

UNIFORM CONVERGENCE RESULTS FOR SINGULARLY PERTURBED FREDHOLM INTEGRO-DIFFERENTIAL EQUATION

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ABSTRACT. The study deals with an initial-value problem for a singularly perturbed Fredholm integro-differential equation. Parameter explicit theoretical bounds on the continuous solution and its derivative are derived. Parameter uniform error estimates for the approximate solution are established. Numerical results are given to illustrate the parameter-uniform convergence of the numerical approximations.

1. INTRODUCTION

Fredholm integro-differential equations play a important role in physics, biology and engineering applications. Some of the the applications are unsteady aerodynamics and aerolastic phenomena, fluid dynamics, electrodynamics of complex medium, many models of population growth, neural network modeling, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, theory of population dynamics, compartmental systems, nuclear reactors, and mathematical modeling of hereditary phenomena, diffraction problems, scattering in quantum mechanics (see, e.g., [1, 6, 18, 20] and the references therein). Over time, especially in recent years, there have been many efforts on studying the solvability of these equations and their properties [4, 23].

Most of the integro-differential equations can not be solved by the well-known exact methods. Hence, it is desirable to introduce numerical methods with high accuracy to solve these equations numerically [7, 14, 17, 21, 24, 25].

We consider the following initial-value problem for a singularly perturbed Fredholm integro-differential equation (SPFIDE):

$$Lu := \varepsilon u' + a(x)u + \lambda \int_0^l K(x, s)u(s)ds = f(x), \quad x \in [0, l], \quad (1.1)$$

$$u(0) = A, \quad (1.2)$$

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where $\varepsilon \in (0, 1]$ is a perturbation parameter, λ is real parameter. We assume that $a(x) \geq \alpha > 0$, $f(x)$, $(x \in [0, l])$, $K(x, s)((x, s) \in [0, l] \times [0, l])$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified. The solution $u(x)$ has general initial layer at $x = 0$ for small values of ε .

Singularly perturbed differential equations are typically characterized by a small parameter ε multiplying some or all of the highest order terms in the differential equation. These equations play a significant role in applied mathematics. The solutions of this type equations display multiscale phenomena. Within certain thin subregions of the domain, the scale of some derivatives is significantly larger than other derivatives. These thin regions of rapid change are called, boundary or interior layers, as appropriate. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modelled by singularly perturbed problems. Typical examples include high Reynold's number flow in the fluid dynamics, heat transport problem, etc. It is also a well known fact that, for small values of ε , standard numerical methods for solving such problems are unstable and do not give accurate results. Therefore, it is important to develop suitable numerical methods for solving these problems, whose accuracy does not depend on the parameter value ε , i.e., methods that are convergent ε -uniformly and reference therein. For more details on singular perturbation problems and their numerical analysis, one may refer to [3],[7]-[10],[15]-[19],[22]. Survey of some existence and uniqueness results of singularly perturbed equations can be found in [5],[15],[16],[19].

In recent years, there has been a growing interest in the numerical solution of Fredholm-Volterra integral equations and SPFIDEs. Zhao and Corless [25] studied on sixth order compact finite difference formula for second order integro-differential equations with different boundary conditions. The least squares approximation method for the solution of Volterra-Fredholm integral equations is investigated in [24]. Micula [14] studied a simple numerical method for approximating solutions of Fredholm-Volterra integral equations of the second kind. By means of the idea of kernel ε -support vector regression machine (ε -SVR), Xu and Fan build an optimization modeling for a class of Volterra-Fredholm integral equations and proposed a new numerical method for solving them in [11]. Darania and Pishbin [8] gave high-order collocation methods for nonlinear delay integral equation. Rohaninasab et al. [21] used the Legendre collocation spectral method for solving the high-order linear Volterra-Fredholm integro-differential equations under the mixed conditions.

The above mentioned papers, related to Fredholm integro-differential equations were concerned only with the regular cases. A SPFIDEs also frequently arise in many scientific applications. The workings for the numerical solution of SPFIDEs have not widespread yet. Various difference schemes for singularly perturbed Volterra integro-differential equations and problems with integral boundary condition were investigated in [12],[13].

In this paper we propose a numerical method for the solution of singularly perturbed Fredholm integro-differential equation (1.1)-(1.2). The difference scheme is constructed by the method of integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [2],[12]. To approximate the integral part of (1.1), the composite right-side rectangle rule with the remainder term in integral form is being used. Section 2 contains results concerning the exact solutions of problem (1.1)-(1.2). In

Section 3, we construct the finite difference discretization on a uniform mesh. The stability and error analysis for the approximate solution are presented in Section 4. Uniform convergence is proved in the discrete maximum norm. Numerical results are given in Section 5 to support the predicted theory. The paper ends with a summary of the main conclusions.

2. THE CONTINUOUS PROBLEM

Here we give useful asymptotic estimates of the exact solution of the problem (1.1)-(1.2) that are needed in later sections.

Lemma 2.1. *Let the following assumptions be fulfilled*

$$a, f \in C^1[0, l], \quad \frac{\partial}{\partial x} K(x, s) \in [0, l]^2 \quad (2.1)$$

and

$$|\lambda| < \frac{\alpha}{\max_{0 \leq x \leq l} \int_0^l |K(x, s)| ds}.$$

Then the solution $u(x)$ of the problem (1.1)-(1.2) satisfies the following estimates

$$\|u\|_\infty \leq C, \quad (2.2)$$

$$|u'(x)| \leq C \left\{ 1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} \right\}, \quad x \in [0, l]. \quad (2.3)$$

Proof. According to the maximum principle for the operator

$$\mathcal{L}v := \varepsilon v' + av,$$

from (1.1)-(1.2) we have

$$\begin{aligned} \|u\|_\infty &\leq |A| + \alpha^{-1} \|f\|_\infty + \alpha^{-1} |\lambda| \max_{0 \leq x \leq l} \int_0^l |K(x, s)| |u(s)| ds, \\ &\leq |A| + \alpha^{-1} \|f\|_\infty + \alpha^{-1} |\lambda| \max_{0 \leq x \leq l} \int_0^l |K(x, s)| ds \|u\|_\infty, \end{aligned}$$

which implies validity of (2.2).

Further we estimate $u'(0)$. From (1.1) we obtain

$$|u'(0)| = \frac{1}{\varepsilon} \left| f(0) - a(0)A - \lambda \int_0^l K(0, s) u(s) ds \right|,$$

which leads to

$$|u'(0)| \leq \frac{C}{\varepsilon}, \quad (2.4)$$

by taking into consideration (2.1) and (2.2).

Next differentiating (1.1) we get

$$\varepsilon v' + a(x)v = F(x) \quad (2.5)$$

with

$$v(x) = u'(x),$$

$$F(x) = f'(x) - a'(x)u(x) - \lambda \int_0^l \frac{\partial}{\partial x} K(x, s)u(s)ds.$$

By virtue of (2.1) and (2.2) evidently

$$|F(x)| \leq C. \quad (2.6)$$

Further, from (2.5) it follows that

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon} \int_0^x a(s)ds} + \frac{1}{\varepsilon} \int_0^x F(\xi)e^{-\frac{1}{\varepsilon} \int_\xi^x a(s)ds} d\xi.$$

From which by using (2.4) and (2.6) we have

$$\begin{aligned} |u'(x)| &\leq \frac{C}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} + \|F\|_\infty \frac{1}{\varepsilon} \int_0^l e^{-\frac{1}{\varepsilon} \alpha(x-\xi)} ds \\ &= \frac{C}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1} \|F\|_\infty \left(1 - e^{-\frac{\alpha l}{\varepsilon}}\right), \end{aligned}$$

which immediately leads to (2.3). \square

3. THE MESH AND DIFFERENCE SCHEME

In what follows, we denote by ω_h the uniform mesh on $[0, l]$:

$$\omega_h = \{x_i : ih, i = 0, 1, 2, \dots, N; h = l/N\},$$

To construct the difference scheme for the problem (1.1)-(1.2), we start with the following identity

$$\chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} Lu(x) \varphi_i(x) dx = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} f(x) \varphi_i(x) dx, \quad 1 \leq i \leq N, \quad (3.1)$$

where

$$\varphi_i(x) = e^{-\frac{a_i(x_i-x)}{\varepsilon}}, \quad i = 1, 2, \dots, N,$$

$$\chi_i = h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i(x) dx = \frac{1 - e^{-a_i \rho}}{a_i \rho}, \quad \rho = \frac{h}{\varepsilon}.$$

We note that the function $\varphi_i(x)$ is the solution of the problem

$$\begin{aligned} -\varepsilon \varphi'(x) + a_i \varphi(x) &= 0, \quad x_{i-1} \leq x \leq x_i, \\ \varphi(x_i) &= 1. \end{aligned}$$

If the method of exact difference schemes(see, e.g., [2],[12]) is applied to identity (3.1), then we obtain the relation

$$\begin{aligned} \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} [\varepsilon u'(x) + a(x)u(x)] \varphi_i(x) dx &= \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} [\varepsilon u'(x) + a(x_i)u(x)] \varphi_i(x) dx \\ &\quad + \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)] u(x) \varphi_i(x) dx \\ &= \varepsilon \theta_i u_{\bar{x},i} + a_i u_i + R_i^{(1)} \end{aligned} \quad (3.2)$$

with

$$\theta_i = \frac{a_i \rho}{1 - e^{-a_i \rho}} e^{-a_i \rho}$$

and the remainder term

$$R_i^{(1)} = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} [a(x) - a(x_i)] u(x) \varphi_i(x) dx. \quad (3.3)$$

Further for the right-side integral in (3.1) we use

$$\chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} f(x) \varphi_i(x) dx = f_i + R_i^{(2)} \quad (3.4)$$

with remainder term

$$R_i^{(2)} = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} [f(x) - f(x_i)] \varphi_i(x) dx. \quad (3.5)$$

For integral term involving kernel function, using right side rectangle rule, we have from (3.1):

$$\begin{aligned} \chi_i^{-1} h^{-1} \lambda \int_{x_{i-1}}^{x_i} dx \varphi_i(x) \int_0^l K(x, s) u(s) ds &= \lambda \int_0^l K(x_i, s) u(s) ds + R_i^{(3)} \\ &= \lambda h \sum_{j=1}^N K_{ij} u_j + R_i^{(3)} + R_i^{(4)} \end{aligned} \quad (3.6)$$

with remainder terms

$$R_i^{(3)} = -\chi_i^{-1} h^{-1} \lambda \int_{x_{i-1}}^{x_i} dx \varphi_i(x) \int_{x_{i-1}}^{x_i} T_0(\xi - x) \left(\int_0^l \frac{\partial}{\partial \xi} K(\xi, s) u(s) ds \right) d\xi, \quad (3.7)$$

$$R_i^{(4)} = -\lambda \sum_{j=1}^N \int_{x_{j-1}}^{x_j} (\xi - x_{j-1}) \frac{\partial}{\partial \xi} \left[K(x_i, \xi) u(\xi) \right] d\xi. \quad (3.8)$$

By (3.1),(3.2),(3.4) and (3.6) from (3.1) we have the following exact relation for $u(x_i)$

$$\varepsilon \theta_i u_{\bar{x},i} + a_i u_i + \lambda h \sum_{j=1}^N K_{ij} u_j + R_i = f_i, \quad i = 1, 2, \dots, N, \quad (3.9)$$

with remainder term

$$R_i = \sum_{k=1}^4 R_i^{(k)},$$

where $R_i^{(k)}$, $(k = 1, 2, 3, 4)$ are defined by (3.3), (3.5), (3.7), (3.8) respectively.

Based on (3.9) we propose the following difference scheme for approximating (1.1)-(1.2)

$$\varepsilon \theta_i y_{\bar{x},i} + a_i y_i + \lambda h \sum_{j=1}^N K_{ij} y_j = f_i, \quad i = 1, 2, \dots, N, \quad (3.10)$$

$$y_0 = A. \quad (3.11)$$

4. THE STABILITY AND CONVERGENCE

Lemma 4.1. *If*

$$|\lambda| < \frac{\alpha}{\max_{1 \leq i \leq N} \sum_{j=1}^N h |K_{ij}|}, \quad (4.1)$$

then for the solution of the difference problem (3.10)-(3.11) the following estimate holds

$$\|y\|_{\infty} \leq c_0 \left(|A| + \|f\|_{\infty} \right), \quad (4.2)$$

where

$$c_0 = \frac{1}{1 - |\lambda| \max_{1 \leq i \leq N} \sum_{j=1}^N h |K_{ij}|}.$$

Proof. By virtue of the maximum principle for the difference operator

$$\mathcal{L}v_i := \varepsilon \theta_i v_{\bar{x},i} + a_i v_i, \quad 1 \leq i \leq N,$$

can be written

$$\|y\|_{\infty} \leq |A| + \alpha^{-1} \|f\|_{\infty} + \alpha^{-1} |\lambda| \max_{1 \leq i \leq N} \sum_{j=1}^N h |K_{ij}| \|y\|_{\infty},$$

which immediately leads to (4.2).

Let $z_i = y_i - u_i$. Then for the error of the approximate the solution z_i from (3.9) and (3.10) we have

$$\varepsilon \theta_i z_{\bar{x},i} + a_i z_i + \lambda h \sum_{j=1}^N K_{ij} z_j = R_i, \quad i = 1, 2, \dots, N,$$

$$z_0 = 0.$$

Using now (4.2) with $A = 0$ and $f = R$ we have the estimate for z_i :

$$\|z\|_\infty \leq c_0 \|R\|_\infty.$$

□

Lemma 4.2. *Under the conditions of Lemma 2.1 and $\frac{\partial K(x,s)}{\partial s} \in [0, l]^2$ the truncation error R_i satisfies*

$$\|R\|_\infty \leq Ch. \quad (4.3)$$

Proof. We estimate $R_i^{(k)}$, ($k = 1, 2, 3, 4$) separately.

For $R_i^{(1)}$, by $a \in C^1[0, l]$ and (2.2) we get

$$\begin{aligned} |R_i^{(1)}| &\leq \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} |a'(\xi_i)| |x - x_i| |u(x)| \varphi_i(x) dx \\ &\leq C \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i(x) dx = Ch. \end{aligned} \quad (4.4)$$

Analogously we have:

$$|R_i^{(2)}| \leq Ch. \quad (4.5)$$

For $R_i^{(3)}$, since $\left| \frac{\partial K}{\partial \xi} \right| \leq C$, $u(x) \leq C$ it follows that

$$|R_i^{(3)}| \leq |\lambda| h \int_0^l \left| \frac{\partial}{\partial \xi} K(\xi, s) u(s) \right| ds \leq Ch. \quad (4.6)$$

Finally for $R_i^{(4)}$, we have the estimate

$$\begin{aligned} |R_i^{(4)}| &\leq |\lambda| \sum_{j=1}^N \int_{x_{j-1}}^{x_j} (\xi, x_{j-1}) \left| \frac{\partial}{\partial \xi} K(x_i, \xi) u(\xi) \right| d\xi \\ &\leq |\lambda| h \int_0^l \left| \frac{\partial}{\partial \xi} K(x_i, \xi) u(\xi) \right| d\xi \\ &\leq |\lambda| h \int_0^l \left\{ \left| \frac{\partial K(x_i, \xi)}{\partial \xi} \right| |u(\xi)| + |K(x_i, \xi)| |u'(\xi)| \right\} d\xi. \end{aligned}$$

From here using (2.3) it follows that

$$\begin{aligned} |R_i^{(4)}| &\leq C |\lambda| h \int_0^l \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha \xi}{\varepsilon}} \right) d\xi \\ &= C |\lambda| \left\{ l + \alpha^{-1} (1 - e^{-\frac{\alpha l}{\varepsilon}}) \right\} \\ &\leq Ch. \end{aligned} \quad (4.7)$$

Now the estimate (4.3) follows from

$$|R_i| \leq \sum_{i=1}^N |R_i^{(k)}|,$$

by using (4.4)-(4.7).

Now we can formulated the convergence result. \square

Theorem 4.3. *Let u be solution of (1.1)-(1.2) and y_i the solution of (3.10)-(3.11). Then under the smoothness conditions above and (4.1)*

$$\|y - u\|_{\infty, \bar{\omega}_h} \leq Ch.$$

Proof. This follows immediately by combining Lemma 4.1 and Lemma 4.2. \square

5. NUMERICAL RESULTS

In this section numerical results are given for the following Fredholm integro-differential equation:

$$\varepsilon u' + u + \frac{1}{2} \int_0^1 xu(s)ds = x + \frac{1}{2}x \left(\frac{1}{2} - \varepsilon + (1 + \varepsilon)\varepsilon \left(1 - e^{-\frac{1}{\varepsilon}} \right) \right),$$

$$u(0) = 1,$$

whose exact solution is given by

$$u(x) = x - \varepsilon + (1 + \varepsilon)e^{-\frac{x}{\varepsilon}}.$$

We define the exact error e_ε^h and the computed ε -uniform maximum pointwise error e^h as follows:

$$e_\varepsilon^h = \|y - u\|_\infty, \quad e^h = \max_\varepsilon e_\varepsilon^h.$$

We also define the computed parameter-uniform rate of convergence to be

$$p^h = \ln \left(e^h / e^{h/2} \right) / \ln 2.$$

The resulting errors e^h and the corresponding numbers p^h for various values ε and h are listed in Table 1.

Table 1 Exact errors e_ε^h , computed ε -uniform errors e^h and convergence rates p^h on ω_h .

ε	$h = 1/32$	$h = 1/64$	$h = 1/128$	$h = 1/256$	$h = 1/512$	$h = 1/1024$
1	0.0318009	0.0180134	0.0099936	0.0054302	0.0027340	0.0013576
	0.82	0.85	0.88	0.99	1.01	
2^{-4}	0.0509429	0.0296675	0.0164591	0.0091313	0.0046616	0.0023308
	0.78	0.85	0.85	0.97	1.00	
2^{-8}	0.0486047	0.0285027	0.0158129	0.0087728	0.0044786	0.0022706
	0.77	0.85	0.85	0.97	0.98	
2^{-12}	0.0464832	0.0272586	0.0151227	0.0084482	0.0043429	0.0022171
	0.77	0.85	0.84	0.96	0.97	
2^{-16}	0.0459176	0.0271142	0.0151472	0.0084619	0.0043499	0.0022207
	0.76	0.84	0.84	0.96	0.97	
e^h	0.0509429	0.0296675	0.0164591	0.0091313	0.0046616	0.0021907
p^h	0.78	0.85	0.85	0.97	1.00	

The obtained results show that the convergence rate of difference scheme is essentially in accord with the theoretical analysis.

6. CONCLUSION

We have considered initial-value problem for a singularly perturbed Fredholm integro-differential equation. For the numerical solution of this problem, we proposed a fitted finite difference scheme on a uniform mesh. The difference scheme is constructed by the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form. It is shown that the method exhibits uniform convergent in terms of perturbation parameter. The numerical results show that the presented method is first-order uniformly accurate and hence can be strongly recommended for singularly perturbed Fredholm integro-differential equations.

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