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STATISTICAL EPI-CONVERGENCE IN SEQUENCES OF FUNCTIONS

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Abstract. In this paper, statistical epi-limit is defined by using epigraphs in order to increase sensitivity by eliminating outliers for mathematical problems. Various characterizations of statistical epi-convergence and their relations are given and it is compared with statistical convergence. Also, its connections with level sets and monotone increasing or decreasing cases are studied. Moreover, statistical equi-lower semicontinuity and its relation with statistical epi-limit is examined.

1. Introduction

In the late of 1960's, epi-convergence is first studied by Wijsman [26, 27] where it is called infimal convergence. After Wijsman's initial contributions, it is studied by Mosco [16] on variational inequalities, by Joly [12] on topological structures compatible with epi-convergence, by Salinetti and Wets [21] on equisemicontinuous families of convex functions, by Attouch [2] on the relationship between the epi-convergence of convex functions and the graphical convergence of their subgradient mappings, and by McLinden and Bergstrom [15] on the preservation of epi-convergence under various operations performed on convex functions. Furthermore, Dal Maso [14] called it Γ-convergence. The term epi-convergence is used by Wets [25] in 1980 for the first time. Epi-convergence is needed to solve some mathematical problems including stochastic optimization, variational problems and partial differential equations.

Statistical convergence was first studied by Zygmund [28] in 1935 and then it was introduced by Steinhaus [23] and Fast [6] and also Schoenberg [22] independently. The definitions of pointwise and uniform statistical convergence of real-valued functions were given by Gökhan and Güngör [10, 11] and by Duman and Orhan [4] independently. Statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [8] and also statistical limit points and cluster points were defined by Fridy [7, 9]. Furthermore statistical lower and upper limits of closed sets were defined and characterized by Talo et al. [24].

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In this part, fundamental definitions and theorems will be given. First of all, let (X, d) be a metric space and f, (f_n) are functions defined on X with $n \in \mathbb{N}$. If it is not mentioned explicitly the symbol d stands for the metric on X.
Let $K \subseteq \mathbb{N}$ and if the limit $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|$ exists then it is called asymptotic density of K where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n (see [1, 17]).
If $\delta(K_1) = \delta(K_2) = 1$, then $\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1$.
If $\delta(K_1) = \delta(K_2) = 0$, then $\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0$.
Statistical convergence of a sequence of scalars was introduced by Fast [6]. Let $x = (x_k)$ be a sequence of real or complex numbers. If for all $\varepsilon > 0$, there exists L such that,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

then the sequence (x_k) is statistically convergent to L.
The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [8]. Let k be a positive integer and x be a real number sequence. Define the sets B_x and A_x as

$$B_x := \{b \in \mathbb{R} : \delta(\{n : x_n > b\}) \neq 0\}, \quad A_x := \{a \in \mathbb{R} : \delta(\{n : x_n < a\}) \neq 0\}.$$

Then statistical limit superior and statistical limit inferior of x is given by

$$st\text{-}\limsup x := \begin{cases} \sup B_x & if \quad B_x \neq \emptyset, \\ -\infty & if \quad B_x = \emptyset. \end{cases}$$

$$st\text{-}\liminf x := \begin{cases} \inf A_x & if \quad A_x \neq \emptyset, \\ +\infty & if \quad A_x = \emptyset. \end{cases}$$

Lemma 1.1. [8] *If $\beta = st\text{-}\limsup x$ is finite, then for every $\varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} : x_k > \beta - \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k > \beta + \varepsilon\}) = 0 \tag{1.1}$$

Conversely, if (1.1) holds for every $\varepsilon > 0$ then $\beta = st\text{-}\limsup x$.

The dual statement for st-$\liminf x$ is as follows:

Lemma 1.2. [8] *If $\alpha = st\text{-}\liminf x$ is finite, then for every $\varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\}) \neq 0 \text{ and } \delta(\{k \in \mathbb{N} : x_k < \alpha - \varepsilon\}) = 0 \tag{1.2}$$

Conversely, if (1.2) holds for every $\varepsilon > 0$ then $\alpha = st\text{-}\liminf x$.

A point $\xi \in X$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $K = k_1 < k_2 < k_3 < ...$ with $\delta(K) \neq 0$ such that $x_{k_n} \to \xi$ as $n \to \infty$. The set of all statistical limit points of a sequence x will be denoted by Λ_x.
A point $\xi \in X$ is called a statistical cluster point of $x = (x_k)$ if for any $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\}) \neq 0.$$

The set of all statistical cluster points of x will be denoted by Γ_x.
Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x; i.e. $\xi \in L_x$ if there exists an infinite set $K = k_1 < k_2 < k_3 < ...$ such that $x_{k_n} \to \xi$ as $n \to \infty$.
Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.
In our study we will be interested much more on sequence of functions. Statistical convergence on sequence of functions is defined by Gökhan and Güngör [10].

Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical epi-limit using sets. In this paper, we deal with Painlevé-Kuratowski [13] convergence and actually its statistical version will be studied here which is defined by Sever and Talo [24]. In set convergence, following collections of subsets of $\mathbb{N}$ play an important role for defining statistical inner and outer limits on sequence of sets.

$$\mathcal{S} := \{N \subset \mathbb{N} : \delta(N) = 1\},$$
$$\mathcal{S}^{\#} := \{N \subset \mathbb{N} : \delta(N) \neq 0\}.$$

Let (X, d) be a metric space. The statistical inner limit and statistical outer limit of a sequence (A_n) of closed subsets of X are defined as follows:

$$\text{st-}\liminf_n A_n := \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}, \forall n \in N : A_n \cap V \neq \emptyset\},$$
$$\text{st-}\limsup_n A_n := \{x \mid \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}^{\#}, \forall n \in N : A_n \cap V \neq \emptyset\}.$$

Proposition 1.3. [24] *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$st\text{-}\liminf_n A_n = \{x \mid \exists N \in \mathcal{S}, \forall n \in N, \exists y_n \in A_n : \lim_n y_n = x\}.$$

Proposition 1.4. [24] *Let (X, d) be a metric space and (A_n) be a sequence of closed subsets of X. Then*

$$st\text{-}\limsup_n A_n = \{x \mid \exists N \in \mathcal{S}^{\#}, \forall n \in N, \exists y_n \in A_n : x \in \Gamma_y\}.$$

Let f be a function defined on X, the epigraph of f is the set $epif := \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \geq f(x)\}$ and its level set is defined by $lev_{\leq \alpha} f := \{x \in X \mid f(x) \leq \alpha\}$. Hence for functions f and g from X to $\mathbb{R}$, if $f \leq g$ for all $x \in X$ it is obvious that

$$epif \supseteq epig. \tag{1.3}$$

For any sequence (f_n) of functions on X, the lower epi-limit, $e\text{-}\liminf_n f_n$, is the function having as its epigraph the outer limit of the sequence of sets $epif_n$:

$$epi(e\text{-}\liminf_n f_n) := \limsup_n (epif_n).$$

The upper epi-limit, $e\text{-}\limsup_n f_n$, is the function having as its epigraph the inner limit of the sequence of sets $epif_n$:

$$epi(e\text{-}\limsup_n f_n) := \liminf_n (epif_n).$$

When these two functions equal to each other, we have $e\text{-}\lim_n f_n = e\text{-}\liminf_n f_n = e\text{-}\limsup_n f_n$. Hence the functions f_n are said to epi convergent to the function f. It is symbolized by $f_n \to_e f$. Moreover, the relation between set convergence and convergence of sequence of functions appears in the following equality.

$$f_n \to_e f \Leftrightarrow epif_n \to epif.$$

Following definition is a sequential characterication of epi-convergence.

Given a sequence (f_n) on a metric space (X, d) is epi-convergent to f, provided at each $x \in X$, if the following two conditions both hold:

(i) for all $x_n \in X$ whenever (x_n) is convergent to x, we have $f(x) \leq \liminf_n f_n(x_n)$,

(ii) there exists a sequence (x_n) convergent to x such that $f(x) = \lim_n f_n(x_n)$.

For every function $f : X \to \overline{\mathbb{R}}$ the lower semicontinuous envelope $sc^- f$ of f is defined for every $x \in X$ by

$$(sc^- f)(x) = \sup_{g \in \mathcal{G}(f)} g(x),$$

where $\mathcal{G}(f)$ is the set of all lower semicontinuous functions g on X such that $g(y) \leq f(y)$ for every $y \in X$.

Proposition 1.5. [14] *Let $f : X \to \overline{\mathbb{R}}$ be a function. Then*

$$(sc^- f)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{y \in V} f(y)$$

for every $x \in X$ where $\mathcal{N}(x)$ is the neighbourhood of x.

We also advise to look at [3, 5, 18, 20] for detailed information about new types of convergence of sequences of real valued functions and statistical convergence.

2. Main Result

In this part, statistical epi-convergence is defined by the help of Kuratowski convergence on sets. The functions will be taken lower semicontinuous in order to use properties on closed sets since epigraphs of lower semicontinuous functions are closed. Set properties will give a new characterization of statistical epi-convergence by using neighbourhoods of the point $x \in X$ in a metric space. After that neighbourhoods will give another characterizations of statistical epi-convergence by using sequences this time. Actually almost all definitions of statistical epi-convergence will be achieved in this paper. Level sets which are important instruments in set theory are also included in our calculations for lower and upper epi-limits. Moreover, statistical epi-convergence and statistical pointwise convergence will be discussed at the end.

Definition 2.1. *Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to $\overline{\mathbb{R}}$. The lower statistical epi-limit, $e_{st}\text{-}\liminf_n f_n$ is defined by the help of the sequence of sets:*

$$epi(e_{st}\text{-}\liminf_n f_n) := st\text{-}\limsup_n (epi f_n).$$

Similarly, the upper statistical epi-limit $e_{st}\text{-}\limsup_n f_n$ is defined:

$$epi(e_{st}\text{-}\limsup_n f_n) := st\text{-}\liminf_n (epi f_n).$$

When these two functions are equal, we get statistical epi-limit function:

$$f = e_{st}\text{-}\lim_n f_n := e_{st}\text{-}\limsup_n f_n = e_{st}\text{-}\liminf_n f_n.$$

As defined in above and by (1.3) *it is obvious that $e_{st}\text{-}\liminf_n f_n \leq e_{st}\text{-}\limsup_n f_n$.*

Here we use statistical Painlevé-Kuratowski convergence. Whenever (f_n) is epi convergent to f we can use the inclusion $st\text{-}\limsup_n (epi f_n) \subset epi f \subset st\text{-}\liminf_n (epi f_n)$. Moreover, following comparisons with e-limit are valid for every function $f : X \to \overline{\mathbb{R}}$.

$$e\text{-}\liminf_n f_n \leq e_{st}\text{-}\liminf_n f_n \ , \qquad e\text{-}\limsup_n f_n \leq e_{st}\text{-}\limsup_n f_n.$$

In the following example, the function is not epi-convergent whereas it has statistical epi-limit.

Example 2.2. *Given a sequence* $f_n : \mathbb{R} \to \mathbb{R}$ *defined as*

$$f_n(x) = \begin{cases} nxe^{nx} & \text{if } n \text{ is square,} \\ nxe^{2nx} & \text{if } n \text{ is nonsquare.} \end{cases}$$

$$e\text{-}\liminf_n f_n(x) = \begin{cases} 0 \ \ if & x < 0, \\ -\frac{1}{e} \ \ if & x = 0, \\ \infty \ \ if & x > 0. \end{cases}$$

$$e\text{-}\limsup_n f_n(x) = \begin{cases} 0 \ \ if & x < 0, \\ -\frac{1}{2e} \ \ if & x = 0, \\ \infty \ \ if & x > 0. \end{cases}$$

$$e_{st}\text{-}\lim_n f_n(x) = \begin{cases} 0 \ \ if & x < 0, \\ -\frac{1}{2e} \ \ if & x = 0, \\ \infty \ \ if & x > 0. \end{cases}$$

In general, statistical epi-convergence is neither stronger nor weaker than statistical pointwise convergence. The obvious difference between these convergence types is obtaining minimums. Next example gives the difference between statistical epi limit and statistical pointwise limit.

Example 2.3. *Given a sequence* $f_n : [-1, 1] \to \mathbb{R}$ *with* $(k \in \mathbb{N})$ *defined as*

$$f_n(x) = \begin{cases} \min\{1, 1 - \frac{x}{2}, 3n|x + \frac{1}{n}| - 2\} & if \ n = k^2, \\ \min\{1, 1 - x, 2n|x + \frac{1}{n}| - 1\} & if \ n \neq k^2. \end{cases}$$

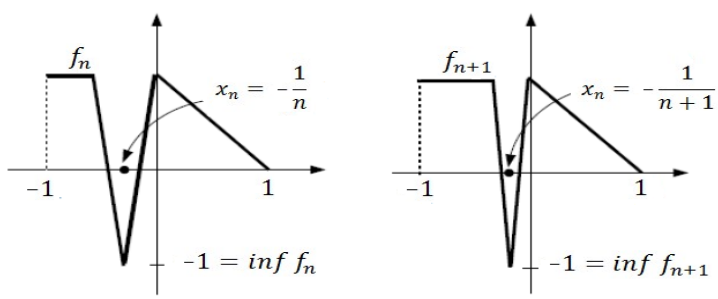

FIGURE 1. the sequence (f_n) when $n \neq k^2$

In Figure 1*, we see the graph of the sequence of* (f_n) *when* $n \neq k^2$*. Obviously, the functions take their infimum at* $x_n = -\frac{1}{n}$ *as* -1*.*
In Figure 2*, it is the graph of the same sequence* (f_n) *when* $n = k^2$ *and the functions take their infimum at* $x_n = -\frac{1}{n}$ *as* -2*.*
It can be seen clearly that, when $n \to \infty$*, the sequence* (f_n) *has not a pointwise limit but it converges statistically to the function* $f(x) = \min\{1, 1-x\}$ *for* $x \in [-1, 1]$*. It takes all its values as bigger than* 0*. Actually infimum of the function* f *is* $f(1) = 0$ *whereas the sequence* (f_n) *takes its infimum at* $x_n = -\frac{1}{n} \neq 0$*. We can see the statistical limit function in Figure* 3 *for* $x \in [-1, 1]$*.*
On the other hand, it has no epi-limit function. Since $e\text{-}\liminf_n f_n(0) = -2$ *and* $e\text{-}\limsup_n f_n(0) = -1$ *are different on* $x = 0$*.*

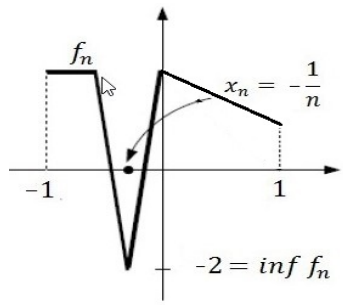

FIGURE 2. the sequence (f_n) when $n = k^2$

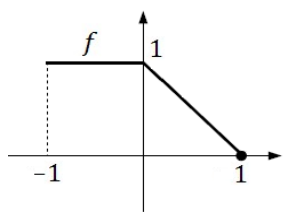

FIGURE 3. statistical limit function $(f_n \xrightarrow{st} f)$

Moreover, it has statistical epi-limit function. If we take $M \in \mathbb{N}$ as a dense set, let $n \in M$ and $-\frac{1}{n} \to 0$ we have $f_n(-\frac{1}{n}) \to -1$. In other words, $st\text{-}\lim_n f_n(-\frac{1}{n}) = -1$. So the statistical epi-limit function of the sequence (f_n) is written as,

$$h(x) = \begin{cases} 1, & x \in [-1, 0), \\ -1, & x = 0, \\ 1 - x, & x \in (0, 1]. \end{cases}$$

We say $f_n \xrightarrow{e_{st}} h$ and we can see it in the following figure.

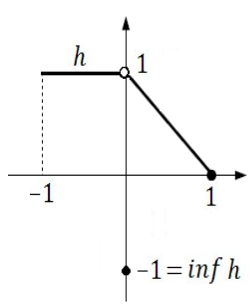

FIGURE 4. statistical epi-limit function $(f_n \xrightarrow{e_{st}} h)$

Lemma 2.4. *Let (X, d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to $\overline{\mathbb{R}}$, for every $x \in X$, define $g : X \to \overline{\mathbb{R}}$ by*

$$g(x) = \sup_{V \in \mathcal{N}(x)} st\text{-}\liminf_n \inf_{y \in V} f_n(y).$$

Then $st\text{-}\limsup_n(epif_n) = epig$.

Proof. We should establish the epigraphical inclusions of the sets $st\text{-}\limsup_n(epif_n) \subset epig$ and $epig \subset st\text{-}\limsup_n(epif_n)$. For the first inclusion, let $(x,\alpha) \in st\text{-}\limsup_n(epif_n)$ be arbitrary. Let $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ be fixed. By definition of the statistical upper limit, $\exists N \in \mathcal{S}^{\#}$ such that $\forall n \in N$ we have

$$V_0 \times (-\infty, \alpha+\varepsilon) \bigcap epif_n \neq \emptyset.$$

As a result,

$$\delta(\{n \in \mathbb{N} : \inf_{y\in V_0} f_n(y) < \alpha+\varepsilon\}) \neq 0$$

By Lemma 1.2 we have,

$$st\text{-}\liminf_n \inf_{y\in V_0} f_n(y) \leq \alpha+\varepsilon.$$

V_0 and ε were arbitrary, we have $g(x) \leq \alpha$ and hence $(x,\alpha) \in epig$.
For the second inclusion let $(x,\alpha) \in epig$, for all $V_0 \in \mathcal{N}(x)$ and for all $\varepsilon > 0$ we have,

$$\alpha+\varepsilon > g(x) \geq st\text{-}\liminf_n \inf_{y\in V_0} f_n(y).$$

Again by Lemma 1.2 we get $\delta(\{n \in \mathbb{N} : \inf_{y\in V_0} f_n(y) < \alpha+\varepsilon\}) \neq 0$. It means, $\exists N \in \mathcal{S}^{\#}$ such that $\forall n \in N$

$$V_0 \times (-\infty, \alpha+\varepsilon) \bigcap epif_n \neq \emptyset.$$

and as epigraphs lie in the vertical direction, we have

$$V_0 \times (\alpha-\varepsilon, \alpha+\varepsilon) \bigcap epif_n \neq \emptyset.$$

Hence $(x,\alpha) \in st\text{-}\limsup_n(epif_n)$. □

Lemma 2.5. *Let (X,d) be a metric space and (f_n) a sequence of lower semicontinuous functions defined from X to $\overline{\mathbb{R}}$, for every $x \in X$, define $h : X \to \overline{\mathbb{R}}$ by*

$$h(x) = \sup_{V\in\mathcal{N}(x)} st\text{-}\limsup_n \inf_{y\in V} f_n(y).$$

Then $st\text{-}\liminf_n(epif_n) = epih$.

Proof. We want to show $st\text{-}\liminf_n(epif_n) \subset epih$ and $epih \subset st\text{-}\liminf_n(epif_n)$. For the first inclusion, let $(x,\alpha) \in st\text{-}\liminf_n(epif_n)$ be arbitrary. Let $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ be fixed. By definition of the statistical lower limit, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$V_0 \times (-\infty, \alpha+\varepsilon) \bigcap epif_n \neq \emptyset.$$

As a result,

$$\delta(\{n \in \mathbb{N} : \inf_{y\in V_0} f_n(y) > \alpha+\varepsilon\}) = 0$$

By Lemma 1.1 we have,

$$\text{st-}\limsup_n \inf_{y\in V_0} f_n(y) \leq \alpha+\varepsilon.$$

V_0 and ε was arbitrary, we have $h(x) \leq \alpha$ and hence $(x,\alpha) \in epih$.
For the second inclusion, fix $(x,\alpha) \in epih$. Given $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$\text{st-}\limsup_n \inf_{y\in V_0} f_n(y) \leq h(x) < \alpha+\varepsilon$$

and it equals to the following equality

$$\delta(\{n \in \mathbb{N} : \inf_{y \in V_0} f_n(y) < \alpha + \varepsilon\}) = 1.$$

Hence,

$$\delta(\{n \in \mathbb{N} : V_0 \times (-\infty, \alpha + \varepsilon) \bigcap epi f_n \neq \emptyset\}) = 1.$$

We conclude that

$$\delta(\{n \in \mathbb{N} : V_0 \times (\alpha - \varepsilon, \alpha + \varepsilon) \bigcap epi f_n \neq \emptyset\}) = 1.$$

It gives $(x, \alpha) \in st\text{-}\liminf_n(epi f_n)$ and concludes the proof. □

Next definition gives us a characterization of epi-limits with the help of Lemma 2.4 and Lemma 2.5.

Definition 2.6. *Let (X, d) be a metric space and (f_n) a sequence of lower semi-continuous functions from X into $\overline{\mathbb{R}}$, for every $x \in X$, lower and upper statistical epi-limit functions are defined by*

$$\left(e_{st}\text{-}\liminf_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} st\text{-}\liminf_n \inf_{y \in V} f_n(y)$$

$$\left(e_{st}\text{-}\limsup_n f_n\right)(x) := \sup_{V \in \mathcal{N}(x)} st\text{-}\limsup_n \inf_{y \in V} f_n(y)$$

If there exists a function $f : X \to \overline{\mathbb{R}}$ such that $e_{st}\text{-}\liminf_n f_n = e_{st}\text{-}\limsup_n f_n = f$, then we write $f = e_{st}\text{-}\lim_n f_n$ and we say that (f_n) is e_{st}-convergent to f on X.

Lemma 2.7. *Let $x = (x_n)$ be a real sequence. Then*

$$st\text{-}\liminf_{n\to\infty} x_n = \inf_{N \in \mathcal{S}^\#} \sup_{n \in N} x_n = \sup_{N \in \mathcal{S}} \inf_{n \in N} x_n$$

$$st\text{-}\limsup_{n\to\infty} x_n = \sup_{N \in \mathcal{S}^\#} \inf_{n \in N} x_n = \inf_{N \in \mathcal{S}} \sup_{n \in N} x_n$$

By lemma 2.7, the statistical epi-limit infimum can be expressed as follows:

$$(e_{st}\text{-}\liminf_n f_n)(x) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}^\#} \sup_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \Omega(x)} \sup_{N \in \mathcal{S}} \inf_{n \in N} \inf_{y \in V} f_n(y).$$

Similarly, the statistical epi-limit supremum can be expressed as follows:

$$(e_{st}\text{-}\limsup_n f_n)(x) = \sup_{V \in \Omega(x)} \sup_{N \in \mathcal{S}^\#} \inf_{n \in N} \inf_{y \in V} f_n(y) = \sup_{V \in \Omega(x)} \inf_{N \in \mathcal{S}} \sup_{n \in N} \inf_{y \in V} f_n(y)$$

Remark. *If the functions $f_n(x)$ are independent of x, for every $n \in \mathbb{N}$ there exists a constant $\alpha_n \in \overline{\mathbb{R}}$ such that $f_n(x) = \alpha_n$ for every $x \in X$,*

$$e_{st}\text{-}\liminf_n f_n(x) = st\text{-}\liminf_n \alpha_n, \qquad e_{st}\text{-}\limsup_n f_n(x) = st\text{-}\limsup_n \alpha_n.$$

If the functions $f_n(x)$ are independent of n, there exists $f : X \to \overline{\mathbb{R}}$ such that $f_n(x) = f(x)$ for every $x \in X$ and for every $n \in \mathbb{N}$,

$$e_{st}\text{-}\liminf_n f_n = e_{st}\text{-}\limsup_n f_n = sc^- f.$$

Proposition 2.8. *In a metric space (X, d) for every $x \in X$, the following inequalities hold:*

$$(e_{st}\text{-}\liminf_n f_n)(x) \leq st\text{-}\liminf_n f_n(x)\ , \qquad (e_{st}\text{-}\limsup_n f_n)(x) \leq st\text{-}\limsup_n f_n(x).$$

Proof. $\forall x \in X$ and $\forall V \in \mathcal{N}(x)$, $\exists N \in \mathcal{S}$ such that $\forall n \in N$ we have

$$\inf_{y\in V} f_n(y) \leq f_n(x)\ , \qquad inf_{y\in V} f_n(y) \leq f_n(x).$$

Since by the choice of our index set ($n \in N$), we get the following inequalities,

$$st\text{-}\liminf_n \inf_{y\in V} f_n(y) \leq st\text{-}\liminf_n f_n(x)\ , \qquad st\text{-}\limsup_n inf_{y\in V} f_n(y) \leq st\text{-}\limsup_n f_n(x).$$

After taking the supremum over all $V \in \mathcal{N}(x)$ we get the desired conclusion. □

Theorem 2.9. *Let (X, d) be a metric space and let (f_n) be a sequence of lower semicontinuous functions. Suppose that for each $\alpha \in \mathbb{R}$, $\exists(\alpha_n)$ of reals statistically convergent to α with $lev_{\leq\alpha} f = st\text{-}\lim_n(lev_{\leq\alpha_n} f_n)$, then $f = e_{st}\text{-}\lim_n f_n$.*

Proof. The condition $lev_{\leq\alpha} f \subset \text{st-}\liminf_n(lev_{\leq\alpha_n} f_n)$ valid for each $\alpha \in \mathbb{R}$ and for some sequence $\alpha_n \overset{st}{\to} \alpha$. Let $(x, \alpha) \in epif$ there exists a sequence α_n statistically convergent to α such that $lev_{\leq\alpha} f \subset \text{st-}\liminf_n(lev_{\leq\alpha_n} f_n)$. Hence $x \in \text{st-}\liminf_n(lev_{\leq\alpha_n} f_n)$. It means there exists a sequence (x_n) statistically convergent to x such that $x_n \in (lev_{\leq\alpha_n} f_n)$. Finally we get $(x_n, \alpha_n) \overset{st}{\to} (x, \alpha)$ and $(x, \alpha) \in \text{st-}\liminf epif_n$.
In order to get $\text{st-}\limsup epif_n \subset epif$, suppose to the contrary that $(x, \beta) \in \text{st-}\limsup epif_n$ but that $(x, \beta) \notin epif$. Then $\beta < f(x)$. We can find $N \in \mathcal{S}^{\#}$ such that $\forall n \in N$ $(x_n, \beta_n) \in epif_n$ such that $(x, \beta) \in \Gamma_{(x_n,\beta_n)}$. Choose a scalar α between β and $f(x)$ and let (α_n) be a sequence statistically convergent to α for which $lev_{\leq\alpha} f \supset \text{st-}\limsup_n(lev_{\leq\alpha_n} f_n)$. We have $\delta(n : \beta_n < \alpha_n) \neq 0$ and $(x_n, \beta_n) \in epif_n$. $\exists N \in \mathcal{S}^{\#}$, $\forall n \in N$, $x_n \in lev_{\leq\alpha_n} f_n$ which means $x \in \text{st-}\limsup_n lev_{\alpha_n} f_n$. By the inclusion $\text{st-}\limsup_n lev_{\leq\alpha_n} f_n \subset lev_{\leq\alpha} f$ we get $x \in lev_{\leq\alpha} f$ and $f(x) \leq \alpha$ which is a contradiction. □

Theorem 2.10. *The following properties hold for any sequence of lower semicontinuous functions (f_n) defined on X.*
(i) The functions $e_{st}\text{-}\liminf_n f_n$ and $e_{st}\text{-}\limsup_n f_n$ are lower semicontinuous and so too is $e_{st}\text{-}\lim_n f_n$ when it exists.
(ii) If the sequence (f_n) is monotone statistically decreasing, then $e_{st}\text{-}\lim_n f_n$ exists and equals $sc^-[\inf_n f_n]$.
(iii) If the sequence (f_n) is monotone statistically increasing, then $e_{st}\text{-}\lim_n f_n$ exists and equals $\sup_n[sc^- f_n]$.

Proof. (i) Let U be a family of open subsets of X, $\alpha : U \to \mathbb{R}$ be an arbitrary function and $f : X \to \mathbb{R}$ be defined by $f(x) = \sup_{U\in N(x)} \alpha(U)$. $\forall U \subseteq X$, $\forall y \in U$ and $\forall U \in N(y)$ it is clear that $f(y) \geq \alpha(U)$. Since the inequality is satisfied by for all $U \in N(x)$ we have

$$\inf_{y\in U} f(y) \geq \alpha(U)$$

Taking supremum of both sides

$$f(x) = \sup_{U\in N(x)} \alpha(U) \leq \sup_{U\in N(x)} \inf_{y\in U} f(y)$$

for every $x \in X$. Since the opposite inequality trivial we get

$$\sup_{U\in N(x)} \alpha(U) = \sup_{U\in N(x)} \inf_{y\in U} f(y)$$

If we write $\alpha(U) = \text{st-}\liminf_n \inf_{y\in U} f_n(y)$ we get the desired conclusion. The proof is similar for functions $e_{st}\text{-}\limsup_n f_n$ and $e_{st}\text{-}\lim_n f_n$.
Now we will prove (ii), the proof of (iii) is similar. Since the sequence (f_n) is statistically decreasing, then there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $f_{k_n} \geq f_{k_{n+1}}$ for all $n \in \mathbb{N}$ and its epigraph $epif_n$ will statistically increase that is $epif_{k_n} \subseteq epif_{k_{n+1}}$. In statistical set convergence theory, we have

$$epi(sc^-[\inf_n f_n]) = cl \bigcup_{n\in\mathbb{N}} epif_{n_k}. \tag{2.1}$$

Moreover, Theorem 2.13 in [24] makes clear the following equality for statistically increasing sequences

$$\text{st-}\lim_n(epif_n) = cl \bigcup_{n\in\mathbb{N}} epif_{n_k}. \tag{2.2}$$

By using (2.1) and (2.2) combining with Definition 2.1,

$$\text{st-}\lim_n(epif_n) = epi(sc^-[\inf_n f_n]) = epi(e_{st}\text{-}\lim_n f_n).$$

Finally we get the desired equation $sc^-[\inf_n f_n] = e_{st}\text{-}\lim_n f_n$. □

Definition 2.11. *The sequence (f_n) is called statistically equi-lower semicontinuous at a point x if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ and $N \subset \mathcal{S}$ such that for all $y \in B(x,\delta)$ we have,*

$$f_n(x) - f_n(y) < \varepsilon$$

for each $n \in M$.

Next theorem gives the basic condition for which statistical convergence and statistical epi-convergence coincide.

Theorem 2.12. *(f_n) and f are functions from X to $\mathbb{R}$, let (f_n) be statistically equi-lower semicontinuous at x. (f_n) is statistically epi-convergent to f at x if and only if (f_n) is statistically convergent to f at x.*

Proof. Assuming (f_n) is statistically equi-lower semicontinuous at x, we have that for all $\varepsilon > 0$, there exists $V \in \mathcal{N}(x)$ and $N \in \mathcal{S}$ such that

$$f_n(x) - \varepsilon < \inf_{y\in V} f_n(y)$$

for all $n \in N$. This implies

$$\text{st-}\liminf_n f_n(x) - \varepsilon \leq \sup_{V\in\mathcal{N}(x)} \text{st-}\liminf_n \inf_{y\in V} f_n(y)$$

for every $\varepsilon > 0$. Combining with Proposition 2.8 we get

$$\text{st-}\liminf_n f_n(x) = \sup_{V\in\mathcal{N}(x)} \text{st-}\liminf_n \inf_{y\in V} f_n(y)$$

which means,

$$\text{st-}\liminf_n f_n(x) = e_{st}\text{-}\liminf_n f_n(x).$$

In similar way, we get $\text{st-}\limsup_n f_n(x) = e_{st}\text{-}\limsup_n f_n(x)$ and finally we reach the desired equality as follows

$$\text{st-}\lim_n f_n(x) = e_{st}\text{-}\lim_n f_n(x).$$

□

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