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ON EQUIVALENCE RESULTS FOR WELL-POSEDNESS OF MIXED HEMIVARIATIONAL-LIKE INEQUALITIES IN BANACH SPACES

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Abstract. In this paper, we devoted to explore conditions of well-posedness for mixed hemivariational-like inequalities in reflexive Banach spaces. By using some equivalent formulations of the mixed hemivariational-like inequality under different η-monotonicity assumptions, we establish two kinds of conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality are equivalent to the existence and uniqueness of its solution, respectively.

1. Introduction

Let X be a real reflexive Banach space with the dual space X^*. Let $A : X \to X^*$ and $\eta : X \times X \to X$ be two mappings and $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper functional. Let $J : X \to \mathbf{R}$ be a locally Lipschitz functional and $J^\circ(\cdot, \cdot)$ stands for its Clarke's generalized directional derivative. Let $f \in X^*$ be some given element. Now, we consider the following mixed hemivariational-like inequality (in short, MHVLI(A, f, J, η, G)): Find $x \in X$ such that

$$\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0,\ \forall y \in X. \tag{1.1}$$

As an important subject in the theorem of optimization problems and their related problems such as variational inequalities, fixed point problems and equilibrium problems, well-posedness has been drawing more and more researchers' attention. The classical concept of well-posedness for a global minimization problem was first introduced by Tykhonov [27]. For more literature, we refer the readers to [6, 8, 10, 11], [14]-[19], [21], [33]-[45] and the references therein.

Hemivariational inequality was introduced by Panagiotopoulos [25] in 1983. In 1995, Goeleven and Mentagui [13] first introduced the well-posedness for a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Later, using the concept of approximating sequence,

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Xiao et al. [29, 30] defined a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. Very recently, Xiao, Yang and Huang [31] studied the conditions of well-posedness for the hemivariational inequality considered in [30]. By using some equivalent formulations of the hemivariational inequality considered under different monotonicity assumptions, they established two kinds of conditions under which the strong well-posedness and the weak well-posedness for the hemivariational inequality considered are equivalent to the existence and uniqueness of its solution, respectively.

This article aims to explore some conditions of well-posedness for the mixed hemivariational-like inequality in reflexive Banach spaces. The paper is structured as follows. In Sect. 2, we recall briefly some preliminary material and introduce the definitions of strong (resp. weak) well-posedness for the mixed hemivariational-like inequality considered. Section 3 introduces a definition of strongly relaxed η-monotonicity for a class of multivalued operators and presents some equivalent formulations of the mixed hemivariational-like inequality considered under the assumptions of strongly relaxed η-monotonicity and relaxed η-monotonicity for the nonconvex and nonsmooth operator involved, respectively. In Sect. 4, we give some conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality are equivalent to the existence and uniqueness of its solution, respectively. Finally, some concluding remarks are provided in Sect. 5.

2. Preliminaries

Let x be a given point and y be a vector in X. The Clarke's generalized directional derivative of J at x in the direction y, denoted by $J^\circ(x,y)$, is defined by

$$J^\circ(x,y) = \lim_{z\to x}\sup_{\lambda\downarrow 0} \frac{J(z+\lambda y) - J(z)}{\lambda}.$$

Let $G : X \to \mathbf{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous functional. We denote by $\partial G(x) : X \to 2^{X^*}\setminus\{\emptyset\}$ and $\overline{\partial} J(x) : X \to 2^{X^*}\setminus\{\emptyset\}$ the subgradient of convex functional G and the Clarke's generalized gradient of a locally Lipschitz functional J, respectively. That is,

$$\partial G(x) = \{\varrho \in X^* : G(y) - G(x) \geq \langle \varrho, y-x\rangle,\ \forall y \in X\}$$

and

$$\overline{\partial} J(x) = \{\xi \in X^* : J^\circ(x,y) \geq \langle \xi, y\rangle,\ \forall y \in X\}.$$

Remark. *([1]). The Clarke's generalized gradient of a locally Lipschitz functional $J : X \to \mathbf{R}$ at a point x is given by*

$$\overline{\partial} J(x) = \partial(J^\circ(x,\cdot))(0).$$

Concerning the subgradient in the sense of convex analysis, the Clarke's generalized directional derivative and the Clarke's generalized gradient, we have the following basic properties (see e.g., [1, 9, 22, 24, 26]).

Proposition 2.1. *Let X be a Banach space and $G : X \to \mathbf{R}\cup\{+\infty\}$ be a convex and proper functional. Then we have*

(i) *$\partial G(x)$ is convex and weak*-closed;*

(ii) *If G is continuous at $x \in \mathrm{dom}G$, then $\partial G(x)$ is nonempty, convex, bounded, and weak*-compact;*
(iii) *If G is Gateaux differentiable at $x \in \mathrm{dom}G$, then $\partial G(x) = \{DG(x)\}$, where $DG(x)$ is the Gateaux derivative of G at x.*

Proposition 2.2. *Let X be a Banach space and $G_1, G_2 : X \to \mathbf{R} \cup \{+\infty\}$ be two convex functionals. If there is a point $x_0 \in \mathrm{dom}G_1 \cap \mathrm{dom}G_2$ at which G_1 is continuous, then the following equation holds:*

$$\partial(G_1 + G_2)(x) = \partial G_1(x) + \partial G_2(x),\ \forall x \in X.$$

Proposition 2.3. *Let X be a Banach space, $x, y \in X$ and J be a locally Lipschitz functional defined on X. Then*

(i) *The function $y \mapsto J^\circ(x, y)$ is finite, positively homogeneous, subadditive and then convex on X;*
(ii) *$J^\circ(x, y)$ is upper semicontinuous as a function of (x, y), as a function of y alone, is Lipschitz continuous on X;*
(iii) *$J^\circ(x, -y) = (-J)^\circ(x, y)$;*
(iv) *$\overline{\partial} J(x)$ is a nonempty, convex, bounded and weak*-compact subset of X^*;*
(v) *For every $y \in X$, one has*

$$J^\circ(x, y) = \max\{\langle \xi, y\rangle : \xi \in \overline{\partial} J(x)\};$$

(vi) *The graph of the Clarke's generalized gradient $\overline{\partial} J(x)$ is closed in $X \times (w^*$-$X^*)$ topology, where $(w^*$-$X^*)$ denotes the space X^* equipped with weak* topology, i.e., if $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ are sequences such that $x_n^* \in \overline{\partial} J(x_n)$, $\quad x_n \to x$ in X and $x_n^* \to x^*$ weakly* in X^*, then $x^* \in \overline{\partial} J(x)$.*

Let $\eta : X \times X \to X$ and $G : X \to \mathbf{R} \cup \{+\infty\}$. A vector $z^* \in X^*$ is called an η-subgradient of G at $x \in \mathrm{dom}G$ if

$$\langle z^*, \eta(y, x)\rangle \le G(y) - G(x),\ \forall y \in X.$$

Each G can be associated with the following η-subdifferential map $\partial_\eta G$ defined by

$$\partial_\eta G(x) = \begin{cases} \{z^* \in X^* : \langle z^*, \eta(y, x)\rangle \le G(y) - G(x),\ \forall y \in X\},\ x \in \mathrm{dom}G, \\ 0,\ x \notin \mathrm{dom}G. \end{cases}$$

Let X be a real Banach space with its dual X^*, $\eta : X \times X \to X$ be a mapping and $T : X \to X^*$ be a single-valued operator.

Definition 2.4. *T is said to be*

(i) *η-monotone, if*

$$\langle Tx - Ty, \eta(x, y)\rangle \ge 0,\ \forall x, y \in X;$$

(ii) *strongly η-monotone with constant $m > 0$, if*

$$\langle Tx - Ty, \eta(x, y)\rangle \ge m\|x - y\|\|\eta(x, y)\|,\ \forall x, y \in X.$$

Definition 2.5. *Let $F : X \to 2^{X^*}$ be a multi-valued operator. F is said to be*

(i) *η-monotone, if*

$$\langle u - v, \eta(x, y)\rangle \ge 0,\ \forall x, y \in X, u \in F(x), v \in F(y);$$

(ii) *strongly η-monotone with constant $k > 0$, if*

$$\langle u - v, \eta(x, y)\rangle \ge k\|x - y\|\|\eta(x, y)\|,\ \forall x, y \in X, u \in F(x), v \in F(y);$$

(iii) *relaxed η-monotone with constant $c > 0$, if*

$$\langle u - v, \eta(x, y)\rangle \geq -c\|x - y\|\|\eta(x, y)\|,\ \forall x, y \in X, u \in F(x), v \in F(y).$$

Definition 2.6. *T is said to be η-hemicontinuous if, for any $x, y \in X$, the function $t \mapsto \langle T(x + t\eta(y, x)), \eta(y, x)\rangle$ from $[0, 1]$ into $\mathbf{R} = (-\infty, \infty)$ is continuous at 0^+.*

Remark. *Clearly, whenever $\eta(x, y) = x - y$ for all $x, y \in X$, then Definitions 2.4, 2.5 and 2.6 reduce to Definitions 2.1, 2.2 and 2.3 in Xiao, Yang and Huang* [31], *respectively. In addition, continuity implies η-hemicontinuity, but the converse is not true in general. For the usual concepts of monotonicity and hemicontinuity of single-valued operators, we refer the readers to* [43].

Definition 2.7. *(*[28]*). The function $G : X \to \mathbf{R}$ is said to be preinvex w.r.t. η iff, for all $x, y \in X$ and $t \in [0, 1]$,*

$$G(x + t\eta(y, x)) \leq (1 - t)G(x) + tG(y).$$

In the sequel, we need to use the following condition introduced by Mohan and Neogy [23].

Hypothesis (A). Let $\eta(\cdot, \cdot) : X \times X \to X$ be a mapping. For for all $x, y \in X$ and $t \in [0, 1]$, the following relations hold:

$$\eta(x, x + t\eta(y, x)) = -t\eta(y, x) \text{ and } \eta(y, x + t\eta(y, x)) = (1 - t)\eta(y, x).$$

Clearly, for $t = 0$, we have $\eta(x, x) = 0$, $\forall x \in X$. Yang et al. [25] have shown that if $\eta : X \times X \to X$ satisfies Hypothesis (A), then

$$\eta(y + t\eta(x, y), y) = t\eta(x, y).$$

Theorem 2.8. *(*[12]*). Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x - y\rangle \geq \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y\rangle \geq \varphi(y) - \varphi(x),\ \forall x \in C.$$

According to the above Theorem 2.8, we naturally introduce the following condition, which will be used in the sequel.

Hypothesis (B). Let $\eta : X \times X \to X$ be a mapping. Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbf{R} \cup \{+\infty\}$ be proper, preinvex w.r.t. η and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that

$$\langle x^*(x), \eta(x, y)\rangle \geq \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, \eta(x, y)\rangle \geq \varphi(y) - \varphi(x),\ \forall x \in C.$$

Remark. *If $\eta(x, y) = x - y$ for all $x, y \in X$, then Hypothesis (B) reduces to Theorem 2.8.*

Based on some concepts of well-posedness in [2, 3, 4, 5, 7, 20, 30, 31], we now introduce some definitions of well-posedness for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G).

Definition 2.9. *A sequence* $\{x_n\} \subset X$ *is said to be an approximating sequence for the mixed hemivariational-like inequality* MHVLI(A, f, J, η, G) *if there exists a nonnegative sequence* $\{\epsilon_n\}$ *with* $\epsilon_n \to 0$ *as* $n \to \infty$ *such that*

$$\langle Ax_n - f, \eta(y, x_n)\rangle + J^\circ(x_n, \eta(y, x_n)) + G(y) - G(x_n) \geq -\epsilon_n \|\eta(y, x_n)\|, \ \forall y \in X, \ n \in \mathbf{N}.$$

Definition 2.10. *The mixed hemivariational-like inequality* MHVLI(A, f, J, η, G) *is said to be strongly (resp. weakly) well-posed if it has a unique solution in* X *and every approximating sequence converges strongly (resp. weakly) to the unique solution.*

Definition 2.11. *The mixed hemivariational-like inequality* MHVLI(A, f, J, η, G) *is said to be strongly (resp. weakly) in the generalized sense if it has a nonempty solution set* S *in* X *and every approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of solution set* S.

3. Strongly relaxed η-monotonicity

In this section, we present equivalent formulations of the mixed hemivariational-like inequality MHVLI(A, f, J, η, G) under the assumptions of strongly relaxed η-monotonicity and relaxed η-monotonicity for the nonconvex and nonsmooth mapping involved, respectively.

Definition 3.1. *Let* X *be a real Banach space with its dual* X^*, $\eta : X \times X \to X$ *be a mapping and* $F : X \to 2^{X^*}$ *a nonempty multi-valued mapping.* F *is said to satisfy the strongly relaxed* η*-monotonicity condition with constant* $c > 0$ *if, for all* $x, y \in X$ *and* $u \in F(x)$ *(or* $v \in F(y)$*), there exists a* $v \in F(y)$ *(or* $u \in F(x)$*) such that*

$$\langle u - v, \eta(x, y)\rangle \geq -c\|x - y\|\|\eta(x, y)\|.$$

Lemma 3.2. *Let* A *be a mapping from a real Banach space* X *to its dual* X^*, $\eta : X \times X \to X$ *be a mapping,* $J : X \to \mathbf{R}$ *be a locally Lipschitz functional and* $G : X \to \mathbf{R} \cup \{+\infty\}$ *be a proper, preinvex w.r.t.* η *and lower semicontinuous functional with the* η*-subdifferential map* $\partial_\eta G$. *Assume that Hypothesis (B) holds. Then,* $x \in X$ *is a solution to the mixed hemivariational-like inequality* MHVLI(A, f, J, η, G) *if and only if* x *is a solution to the following inclusion problem:*

$$\mathrm{IP}(A, f, J, \eta, G) : \text{ Find } x \in X \text{ such that } f \in Ax + \overline{\partial} J(x) + \partial_\eta G(x). \tag{3.1}$$

Proof. The lemma is easily proved by the definitions of the Clarke's generalized gradient for locally Lipschitz functional and the η-subgradient for preinvex functional G w.r.t. η. To this end, let $x \in X$ be a solution to the inclusion problem IP(A, f, J, η, G). Then, there exist $\xi \in \overline{\partial} J(x)$ and $\varrho \in \partial_\eta G(x)$ such that

$$f = Ax + \xi + \varrho. \tag{3.2}$$

For any $y \in X$, multiplying the above Eq. (3.2) by $\eta(y, x)$, we can get by the definitions of the Clarke's generalized gradient for locally Lipschitz functional and the η-subgradient for preinvex functional G w.r.t. η, that

$$\begin{aligned} langle f, \eta(y, x)\rangle &= \langle Ax, \eta(y, x)\rangle + \langle \xi, \eta(y, x)\rangle + \langle \varrho, \eta(y, x)\rangle \\ &\leq \langle Ax, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x), \ \forall y \in X. \end{aligned}$$

Thus, x is a solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G).

On the other hand, let x is a solution to the mixed hemivariational-like inequality MHVLI (A, f, J, η, G), which means

$$\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X. \tag{3.3}$$

From the fact that

$$J^\circ(x, \eta(y, x)) = \max\{\langle \xi, \eta(y, x)\rangle : \xi \in \overline{\partial} J(x)\},$$

we get that there exists a $\xi(x, y) \in \overline{\partial} J(x)$ such that

$$\langle Ax - f, \eta(y, x)\rangle + \langle \xi(x, y), \eta(y, x)\rangle + G(y) - G(x) \geq 0, \ \forall y \in X.$$

By virtue of Proposition 2.3 (iv), $\overline{\partial} J(x)$ is a nonempty, convex, bounded and weak*-compact subset in X^*, which implies that $\{Ax - f + \xi : \xi \in \overline{\partial} J(x)\}$ is a nonempty, convex, bounded and weak*-compact subset in X^*, and hence a nonempty, convex, bounded and weakly closed subset in X^* by virtue of the reflexivity of X. Consequently, it is a nonempty, convex, bounded and closed subset in X^*. Since $G : X \to \mathbf{R} \cup \{+\infty\}$ is a proper, preinvex w.r.t. η and lower semicontinuous functional, it follows from Hypothesis (B) with $\varphi(\cdot) = G(\cdot)$ and the last inequality that there exists $\xi(x) \in \overline{\partial} J(x)$ such that

$$\langle Ax - f, \eta(y, x)\rangle + \langle \xi(x), \eta(y, x)\rangle + G(y) - G(x) \geq 0, \ \forall y \in X.$$

For the sake of simplicity, we denote $\xi = \xi(x)$. Then, by the last inequality we have

$$G(y) - G(x) \geq \langle -(Ax + Tx - f + \xi), \eta(y, x)\rangle, \ \forall y \in X,$$

which together with the definition of the η-subdifferential map $\partial_\eta G$, implies that $-(Ax - f + \xi) \in \partial G_\eta(x)$. Thus, it follows from $\xi \in \overline{\partial} J(x)$ that

$$0 \in Ax - f + \overline{\partial} J(x) + \partial_\eta G(x),$$

which implies that x is a solution to the inclusion problem IP(A, f, J, η, G). This completes the proof. □

Lemma 3.3. *Let $\eta : X \times X \to X$ satisfy Hypothesis (A). Let $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, preinvex w.r.t. η and lower semicontinuous functional with the η-subdifferential map $\partial_\eta G$. Assume that operator $A : X \to X^*$ is η-hemicontinuous and strongly η-monotone with constant $m > 0$ on X and $J : X \to \mathbf{R}$ is a locally Lipschitz functional on X such that the Clarke's generalized gradient $\overline{\partial} J(\cdot)$ satisfies the strongly relaxed η-monotonicity condition with constant $c > 0$. Assume that Hypothesis (B) holds. If $m \geq c$, then the following three statements are equivalent:*

(i) *x is a solution of the mixed hemivariational-like inequality* MHVLI(A, f, J, η, G)*, that is,*

$$\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X;$$

(ii) *x is a solution of the following associated mixed hemivariational-like inequality* AMHVLI (A, f, J, η, G)*: Find $x \in X$ such that*

$$\langle Ay - f, \eta(y, x)\rangle + J^\circ(y, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X;$$

(iii) *x is a solution of the following multi-valued mixed variational-like inequality* MMVLI (A, f, J, η, G)*: Find $x \in X$ such that, for all $y \in X$, there exists a $\zeta \in \overline{\partial} J(y)$ satisfying*

$$\langle Ay + \zeta - f, \eta(y, x)\rangle + G(y) - G(x) \geq 0, \ \forall y \in X.$$

Proof. Firstly, we prove that (i) $\Leftrightarrow$ (ii). To this end, let $x \in X$ be a solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G), which means that

$$\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X.$$

By Lemma 3.2, x be a solution to the inclusion problem IP(A, f, J, η, G), and thus there exist $\xi \in \overline{\partial} J(x)$ and $\varrho \in \partial_\eta G(x)$ such that

$$f = Ax + \xi + \varrho. \tag{3.4}$$

For any $y \in X$, by the strongly relaxed η-monotonicity of $\overline{\partial} J(\cdot)$ on X, there exists a $\zeta \in \partial J(y)$ such that

$$\langle \zeta - \xi, \eta(y, x)\rangle \geq -c\|y - x\|\|\eta(y, x)\|. \tag{3.5}$$

Note that for $\varrho \in \partial_\eta G(x)$ the definition of the η-subgradient of G at x leads to

$$G(y) - G(x) \geq \langle \varrho, \eta(y, x)\rangle,$$

which yields

$$G(y) - G(x) - \langle \varrho, \eta(y, x)\rangle \geq 0.$$

Thus, it follows from the strong η-monotonicity of the operator A, (3.4), (3.5) and the condition $m \geq c$ that

$$\begin{aligned}
&\langle Ay + \zeta - f, \eta(y, x)\rangle + G(y) - G(x) \\
&= \langle Ay + \zeta - (Ax + \xi + \varrho), \eta(y, x)\rangle + G(y) - G(x) \\
&= \langle Ay - Ax, \eta(y, x)\rangle + \langle \zeta - \xi, \eta(y, x)\rangle - \langle \varrho, \eta(y, x)\rangle + G(y) - G(x) \\
&\geq (m - c)\|y - x\|\|\eta(y, x)\| \\
&\geq 0,
\end{aligned}$$

which together with the definition of the Clarke's generalized gradient and $\zeta \in \overline{\partial} J(y)$, implies that

$$\langle f - Ay, \eta(y, x)\rangle + G(x) - G(y) \leq \langle \zeta, \eta(y, x)\rangle \leq J^\circ(y, \eta(y, x)), \ \forall y \in X,$$

i.e., x is a solution to the associated mixed hemivariational-like inequality AMHVLI(A, f, J, η, G). Therefore, (i) $\Rightarrow$ (ii) holds.

On the other hand, utilizing Hypothesis (A), Yang et al. [32] have shown that

$$\eta(x + t\eta(y, x), x) = t\eta(y, x)$$

for all $x, y \in X$ and $t \in [0, 1]$. Let x be a solution to the associated mixed hemivariational-like inequality AMHVLI(A, f, J, η, G), which means that

$$\langle Ay - f, \eta(y, x)\rangle + J^\circ(y, \eta(y, x)) + G(y) - G(x) \geq 0, \ \forall y \in X. \tag{3.6}$$

Given any $y \in X$ we define $y_t = x + t\eta(y, x)$ for all $t \in (0, 1)$. Replacing y by y_t in the above inequality (3.6), we deduce from the preinvexity of G w.r.t. η and the positive homogeneousness of the function $y \mapsto J^\circ(x, y)$ that

$$\begin{aligned}
0 &\leq \langle Ay_t - f, \eta(y_t, x)\rangle + J^\circ(y_t, \eta(y_t, x)) + G(y_t) - G(x) \\
&= \langle Ay_t - f, \eta(x + t\eta(y, x), x)\rangle + J^\circ(y_t, \eta(x + t\eta(y, x), x)) + G(x + t\eta(y, x)) - G(x) \\
&\leq \langle Ay_t - f, t\eta(y, x)\rangle + J^\circ(y_t, t\eta(y, x)) + (1 - t)G(x) + tG(y) - G(x) \\
&= t[\langle Ay_t - f, \eta(y, x)\rangle + J^\circ(y_t, \eta(y, x)) + G(y) - G(x)],
\end{aligned}$$

which hence implies that for each $t \in (0, 1)$,

$$\langle Ay_t - f, \eta(y, x)\rangle + J^\circ(y_t, \eta(y, x)) + G(y) - G(x) \geq 0. \tag{3.7}$$

It is obvious that $y_t = x + t\eta(y, x) \to x$ as $t \to 0^+$ and the η-hemicontinuity of the operator A on X implies that

$$\lim_{t\to 0^+} \langle Ay_t - f, \eta(y, x)\rangle = \lim_{t\to 0^+} \langle A(x+t\eta(y, x)) - f, \eta(y, x)\rangle = \langle Ax - f, \eta(y, x)\rangle. \quad (3.8)$$

Moreover, by Proposition 2.3 (i), (ii), $J^\circ(x, y)$ is positively homogeneous with respect to y and upper semicontinuous with respect to (x, y). Thus, taking the limsup as $t \to 0^+$ at both sides of inequality (3.7), we obtain from (3.8) that

$$\begin{aligned}
&\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x)\\
&\geq \limsup_{t\to 0^+}\{\langle A(x + t\eta(y, x)) - f, \eta(y, x)\rangle + J^\circ(x + t\eta(y, x), \eta(y, x)) + G(y) - G(x)\}\\
&= \limsup_{t\to 0^+}\{\langle Ay_t - f, \eta(y, x)\rangle + J^\circ(y_t, \eta(y, x)) + G(y) - G(x)\}\\
&\geq 0.
\end{aligned}$$

By the arbitrariness of $y \in X$, we conclude that x is a solution of the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). Therefore, (ii) $\Rightarrow$ (i) holds.

Secondly, we prove that (i) $\Leftrightarrow$ (iii). Indeed, let x be a solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). By the same arguments as in the proof of (i) $\Rightarrow$ (ii), from the definition of the η-subgradient of G at x, the strong η-monotonicity of the mapping A, the strongly relaxed η-monotonicity of the Clarke's generalized gradient $\overline{\partial}J(\cdot)$, and the condition $m \geq c$, we know that, for any $y \in X$ there exists a $\zeta \in \overline{\partial}J(y)$ such that

$$\langle Ay + \zeta - f, \eta(y, x)\rangle + G(y) - G(x) \geq 0, \quad (3.9)$$

which actually implies that x is a solution to the multi-valued mixed variational-like inequality MMVLI(A, f, J, η, G). Therefore, (i) $\Rightarrow$ (iii) holds. For (iii) $\Rightarrow$ (i), let x be a solution to the multi-valued mixed variational-like inequality MMVLI(A, f, J, η, G), which means that, for any $y \in X$, there exists a $\zeta \in \overline{\partial}J(y)$ satisfying (3.9). Given any $y \in X$ we define $y_t = x + t\eta(y, x)$ for all $t \in (0, 1)$. Replacing y by y_t in the left side of the above inequality (3.9), we deduce that there exists $\zeta_t \in \overline{\partial}J(y_t)$ such that

$$\langle Ay_t + \zeta_t - f, \eta(y_t, x)\rangle + G(y_t) - G(x) \geq 0, \quad (3.10)$$

which together with the definition of the Clarke's generalized gradient and $\zeta_t \in \overline{\partial}J(y_t)$, implies that $\langle \zeta_t, \eta(y_t, x)\rangle \leq J^\circ(y_t, \eta(y_t, x))$ and hence

$$\langle Ay_t - f, \eta(y_t, x)\rangle + J^\circ(y_t, \eta(y_t, x)) + G(y_t) - G(x) \geq 0.$$

By the same arguments as in the proof of (ii) $\Rightarrow$ (i), from the preinvexity of G w.r.t. η, the η-hemicontinuity of A on X, the positive homogeneousness of $J^\circ(x, y)$ w.r.t. y and the upper semicontinuity of $J^\circ(x, y)$ w.r.t. (x, y), we can conclude that

$$\langle Ax - f, \eta(y, x)\rangle + J^\circ(x, \eta(y, x)) + G(y) - G(x) \geq 0.$$

By the arbitrariness of $y \in X$, we know that x is a solution of the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). This completes the proof. □

4. Equivalence results for well-posedness

In this section, we give some conditions under which the strong well-posedness and the weak well-posedness for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G) are equivalent to the existence and uniqueness of its solution, respectively.

Theorem 4.1. *Let $\eta : X \times X \to X$ be skew, i.e., $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in X$. Let $A : X \to X^*$ be strongly η-monotone with constant $m > 0$, and $J : X \to \mathbf{R}$ be a locally Lipschitz functional such that the Clarke's generalized gradient $\overline{\partial} J(\cdot) : X \to 2^{X^*}$ satisfies the relaxed η-monotonicity condition with constant $c > 0$. Let $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, preinvex w.r.t. η and lower semicontinuous functional with the η-subdifferential map $\partial_\eta G$. Assume that Hypothesis (B) holds. If $m > c$, then the mixed hemivariational-like inequality* MHVLI(A, f, J, η, G) *is strongly well-posed if and only if it has a unique solution in X.*

Proof. Obviously, the necessity follows immediately from Definition 2.10 of the strong α-well-posedness for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). It remains to prove the sufficiency. Assume that the mixed hemivariational-like inequality MHVLI(A, f, J, η, G) has a unique solution $x^* \in X$. We claim that $x_n \to x^*$ in X for any approximating sequence $\{x_n\} \subset X$ for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). Since x^* is the unique solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G), we have that

$$\langle Ax^* - f, \eta(y, x^*)\rangle + J^\circ(x^*, \eta(y, x^*)) + G(y) - G(x^*) \geq 0, \ \forall y \in X.$$

By Lemma 3.2, x^* is also a solution to the inclusion problem

$$f \in Ax + \overline{\partial} J(x) + \partial_\eta G(x),$$

and thus there exist $\xi \in \partial J(x^*)$ and $\varrho \in \partial_\eta G(x^*)$ such that

$$f = Ax^* + \xi + \varrho \tag{4.1}$$

(see the argument process of (i) $\Rightarrow$ (ii) in the proof of Lemma 3.2). Moreover, $\{x_n\} \subset X$ is an approximating sequence for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G), which means that there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\langle Ax_n - f, \eta(y, x_n)\rangle + J^\circ(x_n, \eta(y, x_n)) + G(y) - G(x_n) \geq -\epsilon_n \|\eta(y, x_n)\|, \ \forall y \in X. \tag{4.2}$$

From the fact that

$$J^\circ(x_n, \eta(y, x_n)) = \max\{\langle \nu, \eta(y, x_n)\rangle : \nu \in \overline{\partial} J(x_n)\},$$

we obtain by the inequality (4.2) that there exists a $\xi(x_n, y) \in \overline{\partial} J(x_n)$ such that

$$\langle Ax_n - f, \eta(y, x_n)\rangle + \langle \xi(x_n, y), \eta(y, x_n)\rangle + G(y) - G(x_n) \geq -\epsilon_n \|\eta(y, x_n)\|, \ \forall y \in X. \tag{4.3}$$

Define the functional $Q_n(\cdot) : X \to \mathbf{R}$ as below

$$Q_n(y) = \|\eta(y, x_n)\|, \ \forall y \in X.$$

It is easy to calculate that

$$\partial Q_n(y) = \{y^* \in X^* : \|y^*\| = 1 \text{ and } \langle y^*, \eta(y, x_n)\rangle = \|\eta(y, x_n)\|\},$$

and hence, for each $n \in \mathbf{N}$ there exists a $\zeta(x_n, y) \in \partial Q_n(y)$ with $\|\zeta(x_n, y)\| = 1$ such that

$$\langle \zeta(x_n, y), \eta(y, x_n)\rangle = \|\eta(y, x_n)\|, \ \forall n \in \mathbf{N}.$$

Then (4.3) can be rewritten as

$$\langle Ax_n - f + \xi(x_n, y) + \epsilon_n \zeta(x_n, y), \eta(y, x_n)\rangle \geq G(x_n) - G(y), \ \forall y \in X. \tag{4.4}$$

On the other hand, by virtue of Proposition 2.3 (vi), $\overline{\partial} J(x_n)$ is a nonempty, convex, bounded and weak*-compact subset of X^*. Since X is reflexive, it can be readily seen that the weak topology $\sigma(X^*, X^{**})$ coincides with the weak* topology $\sigma(X^*, X)$. So, it follows that $\overline{\partial}J(x_n)$ is a nonempty, convex, bounded and weakly closed subset of X^*. Note that, for any subset in X, its closed convexity coincides with its weakly closed convexity. Thus, $\overline{\partial}J(x_n)$ is a nonempty, convex, bounded and closed subset of X^*, which immediately implies that $\{Ax_n - f + \xi : \xi \in \overline{\partial}J(x_n)\}$ is a nonempty, convex, bounded and closed subset of X^*. Consequently, we know that

$$\{Ax_n - f + \xi + \zeta : \xi \in \overline{\partial}J(x_n) \text{ and } \zeta \in B(0,1)\}$$

is a nonempty, convex, bounded and closed subset of X^*, where $B(0,1)$ is the closed ball centered at 0 with radius 1. We now set $C = X$ and

$$C^* = \{Ax_n - f + \xi + \zeta : \xi \in \overline{\partial}J(x_n) \text{ and } \zeta \in B(0,1)\}.$$

So, it follows from (4.4) and Hypothesis (B), with $\varphi(\cdot) = G(\cdot)$ which is proper, preinvex w.r.t. η and lower semicontinuous, that there exists $\omega(x_n) \in C^*$ such that

$$\langle \omega(x_n), \eta(y, x_n)\rangle \geq G(x_n) - G(y), \ \forall y \in X. \tag{4.5}$$

From $\omega(x_n) \in C^*$, it follows that there exist $\xi(x_n) \in \overline{\partial}J(x_n)$ and $\zeta(x_n) \in B(0,1)$ such that $\omega(x_n) = Ax_n - f + \xi(x_n) + \epsilon_n\zeta(x_n)$. Then (4.5) can be rewritten as

$$G(y) - G(x_n) \geq \langle -(Ax_n - f + \xi(x_n) + \epsilon_n\zeta(x_n)), \eta(y, x_n)\rangle, \ \forall y \in X. \tag{4.6}$$

For the sake of simplicity, we denote $\xi_n = \xi(x_n)$ and $\zeta_n = \zeta(x_n)$. So, it follows from (4.6) that

$$G(y) - G(x_n) \geq \langle -(Ax_n - f + \xi_n + \epsilon_n\zeta_n), \eta(y, x_n)\rangle, \ \forall y \in X. \tag{4.7}$$

Specially, taking $y = x^*$ in the above inequality (4.7) yields

$$G(x^*) - G(x_n) \geq \langle -(Ax_n - f + \xi_n + \epsilon_n\zeta_n), \eta(x^*, x_n)\rangle,$$

which hence leads to

$$\epsilon_n\langle \zeta_n, \eta(x^*, x_n)\rangle \geq G(x_n) - G(x^*) + \langle f - (Ax_n + \xi_n), \eta(x^*, x_n)\rangle. \tag{4.8}$$

It follows from the strong η-monotonicity of the operator A, the relaxed η-monotonicity of the Clarke's generalized gradient $\overline{\partial}J(\cdot)$, the skew property of η, and the Eqs. (4.1) and (4.8) that

$$\begin{aligned}
\epsilon_n\|\eta(x^*, x_n)\| &\geq \epsilon_n\langle \zeta_n, \eta(x^*, x_n)\rangle \\
&\geq G(x_n) - G(x^*) + \langle f - (Ax_n + \xi_n), \eta(x^*, x_n)\rangle \\
&= G(x_n) - G(x^*) + \langle Ax^* + \xi + \varrho - (Ax_n + \xi_n), \eta(x^*, x_n)\rangle \\
&= G(x_n) - G(x^*) - \langle \varrho, \eta(x_n, x^*)\rangle + \langle Ax^* + \xi - (Ax_n + \xi_n), \eta(x^*, x_n)\rangle \\
&\geq \langle Ax^* - Ax_n + \xi - \xi_n, \eta(x^*, x_n)\rangle \\
&\geq (m - c)\|x^* - x_n\|\|\eta(x^*, x_n)\|,
\end{aligned}$$

which implies from the condition $m > c$ that

$$\|x^* - x_n\| \leq \frac{\epsilon_n}{m - c}. \tag{4.9}$$

Taking the limit at both sides of the above inequality (4.9) yields $x_n \to x^*$ in X. This completes the proof of Theorem 4.1. $\square$

Remark. *By the proof of Theorem 4.1, the condition $m > c$ plays an important role in the proof of the strong convergence of the approximating sequence $\{x_n\}$ for the mixed hemivariational-like inequality* $\mathrm{MHVLI}(A, f, J, \eta, G)$*. It is clear that we cannot obtain the conclusion in Theorem 4.1 when the condition $m > c$ fails to hold. The following theorem gives the conditions under which the existence and uniqueness of solutions of the mixed hemivariational-like inequality* $\mathrm{MHVLI}(A, f, J, \eta, G)$ *is equivalent to its weak well-posedness when $m = c$.*

Theorem 4.2. *Let $\eta : X \times X \to X$ satisfy the conditions:*

(i) $\eta(x, z) = \eta(x, y) + \eta(y, z), \forall x, y, z \in X$;
(ii) $\|\eta(x, y)\| \geq \gamma_0 \|x - y\|, \forall x, y \in X$ *for some* $\gamma_0 > 0$;
(iii) *Hypothesis (A)holds; and*
(iv) η *is weakly continuous in the first variable.*

Let operator $A : X \to X^*$ be η-hemicontinuous and strongly η-monotone with constant $m > 0$, and $J : X \to \mathbf{R}$ be a locally Lipschitz functional such that the Clarke's generalized gradient $\overline{\partial} J(\cdot) : X \to 2^{X^*}$ satisfies the relaxed η-monotonicity condition with constant $c > 0$. Let $G : X \to \mathbf{R} \cup \{+\infty\}$ be a proper, preinvex w.r.t. η and weakly lower semicontinuous functional with the η-subdifferential map $\partial_\eta G$. Assume that Hypothesis (B) holds. If $m = c$, then the mixed hemivariational-like inequality $\mathrm{MHVLI}(A, f, J, \eta, G)$ *is weakly well-posed if and only if it has a unique solution in X.*

Proof. It is easy to see that $\eta : X \times X \to X$ is skew. By Definition 2.10 of weak well-posedness for the mixed hemivariational-like inequality $\mathrm{MHVLI}(A, f, J, \eta, G)$, the necessity is obvious. For the sufficiency, suppose that the mixed hemivariational-like inequality $\mathrm{MHVLI}(A, f, J, \eta, G)$ has a unique solution $x^* \in X$. If the mixed hemivariational-like inequality $\mathrm{MHVLI}(A, f, J, \eta, G)$ is not weakly well-posed, then there exists at least an approximating sequence $\{x_n\} \subset X$ for the mixed hemivariational-like inequality $\mathrm{MHVLI}(A, f, J, \eta, G)$ such that x_n doesn't converge weakly to x^*. We claim that the approximating sequence $\{x_n\}$ is bounded in X. In fact, if x_n is unbounded, we may assume, without loss of generality, that $\|x_n\| \to +\infty$. Utilizing condition (ii) w.r.t. η, we get $\|\eta(x_n, x^*)\| \to +\infty$. Let

$$t_n = \frac{1}{\|\eta(x_n, x^*)\|} \text{ and } z_n = x^* + t_n \eta(x_n, x^*). \tag{4.10}$$

Clearly, $\{z_n\}$ is a bounded sequence in X since $\|z_n\| \leq \|x^*\| + 1$. Thus, without loss of generality, we may assume by the reflexivity of the Banach space X that $\{z_n\}$ converges weakly to some point $z \in X$, which obviously is not equal to x^* by (4.10). Also, since the approximating sequence $\{x_n\}$ is unbounded, we can suppose that $t_n \in (0, 1]$ by (4.10). Now, for any $y \in X$ and $\zeta \in \partial J(y)$, it follows from condition (i) and Hypothesis (A) that

$$\begin{aligned}
\langle Ay + \zeta - f, \eta(y, z)\rangle &= \langle Ay + \zeta - f, \eta(y, x^*)\rangle + \langle Ay + \zeta - f, \eta(x^*, z_n)\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle \\
&= \langle Ay + \zeta - f, \eta(y, x^*)\rangle - t_n \langle Ay + \zeta - f, \eta(x_n, x^*)\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle \\
&= (1 - t_n)\langle Ay + \zeta - f, \eta(y, x^*)\rangle + t_n \langle Ay + \zeta - f, \eta(y, x_n)\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle.
\end{aligned} \tag{4.11}$$

Keep in mind that x^* is the unique solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). By the same arguments as in the proof of Theorem 4.1, there exist $\xi \in \partial J(x^*)$ and $\varrho \in \partial_\eta G(x^*)$ such that

$$f = Ax^* + \xi + \varrho. \tag{4.12}$$

Since the operator A is strongly η-monotone with constant m and the Clarke's generalized gradient $\overline{\partial} J(\cdot)$ of the locally Lipschitz functional J satisfies the relaxed η-monotonicity with constant c, the condition $m = c$ implies that $A + \overline{\partial} J(\cdot)$ is monotone on X. So, it follows from $\zeta \in \overline{\partial} J(y)$, $\xi \in \overline{\partial} J(x^*)$ and (4.12) that

$$\langle Ay + \zeta - f, \eta(y, x^*)\rangle = \langle Ay + \zeta - (Ax^* + \xi), \eta(y, x^*)\rangle - \langle \varrho, \eta(y, x^*)\rangle \ge G(x^*) - G(y). \tag{4.13}$$

Moreover, since $\{x_n\}$ is an approximating sequence for the mixed hemivariational-like inequality MHVLI(A, f, J, η, G), there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that

$$\langle Ax_n - f, \eta(y, x_n)\rangle + J^\circ(x_n, \eta(y, x_n)) + G(y) - G(x_n) \ge -\epsilon_n \|\eta(y, x_n)\|, \ \forall y \in X.$$

Also, by the same argument as in the proof of Theorem 4.1, there exist $\xi_n \in \overline{\partial} J(x_n)$ and $\zeta_n \in B(0, 1)$, which both are independent on y, such that

$$G(y) - G(x_n) \ge \langle -(Ax_n - f + \xi_n + \epsilon_n \zeta_n), \eta(y, x_n)\rangle, \ \forall y \in X.$$

which implies by the strong η-monotonicity of A, the relaxed η-monotonicity of the Clarke's generalized gradient $\overline{\partial} J(\cdot)$, the condition $m = c$ and the last inequality that

$$\langle Ay + \zeta - f, \eta(y, x_n)\rangle \ge \langle Ax_n + \xi_n - f, \eta(y, x_n)\rangle \ge G(x_n) - G(y) - \epsilon_n \langle \zeta_n, \eta(y, x_n)\rangle. \tag{4.14}$$

Therefore, it follows from (4.11), (4.13), (4.14), $t_n = 1/\|\eta(x_n, x^*)\|$ and the preinvexity w.r.t. η that

$$\begin{aligned}
\langle Ay + \zeta - f, \eta(y, z)\rangle &= (1 - t_n)\langle Ay + \zeta - f, \eta(y, x^*)\rangle + t_n \langle Ay + \zeta - f, \eta(y, x_n)\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle \\
&\ge (1 - t_n)[G(x^*) - G(y)] + t_n[G(x_n) - G(y) - \epsilon_n \langle \zeta_n, \eta(y, x_n)\rangle]\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle \\
&\ge G(z_n) - G(y) - \epsilon_n \langle \zeta_n, t_n \eta(y, x_n)\rangle]\rangle \\
&\quad + \langle Ay + \zeta - f, \eta(z_n, z)\rangle.
\end{aligned} \tag{4.15}$$

Since η is weakly continuous in the first variable, G is weakly lower semicontinuous, $z_n \rightharpoonup z$ and $\epsilon_n \to 0$ as $n \to \infty$, we get by taking the limit at both sides of the above inequality (4.15) that

$$\langle Ay + \zeta - f, \eta(y, z)\rangle + G(y) - G(z) \ge 0.$$

By Lemma 3.3, the arbitrariness of $y \in X$ and $\zeta \in \overline{\partial} J(y)$ implies that $z \ne x^*$ is a solution to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G), which reaches a contradiction to the uniqueness of solutions to the mixed hemivariational-like inequality MHVLI(A, f, J, η, G). Thus, our claim that the approximating sequence $\{x_n\}$ is bounded in X is valid.

Since $\{x_n\}$ is bounded in X and Banach space X is reflexive, we let $\{x_{n_k}\}$ be any subsequence of the approximating sequence $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$.

Thus, it follows that

$$\langle Ax_{n_k}-f,\eta(y,x_{n_k})\rangle+J^\circ(x_{n_k},\eta(y,x_{n_k}))+G(y)-G(x_{n_k})\geq -\epsilon_{n_k}\|\eta(y,x_{n_k})\|,\ \forall y\in X. \tag{4.16}$$

By the similar arguments to those of (4.7) in the proof of Theorem 4.1, there exist $\xi_{n_k}\in\overline{\partial}J(x_{n_k})$ and $\zeta_{n_k}\in B(0,1)$ such that

$$\langle Ax_{n_k}+\xi_{n_k}-f,\eta(y,x_{n_k})\rangle\leq G(x_{n_k})-G(y)-\epsilon_{n_k}\langle\zeta_{n_k},\eta(y,x_{n_k})\rangle,\ \forall y\in X.$$

which together with the strong η-monotonicity of A, the relaxed η-monotonicity of the Clarke's generalized gradient $\overline{\partial}J(\cdot)$, $x_{n_k}\rightharpoonup\hat{x}$, the weakly lower semicontinuity of G, the weak continuity of η in the first variable ($\Rightarrow$ the boundedness of $\{\eta(x_{n_k},y)\}$), and $m=c$, implies that for any $y\in X$ and $\zeta\in\partial J(y)$,

$$\begin{aligned}\langle Ay+\zeta-f,\eta(y,\hat{x})\rangle&=\liminf_{k\to\infty}\langle Ay+\zeta-f,\eta(y,x_{n_k})\rangle\\&\geq\liminf_{k\to\infty}\langle Ax_{n_k}+\xi_{n_k}-f,\eta(y,x_{n_k})\rangle\\&\geq\liminf_{k\to\infty}[G(x_{n_k})-G(y)-\epsilon_{n_k}\langle\zeta_{n_k},\eta(y,x_{n_k})\rangle]\\&=\liminf_{k\to\infty}[G(x_{n_k})-G(y)]\\&\geq G(\hat{x})-G(y).\end{aligned}\tag{4.17}$$

By Lemma 3.3, $\hat{x}$ also solves the mixed hemivariational-like inequality MHVLI(A,f,J,η,G) and so we have $\hat{x}=x^*$ in terms of the uniqueness of solutions to the mixed hemivariational-like inequality MHVLI(A,f,J,η,G). Therefore, the whole approximating sequence $\{x_n\}$ converges weakly to x^*. This completes the proof. □

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