

**ON THE ASYMPTOTIC FORMULAS FOR EIGENVALUES AND  
 EIGENFUNCTIONS OF A QUADRATIC DIFFERENTIAL  
 PENCIL PROBLEM**

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ABSTRACT. We consider the quadratic differential pencil

$$\ell(y) = y'' + (\lambda^2 - 2\lambda p(x) - q(x))y$$

with the periodic boundary conditions

$$y(\pi) - y(0) = 0, \quad y'(\pi) - y'(0) = 0$$

where  $p(x) \in C^{(2)}[0, \pi]$ ,  $q(x) \in C^{(1)}[0, \pi]$  are complex-valued functions. We compute new accurate asymptotic expressions of the fundamental solutions of the quadratic differential pencil. We obtain the asymptotic formulas for eigenvalues and eigenfunctions of the differential pencil problem.

1. INTRODUCTION

In this work we consider the quadratic differential pencil

$$\ell(y) = y'' + (\lambda^2 - 2\lambda p(x) - q(x))y \tag{1.1}$$

on the closed interval  $[0, \pi]$  with the periodic boundary conditions

$$U_1(y) := y(\pi) - y(0) = 0, \quad U_2(y) := y'(\pi) - y'(0) = 0, \tag{1.2}$$

where  $p(x) \in C^{(2)}[0, \pi]$ ,  $q(x) \in C^{(1)}[0, \pi]$  are complex-valued functions.

The solution of many mathematical physics problems is reduced to the spectral theory of polynomial differential pencils, such as the quadratic differential pencil of the form (1) (see, [1], [15]). Therefore, to examine the spectral properties of the differential pencil problem (1.1), (1.2) is important (see [5], [6], [7], [8], [9]). Spectral properties of a quadratic differential pencil generated by the periodic or anti-periodic boundary conditions have been investigated by many authors, the results on this direct and references are given details in [2], [3], [4], [16, 17, 18].

In this present paper, we compute accurate asymptotic expressions of solutions of the quadratic differential pencil (1.1). We give the asymptotic formulas for eigenvalues and eigenfunctions of the differential pencil problem (1.1)-(1.2).

In the case of  $p(x) \equiv 0$ , similar results were obtained in [10], [12], [13], [14].

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The paper is organized as follows. In Section 2 we obtain the expression of the fundamental solutions of the quadratic differential pencil (1.1). In Section 3 we get new accurate asymptotic estimates for the eigenvalues, and in Section 4. we give asymptotic formulas for eigenfunctions of the differential pencil problem (1.1)-(1.2).

## 2. THE EXPRESSIONS OF THE FUNDAMENTAL SOLUTIONS

We note that for the solutions  $y_j(x, \lambda)$  ( $j = 1, 2$ ) of the pencil (1.1) the following formulas are valid when  $|\lambda| \rightarrow \infty$  (see [4], [19])

$$y_j(x, \lambda) = e^{\lambda w_j x} \left[ u_{j,0}(x) + \frac{u_{j,1}(x)}{2\lambda w_j} + \frac{u_{j,0}(x)}{(2\lambda w_j)^2} + O\left(\frac{1}{\lambda^3}\right) \right] (j = 1, 2), \quad (2.1)$$

where

$$u_{j,k}(x) = e^{-w_j \beta(x)} \int_0^x L[u_{j,k-1}(t)] e^{w_j \beta(t)} dt, (k = 1, 2), \quad (2.2)$$

$$w_1 = i, w_2 = -i, u_{j,0}(x) = e^{-w_j \beta(x)}, \quad (2.3)$$

and

$$\beta(x) = \int_0^x p(t) dt, \quad L := -\frac{d^2}{dx^2} + q(x). \quad (2.4)$$

Thus, we can find from (2.2), (2.3) and (2.4) that

$$u_{1,0}(x) = e^{-i\beta(x)}, \quad u_{2,0}(x) = e^{i\beta(x)},$$

$$u_{1,1}(x) = e^{-i\beta(x)} [i((p(x) - p(0)) + \int_0^x p^2(t) dt + \int_0^x q(t) dt)],$$

$$\begin{aligned} u_{1,2}(x) &= e^{-i\beta(x)} [2i \int_0^x p^3(t) dt - i(p'(x) + p'(0)) - \frac{5}{2}p^2(x) + p(0)p(x) \\ &\quad + \frac{3}{2}p^2(0) - q(x) + q(0) + 2i \int_0^x p(t)q(t) dt + i(p(x) - p(0)) \int_0^x p^2(t) dt \\ &\quad + \frac{1}{2}(\int_0^x p^2(t) dt)^2 + i(p(x) - p(0)) \int_0^x q(t) dt + \int_0^x q(t) dt \int_0^x p^2(t) dt \\ &\quad + \frac{1}{2}(\int_0^x q(t) dt)^2], \end{aligned}$$

$$u_{2,1}(x) = e^{i\beta(x)}[-i((p(x) - p(0)) + \int_0^x p^2(t)dt + \int_0^x q(t)dt)],$$

and

$$\begin{aligned} u_{2,2}(x) &= e^{i\beta(x)}[i(p'(x) - p'(0)) - \frac{5}{2}p^2(x) + p(0)p(x) + \frac{3}{2}p^2(0) - (q(x) - q(0))] \\ &\quad - i(p(x) - p(0)) \int_0^x p^2(t)dt - 2i \int_0^x p^3(t)dt + +\frac{1}{2}(\int_0^x p^2(t)dt)^2 \\ &\quad - i(p(x) - p(0)) \int_0^x q(t)dt - 2i \int_0^x p(t)q(t)dt + \int_0^x q(t)dt \int_0^x p^2(t)dt \\ &\quad + \frac{1}{2}(\int_0^x q(t)dt)^2]. \end{aligned}$$

If we substitute all these expressions into (2.1), we obtain the expression of the fundamental solutions of the quadratic differential pencil (1.1) as follows:

**Lemma 2.1.** *For sufficiently large  $|\lambda|$ , the expression of the fundamental solutions of the quadratic differential pencil (1.1) are of the following form*

$$\begin{aligned} y_1(x, \lambda) &= e^{i\lambda x} e^{-i\beta x} \left\{ 1 + \frac{1}{2i\lambda} \left[ i(p(x) - p(0)) + \int_0^x p^2(t)dt + \int_0^x q(t)dt \right] \right. \\ &\quad \left. + \frac{1}{(2i\lambda)^2} \left[ -i(p'(x) - p'(0)) - \frac{5}{2}p^2(x) + p(0)p(x) + \frac{3}{2}p^2(0) - q(x) + q(0) \right. \right. \\ &\quad \left. + 2i \int_0^x p^3(t)dt + \int_0^x q(t)dt \int_0^x p^2(t)dt + \frac{1}{2} \left( \int_0^x p^2(t)dt \right)^2 + \frac{1}{2} \left( \int_0^x q(t)dt \right)^2 \right. \\ &\quad \left. + 2i \int_0^x p(t)q(t)dt + i(p(x) - p(0))(\int_0^x p^2(t)dt + \int_0^x q(t)dt) \right] + O\left(\frac{1}{\lambda^3}\right) \right\}, \end{aligned} \quad (2.5)$$

$$\begin{aligned} y_2(x, \lambda) &= e^{-i\lambda x} e^{i\beta x} \left\{ 1 + \frac{1}{2i\lambda} \left[ i(p(x) - p(0)) - \int_0^x p^2(t)dt - \int_0^x q(t)dt \right] \right. \\ &\quad \left. + \frac{1}{(2i\lambda)^2} \left[ i(p'(x) - p'(0)) - \frac{5}{2}p^2(x) + p(0)p(x) + \frac{3}{2}p^2(0) - (q(x) - q(0)) \right. \right. \\ &\quad \left. - 2i \int_0^x p^3(t)dt + \int_0^x q(t)dt \int_0^x p^2(t)dt + \frac{1}{2} \left( \int_0^x p^2(t)dt \right)^2 + \frac{1}{2} \left( \int_0^x q(t)dt \right)^2 \right. \\ &\quad \left. - 2i \int_0^x p(t)q(t)dt - i(p(x) - p(0))(\int_0^x p^2(t)dt + \int_0^x q(t)dt) \right] + O\left(\frac{1}{\lambda^3}\right) \right\}. \end{aligned} \quad (2.6)$$

### 3. THE ASYMPTOTIC FORMULAS FOR THE EIGENVALUES

Now, we give asymptotic formulas for the eigenvalues of the pencil problem.

**Theorem 3.1.** *The eigenvalues of the quadratic differential pencil problem (1.1)-(1.2) form two infinite sequences  $\lambda_{k,1}, \lambda_{k,2}$ ,  $|k| = N, N+1, \dots$  where  $N$  is a positive integer and have the following asymptotic formulas:*

$$\lambda_{k,1} = \frac{\beta(\pi)}{\pi} + 2k - \frac{p(\pi) - p(0) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right), \quad (3.1)$$

$$\lambda_{k,2} = \frac{\beta(\pi)}{\pi} + 2k + \frac{p(\pi) - p(0) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right). \quad (3.2)$$

*Proof.* By derivation of (2.1) with respect to  $x$ , we obtain that

$$y'_j(x, \lambda) = \lambda w_j x e^{\lambda w_j x} [u_{j,0} + \dots + \sum_{k=1}^n \frac{u_{j,k}(x) + u'_{j,k-1}(x)}{(2\lambda w_j)^n} + O\left(\frac{1}{\lambda^{n+1}}\right)] \quad (3.3)$$

where  $j = 1, 2$ . Hence, using the equalities (2.2) and (2.3) we can find all terms  $u'_{j,k-1}(x)$  where  $k = 1, 2, 3, \dots, j = 1, 2$  as follows:

$$u'_{1,0}(x) = -i\beta'(x)e^{-i\beta(x)}, u'_{2,0}(x) = i\beta'(x)e^{i\beta(x)}, \quad (3.4)$$

$$u'_{1,1}(x) = e^{-i\beta(x)}[2p^2(x) - p(0)p(x) + ip'(x) + q(x) - ip(x) \int_0^x p^2(t)dt - ip(x) \int_0^x q(t)dt],$$

$$u'_{2,1}(x) = e^{i\beta(x)}[2p^2(x) - p(0)p(x) - ip'(x) + q(x) + ip(x) \int_0^x p^2(t)dt + ip(x) \int_0^x q(t)dt].$$

Let us substitute all these expressions into the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}, \quad (3.5)$$

where

$$U_1(y) = y(\pi) - y(0) \quad U_2(y) = y'(\pi) - y'(0).$$

By elementary transformation, we find that for sufficiently large  $|\lambda|$ , the following relation is valid:

$$\begin{aligned}
\frac{i\lambda e^{i\beta(\pi)}}{2i\lambda} \Delta(\lambda) &= e^{2i\lambda\pi} \left\{ 1 + \frac{1}{2i\lambda} \left[ -2ip(0) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt \right] \right. \\
&\quad + \frac{1}{(2i\lambda)^2} \left[ -\frac{1}{2}p^2(\pi) + p(0)p(\pi) + \frac{3}{2}p^2(0) + 2q(0) - 2ip(0) \int_0^\pi p^2(t)dt \right. \\
&\quad \left. - 2ip(0) \int_0^\pi q(t)dt + 2i \int_0^\pi p^3(t)dt + \frac{1}{2} \left( \int_0^\pi p^2(t)dt \right)^2 + 2i \int_0^\pi p(t)q(t)dt \right. \\
&\quad \left. + \int_0^\pi q(t)dt \int_0^\pi p^2(t)dt + \frac{1}{2} \left( \int_0^\pi q(t)dt \right)^2 \right] + O\left(\frac{1}{\lambda^3}\right) \Big\} - 2e^{i\lambda\pi} e^{i\beta(\pi)} \\
&\quad \times \left\{ 1 - \frac{2ip(0)}{2i\lambda} + \frac{2p^2(0) + 2q(0)}{(2i\lambda)^2} + O\left(\frac{1}{\lambda^3}\right) \right\} + e^{2i\beta(\pi)} \left\{ 1 - \frac{1}{2i\lambda} [2ip(0) \right. \\
&\quad \left. + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt] + \frac{1}{(2i\lambda)^2} \left[ \frac{3}{2}p^2(0) + p(0)p(\pi) + 2q(0) - \frac{1}{2}p^2(\pi) \right. \right. \\
&\quad \left. + 2ip(0) \int_0^\pi p^2(t)dt + 2ip(0) \int_0^\pi q(t)dt - 2i \int_0^\pi p^3(t)dt + \frac{1}{2} \left( \int_0^\pi p^2(t)dt \right)^2 \right. \\
&\quad \left. - 2i \int_0^\pi p(t)q(t)dt + \int_0^\pi q(t)dt \int_0^\pi p^2(t)dt + \frac{1}{2} \left( \int_0^\pi q(t)dt \right)^2 \right] + O\left(\frac{1}{\lambda^3}\right) \right\}.
\end{aligned} \tag{3.6}$$

Let  $b(\lambda)$  be the coefficient of  $e^{2i\lambda\pi}$  in (3.6). Using the expansion

$$(1-x)^{-1} = 1 + x + x^2 + O(x^3) \quad x \rightarrow 0,$$

we can easily see that for sufficiently large  $|\lambda|$  the following relation holds:

$$\begin{aligned}
b^{-1}(\lambda) &= 1 + \frac{1}{2i\lambda} \left[ 2ip(0) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt \right] \\
&\quad + \frac{1}{(2i\lambda)^2} \left[ \frac{1}{2}p^2(\pi) - p(0)p(\pi) - \frac{11}{2}p^2(0) - 2i \int_0^\pi p^3(t)dt \right. \\
&\quad \left. - 2ip(0) \left( \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt \right) + \int_0^\pi q(t)dt \int_0^\pi p^2(t)dt \right. \\
&\quad \left. + \frac{1}{2} \left( \int_0^\pi p^2(t)dt \right)^2 + \frac{1}{2} \left( \int_0^\pi q(t)dt \right)^2 - 2i \int_0^\pi p(t)q(t)dt \right. \\
&\quad \left. - 2q(0) \right] + O\left(\frac{1}{\lambda^3}\right).
\end{aligned} \tag{3.7}$$

Thus, the equation  $\Delta(\lambda) = 0$  is equivalent to the equation

$$b^{-1}(\lambda) \frac{e^{i\lambda\pi} e^{i\beta(\pi)}}{2i\lambda} \Delta(\lambda) = 0. \tag{3.8}$$

In view of (3.6), (3.7), Eq. (3.8) can easily be transformed to the form

$$\begin{aligned}
e^{i\lambda\pi} - e^{i\beta(\pi)} \left\{ 1 - \frac{1}{2i\lambda} \left[ \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt \right] + O\left(\frac{1}{\lambda^2}\right) \right\} &= \\
&= \mp e^{i\beta(\pi)} \frac{1}{2i\lambda} [p(\pi) - p(0)] + O\left(\frac{1}{\lambda^2}\right).
\end{aligned} \tag{3.9}$$

Thus, Eq.(3.9) splits into two equations :

$$e^{i(\lambda\pi-\beta(\pi))} - 1 = \frac{p(\pi) - p(0) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt}{2i\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad (3.10)$$

$$e^{i(\lambda\pi-\beta(\pi))} - 1 = -\frac{p(\pi) - p(0) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt}{2i\lambda} + O\left(\frac{1}{\lambda^2}\right). \quad (3.11)$$

By Rouche's Theorem , we obtain the asymptotic expressions for roots  $\lambda_{k,1}$  and  $\lambda_{k,2}$   $|k| = N, N+1, \dots$  ( $N$  being positive integer), of Eq. (3.10), (3.11), respectively:

$$\lambda_{k,1} = \frac{\beta(\pi)}{\pi} + 2k - \frac{p(\pi) - p(0) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right),$$

$$\lambda_{k,2} = \frac{\beta(\pi)}{\pi} + 2k + \frac{p(\pi) - p(0) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right).$$

Thus, the asymptotic formulas (3.1),(3.2) are valid and the proof of the theorem is completed.

□

#### 4. THE ASYMPTOTIC FORMULAS FOR THE EIGENFUNCTIONS

We give the asymptotic formulas for the eigenfunctions of the quadratic differential pencil problem (1.1), (1.2).

**Theorem 4.1.** *The eigenfunctions of the boundary-value problem (1.1), (1.2) corresponding the eigenvalues  $\lambda_{k,1}$  and  $\lambda_{k,2}$  are of the asymptotic form, respectively,*

$$\begin{aligned} \phi_{k,1}(x) &= \cos \beta(x) [(\cos A_k x)(\cos 2kx + \sin 2kx) + (\sin A_k x) \times \\ &\quad \times (\cos 2kx - \sin 2kx)] + \sin \beta(x) [(\cos A_k x)(\cos 2kx - \sin 2kx) + \\ &\quad + (\sin A_k x)(\cos 2kx + \sin 2kx)] + O\left(\frac{1}{k^2}\right), \end{aligned} \quad (4.1)$$

$$\begin{aligned} \phi_{k,2}(x) &= \cos \beta(x) [(\cos B_k x)(\cos 2kx - \sin 2kx) - (\sin B_k x) \times \\ &\quad \times (\cos 2kx + \sin 2kx)] + \sin \beta(x) [(\cos B_k x)(\sin 2kx + \cos 2kx) - \\ &\quad - (\sin B_k x)(\sin 2kx - \cos 2kx)] + O\left(\frac{1}{k^2}\right), \end{aligned} \quad (4.2)$$

where  $k$  is a sufficiently large positive integer and

$$A_k = \frac{\beta(\pi)}{\pi} - \frac{p(\pi) - p(0) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right),$$

$$B_k = \frac{\beta(\pi)}{\pi} + \frac{p(\pi) - p(0) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt}{4k\pi} + O\left(\frac{1}{k^2}\right).$$

*Proof.* Let us calculate  $U_1(y_j(x, \lambda_{k,1}))$  and  $U_1(y_2(x, \lambda_{k,1}))$  up to order  $O(k^{-2})$ . It is clear that from formula (2.5) we have

$$\begin{aligned} y_1(x, \lambda_{k,1}) = & e^{i\lambda_{k,1}x} e^{-i\beta(x)} \left\{ 1 + \frac{1}{2i\lambda_{k,1}} \left[ i(p(x) - p(0)) + \int_0^x p^2(t)dt + \int_0^x q(t)dt \right] \right. \\ & \left. + O\left(\frac{1}{\lambda_{k,1}^2}\right) \right\}. \end{aligned} \quad (4.3)$$

Therefore by using Eq.(4.3), we get

$$\begin{aligned} U_1(y_1(x, \lambda_{k,1})) = & y_1(\pi, \lambda_{k,1}) - y_1(0, \lambda_{k,1}) \\ = & e^{i\lambda_{k,1}\pi} e^{-i\beta(\pi)} \left( 1 + \frac{i(p(\pi) - p(0)) + \int_0^\pi p^2(t)dt + \int_0^\pi q(t)dt}{2i\lambda_{k,1}} \right) \\ & - \left( 1 + O\left(\frac{1}{\lambda_{k,1}^2}\right) \right) \\ = & \frac{(i+1)(p(\pi) - p(0))}{2i\lambda_{k,1}} + O\left(\frac{1}{\lambda_{k,1}^2}\right). \end{aligned} \quad (4.4)$$

Similarly, from Eq. (2.6), we have

$$\begin{aligned} y_2(x, \lambda_{k,1}) = & e^{-i\lambda_{k,1}x} e^{i\beta(x)} \left\{ 1 + \frac{1}{2i\lambda_{k,1}} \left[ i(p(x) - p(0)) - \int_0^x p^2(t)dt - \int_0^x q(t)dt \right] \right. \\ & \left. + O\left(\frac{1}{\lambda_{k,1}^2}\right) \right\} \end{aligned} \quad (4.5)$$

So, by using (4.5), we obtain

$$\begin{aligned} U_1(y_2(x, \lambda_{k,1})) = & y_2(\pi, \lambda_{k,1}) - y_2(0, \lambda_{k,1}) \\ = & e^{-i\lambda_{k,1}\pi} e^{i\beta(\pi)} \left( 1 + \frac{i(p(\pi) - p(0)) - \int_0^\pi p^2(t)dt - \int_0^\pi q(t)dt}{2i\lambda_{k,1}} \right) \\ & - \left( 1 + O\left(\frac{1}{\lambda_{k,1}^2}\right) \right) \\ = & \frac{(i-1)(p(\pi) - p(0))}{2i\lambda_{k,1}} + O\left(\frac{1}{\lambda_{k,1}^2}\right). \end{aligned} \quad (4.6)$$

We can seek the eigenfunctions  $\phi_{k,1}(x)$  corresponding to the eigenvalue  $\lambda_{k,1}$  in form:

$$\phi_{k,1}(x) = \frac{-2i\lambda_{k,1}}{2(p(\pi) - p(0))} \begin{vmatrix} y_1(x, \lambda_{k,1}) & y_2(x, \lambda_{k,1}) \\ U_1(y_1(x, \lambda_{k,1})) & U_1(y_2(x, \lambda_{k,1})) \end{vmatrix}. \quad (4.7)$$

Here, we assume that  $p(\pi) - p(0) \neq 0$  for normalized the eigenfunctions.

Hence, from formulas (4.3)-(4.7) for sufficiently large  $|k|$ , we get

$$\begin{aligned}\phi_{k,1}(x) = & \cos \beta(x) [(\cos A_k x)(\cos 2kx + \sin 2kx) + (\sin A_k x) \times \\ & \times (\cos 2kx - \sin 2kx)] + \sin \beta(x) [(\cos A_k x)(\cos 2kx - \sin 2kx) + \\ & + (\sin A_k x)(\cos 2kx + \sin 2kx)] + O\left(\frac{1}{k^2}\right).\end{aligned}$$

In similar way, we can seek the eigenfunctions  $\phi_{k,2}(x)$  corresponding to the eigenvalue  $\lambda_{k,2}$  in form:

$$\phi_{k,2}(x) = \frac{2i\lambda_{k,2}}{2(p(\pi) - p(0))} \begin{vmatrix} y_1(x, \lambda_{k,2}) & y_2(x, \lambda_{k,2}) \\ U_1(y_1(x, \lambda_{k,2})) & U_1(y_2(x, \lambda_{k,2})) \end{vmatrix}. \quad (4.8)$$

Therefore, for sufficiently large integer  $k$ , we obtain formula (4.2). This completes the proof of the theorem.  $\square$

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