Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 9 Issue 6 (2018), Pages 28-39.

ON FIXED POINT THEOREMS VIA PROXIMINAL MAPS IN PARTIAL METRIC SPACES WITH APPLICATION

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Abstract. In this manuscript, we present a common fixed point theorem for a pair of multivalued $F-\Psi-$proximinal mappings satisfying Ciric-Wardowski type contraction in partial metric spaces. An example and application to system of integral equations are given to support our results

1. Introduction

In 1922, Banach established the most famous fundamental fixed point theorem (the so-called the Banach contraction principle [10]) which has played an important role in various fields of applied mathematical analysis. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle.

In 2012, Wardowski [34] introduced a new type of contraction called $F-$contraction and proved a new fixed point theorem concerning $F-$contraction. He generalized the Banach contraction principle in a different aspect from the well-known results from the literature . Acar et al.[4] introduced the concept of generalized multivalued $F-$contraction mappings. Further Acar et al. [3] extended multivalued mappings with $\delta-$Distance and established fixed point results in complete metric space. Sgroi et al. [30] established fixed point theorems for multivalued $F-$contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler's. Ahmad et al. [5] recalled the concept of $F-$contraction to obtain some fixed point and common fixed point results in the context of complete metric spaces. Recently, Nazam et al.,[21] proved a common fixed point theorem for a pair of self-mappings satisfying F-contraction of rational type in complete metric spaces.

2000 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.
Key words and phrases. Fixed point; partial metric space; multivalued mappings; $F-$contraction.

Submitted September 22, 2018. Published October 11, 2018.
Communicated by M. Postolache.

2. Preliminaries

In this section, $\mathbb{R}, \mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{N}^+$ will represent the set of all real numbers, nonnegative real numbers, natural numbers and positive integers, respectively. Some elementary definitions and relevant results of partial metric spaces and $F-$contraction are borrowed, which are necessary in our subsequent discussion.

Definition 2.1. [19] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

(P1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$;
(P2) $p(x,x) \leq p(x,y)$;
(P3) $p(x,y) = p(y,x)$;
(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric spaces is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

Example 2.2. [19] *If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x,y) = \max\{x,y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.*

For more examples of partial metric spaces, we refer the reader to [8] and the references therein.

Each partial metric p on X generates a T_0 topology $\tau(p)$ on X which has a base topology of open $p-$balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$ and $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon + p(x,x)\}$.

A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to $\tau(p)$ to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to $\tau(p)$ to $fx \in X$.

Let (X,p) be a partial metric space.

(i) A sequence $\{x_n\}$ in partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

(ii) A sequence $\{x_n\}$ in partial metric space (X,p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$. The space $(X,\ p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iii) A sequence $\{x_n\}$ in partial metric space (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges (in $\tau(p)$) to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 2.3. *Let (X,p) be a partial metric space.*

(a) [2], *If $p(x_n,z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n,y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*
(b) [28] *If (X,p) is complete, then it is $0-$complete.*

It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete. The following example shows that the converse assertion of (b) need not hold.

Example 2.4. [28] *The space $X = [0,+\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = \max\{x,y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with*

$x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p), but it is not a 0-Cauchy sequence.

Definition 2.5. [18] *Let (X, d) be a metric space and $f : X \to X$ be a mapping. Then it is said that f satisfies the orbital condition if there exists a constant $k \in (0, 1)$ such that*

$$d\left(fx, f^2x\right) \leq k \ d\left(x, fx\right), \tag{2.1}$$

for all $x \in X$.

Theorem 2.6. [1] *Let (X, p) be a $0-$complete partial metric space and $f : X \to X$ be continuous such that*

$$p\left(fx, f^2x\right) \leq k \ p\left(x, fx\right) \tag{2.2}$$

holds for all $x \in X$, where $k \in (0, 1)$. Then there exists $z \in X$ such that $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$.

Definition 2.7. [18] *Let (X, p) be a partial metric space and $f : X \to X$ be a mapping with fixed point set $Fix(f) \neq \phi$. Then f has property (P) if $Fix(f^n) = Fix(f)$, for each $n \in N$.*

Lemma 2.8. [18] *Let (X, p) be a partial metric space, $f : X \to X$ be a self map such that $Fix(f) \neq \phi$. Then f has the property (P) if (2.2) holds for some $k \in (0, 1)$ and either (i) for all $x \in X$, or (ii) for all $x \neq fx$.*

Definition 2.9. [9] *Let K be a nonempty setand let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if*

$$d(x, K) = d(x, y_0), \ \text{where } d(x, K) = \inf_{y \in K} d(x, y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

Definition 2.10. [9] *The function $H : P(X) \times P(X) \to R^+$, defined by*

$$H_p(A, B) = \max\{\sup_{a \in A} p(a, B), \ \sup_{b \in B} p(A, b)\}$$

is called partial Hausdorff metric on $P(X)$.

Lemma 2.11. *Let $(P(X), H_p)$ be a partially Hausdorff metric space on $P(X)$. Then for all $A, B \in P(X)$ and for each $a \in A$ there exists $b_a \in B$ satisfies $d(a, B) = d(a, b_a)$ then $H_p(A, B) \geq d(a, b_a)$.*

Let Φ be the set of functions $\varphi : [0, \infty) \to [0, \infty)$ such that
1. φ is upper semi-continuous.
2. $\varphi(t) < t$, for each $t > 0$.

Let Ψ denote the set of all decreasing function $\psi : (0, \infty) \to (0, \infty)$.

Wardowski [34] defined $F-$contraction as fallows:

Definition 2.12. [34] *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F contraction if there exists $\tau > 0$ such that*

$$\forall x, y \in X, \ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right) \tag{2.3}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if
$\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0,1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.
We denote by $\triangle_F$, the set of all functions satisfying the conditions (F1)-(F3) .

Example 2.13. [34] *The Family of $\mathcal{F}$ is not empty.*

(1) $F(x) = \ln(x); x > 0$.
(2) $F(x) = x + \ln(x); x > 0$.
(3) $F(x) = \ln(x^2 + x); x > 0$.
(4) $F(x) = \frac{-1}{\sqrt{x}}; x > 0$.

Remark. *From (F1) and* (1.3) *it is easy to conclude that every F-contraction is necessarily continuous.*

Wardowski [34] stated a modified version of the Banach contraction principle as follows.

Theorem 2.14. [34] *Let (X, d) be a complete metric space and let $T : X \to X$ be an F contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

3. Main Results

In this section we present our essential results.

Let (X, p) be a partial metric space, $x_0 \in X$ and $S, T : X \to P(X)$ be the multifunctions on X. Let $x_1 \in Sx_0$ be an element such that $p(x_0, Sx_0) = p(x_0, x_1)$. Let $x_2 \in Tx_1$ be such that $p(x_1, Tx_1) = p(x_1, x_2)$. Let $x_3 \in Sx_2$ be such that $p(x_2, Sx_2) = p(x_2, x_3)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, ...$ Also $p(x_{2n}, Sx_{2n}) = p(x_{2n}, x_{2n+1})$, $p(x_{2n+1}, Tx_{2n+1}) = p(x_{2n+1}, x_{2n+2})$. We denote this iterative sequence by $\{TS(x_n)\}$. We say that $\{TS(x_n)\}$ is a sequence in X generated by x_0.

We begin with the following definition.

Definition 3.1. *Let (X, p) be a complete partial metric space. The mappings $S, T : X \to P(X)$ are said to be a pair of Ćirić-Wardowski type generalized multivalued $F - \Psi-$proximinal contraction, if there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$ with $p(Tx, Ty) > 0$,*

$$\psi\left(p\left(x,y\right)\right) + F(H_p(Sx,Ty)) \leq F\left(\phi\left(M\left(x,y\right)\right)\right) \tag{3.1}$$

where $F \in \triangle_F$ and $\tau > 0$, and

$$M(x,y)) = \max\left\{p(x,y), \frac{p\left(x,Sx\right).p\left(y,Ty\right)}{1+p\left(x,y\right)}, \frac{p\left(x,Sx\right).p\left(y,Ty\right)}{1+p\left(Sx,Ty\right)}, p(x,Sx), p(y,Ty)\right\}. \tag{3.2}$$

The following theorem is one of our main results.

Theorem 3.2. *Let (X, p) be a partial metric space and $S, T : X \to P(X)$ are said to be a pair of multivalued mappings such that*

(1). (S,T) are pair of continuous mappings,

(2). (S,T) are pair of Ciric-Wardowski type generalized multivalued $F-\Psi-$proximinal contraction .

Then the pair (S,T) have a common fixed point u in X and $p(u,u)=0$.

Proof. We begin with the following observation: □

If, $M(y,x)=0$, then clearly $x=y$ is a common fixed point of (S,T) .and there is nothing to prove and our proof is complete. In ordered to find common fixed point of both S and T for the situation when $M(y,x)>0$ for all $x,y\in X$ with $x\neq y$, we construct an iterative sequence $\{TS(x_n)\}$ generated by x_0.Then from contractive condition (3.1), and Lemma 2.11, we get

$$\psi\left(p\left(x_{2i+1},x_{2i+2}\right)\right)+F(H_p(Sx_{2i},Tx_{2i+1}))\leq F\left(\phi(M(x_{2i},x_{2i+1}))\right) \tag{3.3}$$

for all $i\in\mathbb{N}\cup\{0\}$, where

$$\begin{aligned} M(x_{2i},x_{2i+1}) &= \max\left\{\begin{array}{c} p(x_{2i},x_{2i+1}), \frac{p(x_{2i},Sx_{2i}).p(x_{2i+1},Tx_{2i+1})}{1+p(x_{2i},x_{2i+1})}, \frac{p(x_{2i},Sx_{2i}).p(x_{2i+1},Tx_{2i+1})}{1+p(Sx_{2i},Tx_{2i+1})} \\ ,p(x_{2i},Sx_{2i}),p(x_{2i+1},Tx_{2i+1}) \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p(x_{2i},x_{2i+1}), \frac{p(x_{2i},x_{2i+1}).p(x_{2i+1},x_{2i+2})}{1+p(x_{2i},x_{2i+1})}, \frac{p(x_{2i},x_{2i+1}).p(x_{2i+1},x_{2i+2})}{1+p(x_{2i+1},x_{2i+2})} \\ ,p(x_{2i},x_{2i+1}),p(x_{2i+1},x_{2i+2}) \end{array}\right\} \\ &= \max\{p(x_{2i},x_{2i+1}),p(x_{2i+1},x_{2i+2})\}. \end{aligned}$$

If for some $i\in\mathbb{N}^+$, $M(x_{2i},x_{2i+1})=p(x_{2i+1},x_{2i+2})$, then taking (**??**) into account, we get that

$$\psi\left(p\left(x_{2i+1},x_{2i+2}\right)\right)+F(H_p(Sx_{2i},Tx_{2i+1}))\leq F\left(\phi(M(x_{2i+1},x_{2i+2}))\right)$$

On using the property of ϕ and from $(F1)$, we get

$$\psi\left(p\left(x_{2i+1},x_{2i+2}\right)\right)+F(H_p(Sx_{2i},Tx_{2i+1}))\leq F\left(p(x_{2i+1},x_{2i+2})\right)$$

for all $i\in\mathbb{N}\cup\{0\}$. Since,$\psi\left(p\left(x_{2i+1},x_{2i+2}\right)\right)>0$, which give contradiction, yielding thereby

$$M(x_{2i+1},x_{2i+2})=p\left(x_{2i+1},x_{2i+2}\right),\ \text{for all } i\in\mathbb{N}^+.$$

Therefore from (3.3) and by the property of F, ϕ and ψ, we get

$$\begin{aligned} F(p(x_{2i+1},x_{2i+2})) &\leq F\left(\phi\left(p(x_{2i},x_{2i+1})\right)\right)-\psi\left(p(x_{2i},x_{2i+1})\right) \\ &\leq F\left(\phi\left(p(x_{2i},x_{2i+1})\right)\right), \\ F(p(x_{2i+1},x_{2i+2})) &< F\left(p(x_{2i},x_{2i+1})\right). \end{aligned} \tag{3.4}$$

It fallows from the above inequality and property of $(F1)$ that

$$p(x_{2i+1},x_{2i+2})<p(x_{2i},x_{2i+1})\ \text{ for all } i\in\mathbb{N}^+$$

Thus $\{p(x_{2i+1},x_{2i+2})\}$ is a decreasing sequence of positive real numbers . Consequently from (3.4), we have

$$\begin{aligned} F\left(p\left(x_{2i+1},x_{2i+2}\right)\right) &< F\left(p(x_{2i},x_{2i+1})\right)-\psi\left(p(x_{2i},x_{2i+1})\right) \\ &< F\left(p(x_{2i-1},x_{2i})\right)-\psi\left(p(x_{2i-1},x_{2i})\right)-\psi\left(p(x_{2i},x_{2i+1})\right). \end{aligned}$$

As ψ is a decreasing function, we get

$$F\left(p\left(x_{2i+1},x_{2i+2}\right)\right)<F\left(p(x_{2i-1},x_{2i})\right)-2\psi\left(p(x_{2i-1},x_{2i})\right)$$

Repeating the same process, we get

$$F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) < F\left(p(x_0, x_1)\right) - n\psi\left(p(x_0, x_1)\right) \tag{3.5}$$

Since, $F \in \triangle_F$, letting the limit as $i \to \infty$ in (3.5) we must have

$$\lim_{i\to\infty} F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) = -\infty \Longleftrightarrow \lim_{i\to\infty} p\left(x_{2i+1}, x_{2i+2}\right) = 0. \tag{3.6}$$

Further, by $(P2)$ we have the following equality

$$\lim_{i\to\infty} p\left(x_i, x_i\right) = 0. \tag{3.7}$$

.Next, we will show that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in X. Suppose, to contrary that, $\{x_i\}_{i=1}^{\infty}$ is not a Cauchy sequence in a complete partial metric space (X, p). Then there exisit $\varepsilon > 0$ and two sub-sequences $\{x_{i(k)}\}$ and $\{x_{j(k)}\}$ of $\{x_i\}_{i=1}^{\infty}$ such that $i\left(k\right) > j\left(k\right) \geq k$ and

$$p\left(x_{j(k)}, x_{i(k)}\right) \geq \varepsilon,$$

which yields

$$p\left(x_{j(k)}, x_{i(k)-1}\right) < \varepsilon. \tag{3.8}$$

Applying the property $(P4)$ and inequality (3.8), we get

$$\begin{aligned}
\varepsilon &\leq p\left(x_{j(k)}, x_{i(k)}\right) \leq p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{i(k)}\right) - p\left(x_{j(k)+1}, x_{j(k)+1}\right) \\
&\leq p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) + p\left(x_{j(k)}, x_{i(k)}\right) - p\left(x_{j(k)}, x_{j(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)-1}\right) + p\left(x_{i(k)-1}, x_{i(k)}\right) - p\left(x_{i(k)-1}, x_{i(k)-1}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)-1}\right) + p\left(x_{i(k)-1}, x_{i(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + \varepsilon + p\left(x_{i(k)-1}, x_{i(k)}\right),
\end{aligned}$$

which on making $k \to \infty$, yields

$$\lim_{k\to\infty} p\left(x_{j(k)}, x_{i(k)}\right) = \varepsilon. \tag{3.9}$$

Furthermore, from $(P4)$, (3.6), (3.7) and (3.9), we can get

$$\begin{aligned}
\lim_{k\to\infty} p\left(x_{j(k)}, x_{i(k)+1}\right) &= \varepsilon, \\
\lim_{k\to\infty} p\left(x_{j(k)+1}, x_{i(k)}\right) &= \varepsilon,
\end{aligned}$$

and

$$\lim_{k\to\infty} p\left(x_{j(k)+1}, x_{i(k)+1}\right) = \varepsilon. \tag{3.10}$$

Also from (3.7) there exists a natural number $i_0 \in \mathbb{N}$ such that

$$p\left(x_{i(k)}, x_{i(k)+1}\right) = \frac{\varepsilon}{4} \text{ and } p\left(x_{j(k)}, x_{j(k)+1}\right) = \frac{\varepsilon}{4},$$

forl all $i, k \geq i_0$. Now we claim that

$$p\left(Tx_{i(k)}, Tx_{j(k)}\right) = p\left(x_{i(k)+1}, x_{j(k)+1}\right) > 0. \tag{3.11}$$

Suppose on contrary that, $p\left(x_{i(k)+1}, x_{j(k)+1}\right) = 0$. Then

$$\begin{aligned}
\varepsilon &\leq p\left(x_{i(k)}, x_{j(k)}\right) \leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)}\right) - p\left(x_{i(k)+1}, x_{i(k)+1}\right) \\
&\leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) - p\left(x_{j(k)+1}, x_{j(k)+1}\right) \\
&\leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) \\
&\leq \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}$$

which yields a contradiction, thus (3.11) holds. Then it follows by the contractive condition (3.1) and the property of ψ that

$$\begin{aligned} \psi\left(p\left(x_{i(k)}, x_{j(k)}\right)\right)+F\left(H_p\left(x_{i(k)+1}, x_{j(k)+1}\right)\right) &= \psi\left(p\left(x_{i(k)}, x_{j(k)}\right)\right)+F\left(H_p\left(Sx_{i(k)}, Tx_{j(k)}\right)\right) \\ &\leq F\left(\phi\left(M\left(x_{i(k)}, x_j\left(k\right)\right)\right)\right), \text{ i.e.,} \qquad (3.12) \\ F\left(H_p\left(x_{i(k)+1}, x_{j(k)+1}\right)\right) &\leq F\left(\phi\left(M\left(x_{i(k)}, x_j\left(k\right)\right)\right)\right). \end{aligned}$$

By the definition of $M(x,y)$, (3.9), (**??**) and after repeating the same process, we get

$$\lim_{k\to\infty} M\left(x_{i(k)}, x_{j(k)}\right)=\varepsilon. \tag{3.13}$$

Letting $k \to \infty$ in (3.12) and taking into account (3.6), (3.9), (3.10), (3.14), property $\left(F3^{'}\right)$ and upper semi continuity of ϕ, we find that $F(\varepsilon) \leq F(\phi(\varepsilon)) \leq F(\varepsilon)$, which gives a contradiction. Thus, we conclude that

$$\lim_{i,j\to\infty} p\left(x_i, x_j\right)=0,$$

i.e., the sequence $\{TS(x_i)\}_{i=1}^{\infty}$ is a 0-Cauchy sequence. Therefore, the 0-completeness of X ensures that there exists a point $u \in X$ such that $\{TS(x_i)\} \to u$ that is

$$\lim_{i\to\infty} p(u, x_i)=0. \tag{3.14}$$

Now,

$$\begin{aligned} F(p(u,Tu)) &\leq F(p(u,x_{2i+1})+p(x_{2i+1},Tu)) \\ &\leq F(p(u,x_{2i+1})+H_p(Sx_{2i},Tu)) \text{ by Lemma 1.11} \\ &\leq F(p(u,x_{2i+1}))+F(H_p(Sx_{2i},Tu)) \qquad (3.15) \end{aligned}$$

By using inequality (3.1), we have

$$\psi\left(p\left(u,Tu\right)\right)+F(H_p(Sx_{2i},Tu) \leq F(\phi M(x_{2i},u)) \tag{3.16}$$

where,

$$\begin{aligned} M(x_{2i},u) &= \max\left\{\begin{array}{c} p(x_{2i},u), \frac{p(x_{2i},Sx_{2i}).p(u,Tu)}{1+p(x_{2i},u)}, \frac{p(x_{2i},Sx_{2i}).p(u,Tu)}{1+p(Sx_{2i},Tu)}, \\ p(x_{2i},Sx_{2i}), p(u,Tu) \end{array}\right\} \\ &= \max\left\{\begin{array}{c} p(x_{2i},u), \frac{p(x_{2i},x_{2i+1}).p(u,Tu)}{1+p(x_{2i},x_{2i+1})}, \frac{p(x_{2i},x_{2i+1}).p(u,Tu)}{1+p(Sx_{2i},Tu)}, \\ p(x_{2i},x_{2i+1}), p(u,Tu) \end{array}\right\}. \end{aligned}$$

Taking limit $i \to \infty$, and by using (3.14), we get

$$p(x_{2i},u)=p(u,Tu). \tag{3.17}$$

It fallows from the above inequality that

$$F(p(Tu,Tu)) \leq F\left(\phi\left(p\left(u,Tu\right)\right)\right)-\psi\left(p\left(u,u\right)\right),$$

which implies that

$$p(u,Tu) < p(u,Tu).$$

Which is a contradiction, hence $p(Tu,Tu)=p(u,Tu)=0$ or $u \in Tu$. Similarly by using (3.14), Lemma 1.11 and the inequality

$$\psi\left(p\left(u,Su\right)\right)+F(p(Su,u)) \leq F(p(Su,x_{2n+2})+p(x_{2n+2},Su))$$

we can show that $p(Su,u)=0$ or $u\in Su$. Hence the pair (S,T) have a common fixed point u in (X,p). Now,

$$p(u,u)\leq p(u,Tu)+p(Tu,u)\leq 0.$$

This implies that $p(u,u)=0$.

Example 3.3. *Let $X=\mathbb{R}$ be equipped with a usual partial metric $p\left(x,y\right)=\max\left\{\left(\left|x\right|,\left|y\right|\right)\right\}$, It is obvious that (X,p) is $0-$complete partial metric space. Define the mappings $S,T:X\to P(X)$ as follows:*

$$S(x)=\left[\frac{1}{3}x,\frac{2}{3}x\right]\ \textit{and}\ T(x)=\left[\frac{1}{5}x,\frac{2}{5}x\right]\ \textit{for all}\ x\in X.$$

Then S,T is a pair of continuous mappings. Define the function $F:R^{+}\to R$ by $F(x)=\ln(x)$ and for $x=2$ and $y=3$, we have

$$\left(H_p(S(2),T(3))\right)\leq\left\{p\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{2}{5},\frac{4}{5}\right]\right)\right\}=\frac{27}{15}$$

Define $\psi:(0,\infty)\to(0,\infty)$ by $\psi\left(t\right)=\frac{1}{30(t+1)}$ and let $\phi:[0,\infty)\to[0,\infty)$ be given by $\phi\left(t\right)=\frac{50t+1}{70}$. It is easy to see that S,T is a Ćirić-Wardowski type generalized multivalued $F-\Psi-$proximinal contraction on X. In short we proceed as follows:

$$\begin{aligned} L.H.S &= \psi\left(p\left(x,y\right)\right)+F(H_p(Sx,Ty))=\frac{1}{30\left(x+1\right)}+F\left(\frac{27}{15}\right)\\ &= \frac{1}{30\left(x+1\right)}+\log\left(1.8\right)=\frac{1}{90}+0.25552=0.26666.\end{aligned}$$

R.H.S=$F\left(\phi\left(M\left(x,y\right)\right)\right)$ where

$$M(x,y))=\max\left\{p(x,y),\frac{p\left(x,Sx\right).p\left(y,Ty\right)}{1+p\left(x,y\right)},\frac{p\left(x,Sx\right).p\left(y,Ty\right)}{1+p\left(Sx,Ty\right)},p(x,Sx),p(y,Ty)\right\}.$$

Now for $x=2$ and $y=3$, we have

$$\begin{aligned} M(2,3) &= \max\left\{\begin{array}{c} p(2,3),\frac{p(2,S(2)).p(3,T(3))}{1+p(2,3)},\frac{p(2,S(2)).p(3,T(3))}{1+p(S(2),T(3))},\\ p(2,S(2)),p(3,T(3))\end{array}\right\}\\ &= \max\left\{\begin{array}{c} p(2,3),\frac{p(2,[\frac{2}{3},\frac{4}{3}]).p(3,[\frac{3}{5},\frac{6}{5}]))}{1+p(2,3)},\frac{p(2,[\frac{2}{3},\frac{4}{3}]).p(3,[\frac{3}{5},\frac{6}{5}]))}{1+\left\{p\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{2}{5},\frac{4}{5}\right]\right)\right\}},\\ p(2,[\frac{2}{3},\frac{4}{3}]),p(3,[\frac{3}{5},\frac{6}{5}])\end{array}\right\}\\ &= \max\left\{5,\frac{8}{5},\frac{10}{7},\frac{24}{7},\frac{8}{3},\frac{18}{5}\right\}=5.\end{aligned}$$

Thus,

$$\ln(p(2,3))=\ln(5)=1.6094$$

Hence

$$0.2666\leq 1.6094$$

Hence all the hypothesis of Theorem 2.2 are satisfied. So (S,T) have a common fixed point.

Corollary 3.4. *Let (X, p) be a complete partial metric space. The mappings $S, T : X \to P(X)$ are said to be a pair of multivalued $F - \Psi -$proximinal contraction, if there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that for all $x, y \in X$ with $p(Tx, Ty) > 0$,*

$$\psi(p(x, y)) + F(H_p(Sx, Ty)) \leq F(\phi(M(x, y)))$$

where $F \in \triangle_F$ and $\tau > 0$, and

$$M(x, y)) = \max\left\{p(x, y), \frac{p(x, Sx) .p(y, Ty)}{1 + p(x, y)}, \frac{p(x, Sx) .p(y, Ty)}{1 + p(Sx, Ty)}, p(x, Sx)\right\}.$$

Then the pair (S, T) have a common fixed point u in X and $p(u, u) = 0$.

Corollary 3.5. *Let (X, p) be a partial metric space and $S, T : X \to P(X)$ are said to be a pair of multivalued mappings such that*

(1). (S, T) are pair of upper semi-continuous mappings,
(2). (S, T) are pair of Ciric-Wardowski type generalized multivalued $F - \Psi -$proximinal contraction .

Then the pair (S, T) have a common fixed point u in X and $p(u, u) = 0$.

4. Application to a system of integral equations

In this section, we discuss the application of Theorem 2.2 in form of following Volterra type integral equations

$$\gamma(t) = \int_0^t K_1(t, s, \gamma(s))ds + f(t), \tag{4.1}$$

$$\zeta(t) = \int_0^t K_2(t, s, v(s))ds + g(t) \tag{4.2}$$

for all $t \in [0, 1]$. We find the solution of (4.1) and (3.2). Let $X = C([0, 1])$ be the set of all real continuous functions on $[0, 1]$, endowed with the complete partial metric spaces. For $\gamma \in C([0, 1], \mathbb{R})$, define supremum norm as:

$$\max \|\gamma, v\|_\tau = \max\left\{\sup_{t\in[0,1]} \{\gamma(t), v(t)\}e^{-\tau(t)t}\right\},$$

where $\tau > 0$ is taken arbitrary. Then

$$p_\tau(\gamma, v) = \max\left\{\sup_{t\in[0,1]} \| |\gamma(t), v(t)| e^{-\tau t}\|_\tau\right\}$$

for all $\gamma, v \in C([0, 1], \mathbb{R})$. With these setting, $C([0, 1], \mathbb{R}, \| \cdot \|_\tau)$ becomes complete partial metric space.

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 4.1. *Assume the following conditions are satisfied:*
(i) $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0, 1] \to \mathbb{R}$ are continuous;

(ii) Define

$$\begin{aligned} S\gamma(t) &= \int_0^t K_1(t,s,\gamma(s))ds + f(t), \\ Tv(t) &= \int_0^t K_2(t,s,v(s))ds + g(t). \end{aligned}$$

Suppose there exist $\tau > 1$*, such that*

$$\max |K_1(t,s,\gamma), K_2(t,s,v)| \leq \tau e^{-\tau}[M(\gamma,v)]$$

for all $t,s \in [0,1]$ *and* $\gamma, v \in C([0,1],\mathbb{R})$*, where*

$$M(\gamma,v) = \max \left\{ \begin{array}{c} \max|\gamma(t),v(t)|\,, \frac{\max|\gamma(t),S\gamma(t)|.\max|v(t),Tv(t)|}{1+\max|\gamma(t),v(t)|}, \frac{\max|\gamma(t),S\gamma(t)|.\max|v(t),Tv(t)|}{1+\max|S\gamma(t),Tv(t)|} \\ \max|\gamma(t),S\gamma(t)|\,, \max|v(t),Tv(t)| \end{array} \right\},$$

Then integral equations (4.1) *and* (4.2) *has a solution.*

Proof. By assumption (ii)

$$\begin{aligned} \max|S\gamma(t),Tv(t)| &= \int_0^t \max|K_1(t,s,\gamma(s),K_2(t,s,v(s)))|\,ds, \\ &\leq \int_0^t \tau e^{-\tau}([M(\gamma,v)]e^{-\tau s})e^{\tau s}ds, \\ &\leq \int_0^t \tau e^{-\tau}\|M(\gamma,v)\|_\tau e^{\tau s}ds, \\ &\leq \tau e^{-\tau}\|M(\gamma,v)\|_\tau \int_0^t e^{\tau s}ds, \\ &\leq \tau e^{-\tau}\|M(\gamma,v)\|_\tau \frac{1}{\tau}e^{\tau t}, \\ &\leq e^{-\tau}\|M(\gamma,v)\|_\tau e^{\tau t}. \end{aligned}$$

This implies

$$\max|S\gamma(t),Tv(t)|\,e^{-\tau t} \leq e^{-\tau}\|M(\gamma,v)\|_\tau.$$

That is

$$\max\|S\gamma(t),Tv(t)\|_\tau \leq e^{-\tau}\|M(\gamma,v)\|_\tau,$$

which further implies

$$\tau + \ln\{\max\|S\gamma(t),Tv(t)\|_\tau\} \leq \ln\|M(\gamma,v)\|_\tau.$$

So all the conditions of Theorem 3.2 are satisfied. Hence integral equations given in (4.1)and (4.2) has a unique common solution. □

4.1. **Conclusions.** In this paper we have studied common fixed point theorem for a pair of multivalued $F-\Psi-$proximinal maps satisfying Ciric-Wardowski type contraction in partial metric spaces. This will lead towards a new generalization of F-contractions by using the idea of proximinal maps for Kannan-Wardowski and Chatterjea-Wardowski type mappings.

4.2. **Competing interests.** The author declares to have no competing interests.

4.3. **Authors' contributions.** The author contributed to each part of this work seriously and read and approved the final version of the manuscript.

Acknowledgments. The author is grateful to anonymous referees and the associate editor for their careful reading, helpful comments and constructive suggestions, which improved the presentation of results.

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