left( (B(n+1,\alpha-n)) f(a) + \int_0^1 t^{\alpha-n-1} (1-t)^n f \left( \frac{ab}{ta+(1-t)b} \right) dt \right) \end{array} \right] \\
&= \left[ \begin{array}{l} \frac{\alpha-n}{\alpha+1} (f(b) - f(a)) + f(a) \\ -\frac{1}{B(n+1,\alpha-n)} \int_0^1 t^{\alpha-n-1} (1-t)^n f \left( \frac{ab}{ta+(1-t)b} \right) dt \end{array} \right] \\
&= \left[ \begin{array}{l} \frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} \\ -\frac{1}{B(n+1,\alpha-n)} \int_0^1 t^{\alpha-n-1} (1-t)^n f \left( \frac{ab}{ta+(1-t)b} \right) dt \end{array} \right] \\
&= \frac{(n+1)f(a) + (\alpha-n)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} {}_b^I \! I_\alpha (f \circ h)(\frac{1}{\alpha}). 
\end{aligned}$$

This completes the proof.  $\square$

**Remark.** In Lemma 3.2,

- (1) If one takes  $\alpha = n+1$ , one has the inequality proved in [18, Lemma 2],
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, 2.5. Lemma].

#### 4. SOME NEW CONFORMABLE FRACTIONAL TRAPEZOID TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS

In this section, we will prove some new conformable fractional trapezoid type inequalities for harmonically convex functions by using Lemma 3.1 and Lemma 3.2.

**Theorem 4.1.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$  and  $|f'|^q$  harmonically convex on  $[a, b]$  for  $q \geq 1$ , then the following left conformable fractional integral inequality holds:

$$\begin{aligned}
&\left| \frac{(\alpha-n)f(a) + (n+1)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} {}_a^I \! I_\alpha^{\frac{1}{\alpha}} (f \circ h)(\frac{1}{b}) \right| \quad (4.1) \\
&\leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} Z_1^{1-\frac{1}{q}}(\alpha,n) (|f'(b)|^q Z_2(\alpha,n) + |f'(a)|^q Z_3(\alpha,n))^{\frac{1}{q}}
\end{aligned}$$

where

$$\begin{aligned}
Z_1(\alpha,n) &= \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \frac{1}{(ta+(1-t)b)^2} dt, \\
Z_2(\alpha,n) &= \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \frac{1}{(ta+(1-t)b)^2} t dt,
\end{aligned}$$

$$Z_3(\alpha, n) = Z_1(\alpha, n) - Z_2(\alpha, n),$$

with  $n = 0, 1, 2, \dots$  and  $\alpha \in (n, n+1]$ .

*Proof.* By using Lemma 3.1, power mean inequality and harmonically convexity of  $|f'|^q$ , we have,

$$\begin{aligned} & \left| \frac{(\alpha - n)f(a) + (n+1)f(b)}{\alpha + 1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} I_\alpha^{\frac{1}{\alpha}}(f \circ h)(\frac{1}{b}) \right| \\ & \leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} \int_0^1 \left| \frac{(\alpha+1)B_t(n+1,\alpha-n)}{-(\alpha-n)B(n+1,\alpha-n)} \right| \\ & \quad \times \frac{1}{(ta+(1-t)b)^2} \left| f' \left( \frac{ab}{ta+(1-t)b} \right) \right| dt \\ & \leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \times \frac{1}{(ta+(1-t)b)^2} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right|^q \times \frac{1}{(ta+(1-t)b)^2} \left| f' \left( \frac{ab}{ta+(1-t)b} \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \times \frac{1}{(ta+(1-t)b)^2} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right|^q \times \frac{1}{(ta+(1-t)b)^2} [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{\frac{1}{q}} \\ & \leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \times \frac{1}{(ta+(1-t)b)^2} dt \right)^{1-\frac{1}{q}} \\ & \quad \times \left( |f'(b)|^q \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \times \frac{1}{(ta+(1-t)b)^2} t dt \right)^{\frac{1}{q}} \\ & \quad + |f'(a)|^q \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right| \times \frac{1}{(ta+(1-t)b)^2} (1-t) dt \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof.  $\square$

**Remark.** In Theorem 4.1,

- (1) If one takes  $\alpha = n+1$ , one has the inequality proved in [18, Theorem 9],
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, Theorem 2.6].

**Theorem 4.2.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$  and  $|f'|^q$  harmonically convex on  $[a, b]$  for  $q > 1$  and  $\frac{1}{q} + \frac{1}{p} = 1$ , then the following left conformable fractional integral inequality holds

$$\left| \frac{(\alpha - n)f(a) + (n + 1)f(b)}{\alpha + 1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n)} I_{\alpha}^{\frac{1}{\alpha}}(f \circ h)(\frac{1}{b}) \right| \quad (4.2)$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} Z_4^{\frac{1}{p}}(\alpha, n) (|f'(b)|^q Z_5(a, b) + |f'(a)|^q Z_6(a, b))^{\frac{1}{q}}$$

where

$$Z_4(\alpha, n) = \int_0^1 |(\alpha - n)B(n+1, \alpha - n) - (\alpha + 1)B_t(n+1, \alpha - n)|^p dt,$$

$$Z_5(a, b) = \frac{b^{-2q}}{2} {}_2F_1(2q, 2; 3; 1 - \frac{a}{b}),$$

$$Z_6(a, b) = \frac{b^{-2q}}{2} {}_2F_1(2q, 1; 3; 1 - \frac{a}{b}),$$

with  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $n = 0, 1, 2, \dots$  and  $\alpha \in (n, n+1]$ .

*Proof.* By using Lemma 3.1, Hölder inequality and harmonically convexity of  $|f'|^q$ , we have

$$\left| \frac{(\alpha - n)f(a) + (n + 1)f(b)}{\alpha + 1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - n)} I_{\alpha}^{\frac{1}{\alpha}}(f \circ h)(\frac{1}{b}) \right| \quad (4.3)$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} \int_0^1 \left| \frac{(\alpha - n)B(n+1, \alpha - n)}{-(\alpha + 1)B_t(n+1, \alpha - n)} \right| \times \frac{1}{(ta + (1-t)b)^2} \left| f' \left( \frac{ab}{ta + (1-t)b} \right) \right| dt$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} \left( \int_0^1 \left| \frac{(\alpha - n)B(n+1, \alpha - n)}{-(\alpha + 1)B_t(n+1, \alpha - n)} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1}{(ta + (1-t)b)^{2q}} \left| f' \left( \frac{ab}{ta + (1-t)b} \right) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} \left( \int_0^1 \left| \frac{(\alpha - n)B(n+1, \alpha - n)}{-(\alpha + 1)B_t(n+1, \alpha - n)} \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \frac{1}{(ta + (1-t)b)^{2q}} [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} Z_4^{\frac{1}{p}}(\alpha, n) (|f'(b)|^q Z_5(a, b) + |f'(a)|^q Z_6(a, b))^{\frac{1}{q}}$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} \left( \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_t(n+1,\alpha-n)} \right|^p dt \right)^{\frac{1}{p}} \\ \times \left( \begin{array}{l} \int_0^1 \frac{1}{(ta+(1-t)b)^{2q}} t |f'(b)|^q dt \\ + \int_0^1 \frac{1}{(ta+(1-t)b)^{2q}} (1-t) |f'(a)|^q dt \end{array} \right)^{\frac{1}{q}}.$$

Calculating the appearing integrals in (4.3), we have

$$\begin{aligned} \int_0^1 \frac{t}{(ta+(1-t)b)^{2q}} dt &= b^{-2q} \int_0^1 t \left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q} dt \\ &= \frac{b^{-2q}}{2} {}_2F_1(2q, 2; 3; 1 - \frac{a}{b}) \\ &= Z_5(a, b) \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \int_0^1 \frac{t}{(ta+(1-t)b)^{2q}} dt &= b^{-2q} \int_0^1 (1-t) \left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q} dt \\ &= \frac{b^{-2q}}{2} {}_2F_1(2q, 1; 3; 1 - \frac{a}{b}) \\ &= Z_6(a, b), \end{aligned} \quad (4.5)$$

This completes the proof.  $\square$

**Remark.** In Theorem 4.2,

- (1) If one takes  $\alpha = n+1$ , one has the inequality proved in [18, Theorem 10],
- (2) If one takes  $\alpha = n+1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, Theorem 2.7].

**Theorem 4.3.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$  and  $|f'|^q$  harmonically convex on  $[a, b]$  for  $q \geq 1$ , then the following right conformable fractional integral inequality holds:

$$\begin{aligned} &\left| \frac{(n+1)f(a) + (n-\alpha)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} {}_{\frac{1}{b}}I_\alpha(f \circ h)(\frac{1}{a}) \right| \quad (4.6) \\ &\leq \frac{ab(b-a)}{(\alpha+1)B(n+1,\alpha-n)} Z_7^{1-\frac{1}{q}}(\alpha, n) (|f'(b)|^q Z_8(\alpha, n) + |f'(a)|^q Z_9(\alpha, n))^{\frac{1}{q}} \end{aligned}$$

where

$$\begin{aligned} Z_7(\alpha, n) &= \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_{1-t}(n+1,\alpha-n)} \right| \frac{dt}{(ta+(1-t)b)^2}, \\ Z_8(\alpha, n) &= \int_0^1 \left| \frac{(\alpha-n)B(n+1,\alpha-n)}{-(\alpha+1)B_{1-t}(n+1,\alpha-n)} \right| \frac{tdt}{(ta+(1-t)b)^2}, \end{aligned}$$

$$Z_9(\alpha, n) = Z_7(\alpha, n) - Z_8(\alpha, n),$$

with  $n = 0, 1, 2, \dots$  and  $\alpha \in (n, n+1]$ .

*Proof.* Similarly the proof of the Theorem 4.1, by using Lemma 3.2, power mean inequality and harmonically convexity of  $|f'|^q$ , we have (4.6).  $\square$

**Remark.** In Theorem 4.3,

- (1) If one takes  $\alpha = n+1$ , one has the inequality proved in [18, Theorem 7],

- (2) If one takes  $\alpha = n + 1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, Theorem 2.6].

**Theorem 4.4.** Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$ ,  $a, b \in I^\circ$  and  $a < b$ . If  $f' \in L[a, b]$  and  $|f'|^q$  harmonically convex on  $[a, b]$  for  $q > 1$  and  $\frac{1}{q} + \frac{1}{p} = 1$ , then the following right conformable fractional integral inequality holds

$$\left| \frac{(n+1)f(a) + (n-\alpha)f(b)}{\alpha+1} - \left( \frac{ab}{b-a} \right)^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n)} {}_{\frac{1}{b}}I_\alpha(f \circ h)(\frac{1}{a}) \right| \quad (4.7)$$

$$\leq \frac{ab(b-a)}{(\alpha+1)B(n+1, \alpha-n)} Z_4^{\frac{1}{p}}(\alpha, n) (|f'(b)|^q Z_5(a, b) + |f'(a)|^q Z_6(a, b))^{\frac{1}{q}}$$

where  $Z_4(\alpha, n)$ ,  $Z_5(a, b)$  and  $Z_6(a, b)$  are the same as in Theorem 4.2 and  $\alpha \in (n, n+1]$ .

*Proof.* Similarly the proof of the Theorem 4.2, by using Lemma 3.2, Hölder inequality and harmonically convexity of  $|f'|^q$ , we have (4.7).  $\square$

**Remark.** In Theorem 4.4,

- (1) If one takes  $\alpha = n + 1$ , one has the inequality proved in [18, Theorem 8],  
 (2) If one takes  $\alpha = n + 1$ , after that if one takes  $\alpha = 1$ , one has the inequality proved in [5, Theorem 2.7].

## 5. COMPETING INTERESTS

The authors declare that they have no competing interests.

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