Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 9 Issue 6 (2018), Pages 1-16.

ON SOLUTIONS OF INFINITE SYSTEMS OF INTEGRAL EQUATIONS IN N-VARIABLES IN SPACES OF TEMPERED SEQUENCES c_0^β AND l_1^β

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Abstract. The aim of the present paper is to establish the existence of solution of infinite systems of integral equations in N-variables in spaces of tempered sequences c_0^β and l_1^β by applying Meir-Keeler condensing operators. Our theorems improve the results of Hazarika et al. (Journal of Computational and Applied Mathematics 326 (2017) 183-192). The results we have established are illustrated with some examples which also show that the improvements are actual.

1. Introduction

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness. The first measure of noncompactness, the function α, was defined and studied by Kuratowski [14] for purely topological considerations. In 1955, Darbo [10] used this measure to generalize Banach's contraction mapping principle for so-called condensing operators. The Hausdorff measure of noncompactness χ was introduced by Goldenstein et al. [11] in the year 1957, and it was further studied by Goldenstein and Markus [12]. Measures of noncompactness are very useful tools widely used in fixed point theory, differential equations, integral and integro-differential equations, and optimization, etc. They have also been used in defining geometric properties of Banach spaces and in characterizing compact operators between sequence spaces.

The study of sequence spaces has been of great interest recently. A number of books have been published in this area over the last few years (see, for example [8]). Sequence spaces have various applications in several branches of functional analysis, in particular, the theory of locally convex spaces, matrix transformations, as well as the theory of summability invariably depends upon the study of sequences and series.

In recent years, a lot of scholars (see e.g. [1, 9, 15]) studied the existence of solutions of integral equations in one or two variables on some spaces. Mursaleen and Mohiuddine [18] proved existence theorems for the infinite system of differential

2000 *Mathematics Subject Classification.* 47H09, 47H10, 34A12.
Key words and phrases. Hausdorff measure of noncompactness; Spaces of tempered sequences; System of integral equations; Meir-Keeler condensing operator.

Submitted May 5, 2018. Published December 23, 2018.
Communicated by M. Mursaleen.

equations in the space l_p. Furthermore, existence theorems for the infinite systems of linear equations in l_1 and l_p were given by Alotaibi et al. [4]. Mursaleen and Rizvi [19] studied solvability of infinite system of second order differential equations in c_0 and l_1 by Meir-Keeler condensing operators. Mursaleen and Alotaibi [17] proved existence theorems for the infinite system of differential equations in some BK-spaces. Arab et al. [5] investigated the existence of solutions of system of integral equations in two variables. Hazarika et al. [13] studied solvability of the infinite systems of integral equations in two variables in the sequence spaces c_0 and l_1.
Classical sequence spaces are not always suitable to consider initial value problems for infinite systems of differential equations. Therefore, in order to consider those initial value problems we are frequently forced to treat the problems in question in enlarged sequence spaces. Such sequence spaces can be obtained if we consider the so-called tempered sequence spaces.
To define the mentioned spaces let us fix a real sequence $\beta = (\beta_n)$ such that β_n is positive for $n = 1, 2, \ldots$ and the sequence (β_n) is nonincreasing. Such a sequence β will be called the tempering sequence. Next, consider the set X consisting of all real (or complex) sequences $x = (x_n)$ such that $\beta_n x_n \to 0$ as $n \to \infty$. It is easily seen that X forms a linear space over the field of real (or complex) numbers. We will denote this space by the symbol c_0^β. It is easy to check that c_0^β is a Banach space under the norm

$$\|x\|_{c_0^\beta} = \|(x_n)\|_{c_0^\beta} = \sup\{\beta_n |x_n| : n = 1, 2, \ldots\}.$$

In a similar way we may consider the space l_1^β consisting of real (complex) sequences (x_n) such that the sequence $(\beta_n x_n)$ converges to a finite limit. Obviously l_1^β forms a linear space and it becomes a Banach space if we normed it by norm

$$\|x\|_{l_1^\beta} = \|(x_n)\|_{l_1^\beta} = \sum_{n=1}^{\infty} \beta_n |x_n|.$$

Let us pay attention to the fact that taking $\beta_n = 1$ for $n = 1, 2, \ldots$ we obtain spaces $c_0^\beta = c_0$, and $l_1^\beta = l_1$. Similarly, if the sequence (β_n) is bounded from below by a positive constant m i.e., if $\beta_n \geq m > 0$ for $n = 1, 2, \ldots$, then the norm in the tempered sequence space c_0^β, (l_1^β) is equivalent to the classical supremum $(\sum)$ norm in the space c_0 (l_1). Thus, to obtain an essential enlargement of the spaces c_0 we should to assume that the tempering sequence (β_n) converges to zero. The pairs of the spaces (c_0, c_0^β) and (l_1^β, l_1) are isometric [7].
Now, in this paper we study the existence of solutions of infinite system of integral equations in N-variables in the spaces c_0^β and l_1^β by applying Meir-Keeler condensing operators. The results obtained in this paper generalize and extend earlier results due to Hazarika, Das, Arab and Mursaleen (see [13]).

The rest of the paper is organized as follows. In Section 2, we provide some notations, definitions and preliminary facts which will be needed further on. In Section 3, we construct the Hausdorff measures of noncompactness in both sequence spaces c_0^β and l_1^β. Sections 4 and 5 are devoted to applications of the results obtained to infinite systems of integral equations in N-variables in theses sequence spaces. We also give some examples to verify the effectiveness and applicability of our results.

2. Notation and auxiliary facts

Suppose $(E, \|\cdot\|)$ is a real Banach space with zero element 0. The symbol $B(x,r)$ stands for the ball centered at x with radius r. For a nonempty subset X of E, we denote by $\overline{X}$ and Conv$X$ the closure and closed convex hull of X, respectively. Moreover, let $\mathfrak{M}_E$ indicate the family of nonempty and bounded subsets of E and $\mathfrak{N}_E$ indicate the family of all nonempty and relatively compact subsets of E.

Definition 2.1. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, +\infty)$ is said to be a measure of noncompactness in E if it fulfils the following conditions:*

1° *The family* $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subseteq \mathfrak{N}_E$,
2° *$X \subset Y$ implies that $\mu(X) \le \mu(Y)$,*
3° $\mu(\overline{X}) = \mu(X)$,
4° $\mu(ConvX) = \mu(X)$,
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$,
6° *If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X_n}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.*

In the following, we denote by $\mathfrak{M}_X$, the collection of all bounded subsets of a metric space (X, d).

Definition 2.2. [6] *Let (X, d) be a metric space and $Q \in \mathfrak{M}_X$. Then the Kuratowski measure of noncompactness of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with diameters ε, that is*

$$\alpha(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i, S_i \subset X, diam(S_i) < \varepsilon \ (i = 1, 2, \ldots, n);\ n \in \mathbb{N}\Big\},$$

where $diam(S_i) = \sup\{d(x,y) : x, y \in S_i\}$.

The Hausdorff measure of noncompactness for a bounded set Q is defined by

$$\chi(Q) = \inf\Big\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon \ (i = 1, 2, \ldots, n);\ n \in \mathbb{N}\Big\}.$$

The Hausdorff measure of noncompactness is often called ball measure of noncompactness .

Lemma 2.3. [6] *Let Q, Q_1 and Q_2 be bounded subsets of a metric space (X, d). Then*

1° *$\chi(Q) = 0$ if and only if Q is totally bounded,*
2° *$Q_1 \subset Q_2$ implies that $\chi(Q_1) \le \chi(Q_2)$,*
3° $\chi(\overline{Q}) = \chi(Q)$,
4° $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$.

In the case of a normed space $(X, \|\cdot\|)$, the function $\chi : \mathfrak{M}_X \to \mathbb{R}_+$ has some additional properties connected with the linear structure for example, we have
i) $\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2)$,
ii) $\chi(Q + x) = \chi(Q)$ for all $x \in X$,
iii) $\chi(\lambda Q) = |\lambda|\chi(Q)$ for all $\lambda \in \mathbb{C}$,
iv) $\chi(Q) = \chi(\text{Conv}Q)$.

Definition 2.4. [3] *Suppose that E_1 and E_2 are two Banach spaces and μ_1 and μ_2 are arbitrary measures of noncompactness on E_1 and E_2, respectively. Also, suppose $T : E_1 \to E_2$ is a continuous operator satisfies the following condition:*

$$\mu_2(T(C)) < \mu_1(C)$$

for every bounded noncompact set $C \subset E_1$, then T is called a (μ_1, μ_2)-condensing operator.

Remark. *If in Definition 2.4 $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$, then T is called a μ-condensing operator.*

Theorem 2.5. *(Darbo* [10]*) Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \to C$ be a continuous mapping. Assume that a constant $k \in [0, 1)$ exists such that*

$$\mu(T(X)) \leq k\mu(X)$$

for any nonempty subset X of C, where μ is a measure of noncompactness defined in E. Then T has a fixed point in the set C.

The contractive maps and the compact maps are condensing if we take as measure of noncompactness the diameter of a set and the indicator function of a family of non-relatively compact sets, respectively [2]. In 1969, Meir and Keeler [16] introduced the concept of Meir-Keeler contractive mapping and proved some fixed point theorems for this kind of mappings. Thereafter, Aghajani et al. [2] generalized some fixed point and coupled fixed point theorems for Meir-Keeler condensing operators via measures of noncompactness.

Definition 2.6. [16] *Let (X, d) be a metric space. Then, a mapping T on X is said to be a Meir-Keeler contraction if for any $\varepsilon > 0$, $\delta > 0$ exists such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

for all $x, y \in X$.

Theorem 2.7. [16] *Let (X, d) be a complete metric space. If $T : X \to X$ is a Meir-Keeler contraction, then T has a unique fixed point.*

Definition 2.8. [2] *Let C be a nonempty subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. An operator $T : C \to C$ is called a Mier-Keeler condensing operator if for any $\varepsilon > 0$, $\delta > 0$ exists such that*

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \quad \textit{implies} \quad \mu(T(X)) < \varepsilon$$

for any bounded subset X of C.

Theorem 2.9. [2] *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and μ be an arbitrary measure of noncompactness on E. If $T : C \to C$ is a continuous and Meir-Keeler condensing operator, then T has at least one fixed point and the set of all fixed points of T in C is compact.*

3. Hausdorff measure of noncompactness in spaces of tempered sequences

In this section, we formulate the Hausdorff measures of noncompactness χ in the Banach spaces $(c_0^\beta, \|\cdot\|_{c_0^\beta})$ and $(l_1^\beta, \|\cdot\|_{l_1^\beta})$ in N-variables.

Let Q be a bounded subset of the normed space $(c_0^\beta, \|\cdot\|_{c_0^\beta})$, then the Huasdorff measure of noncompactness χ in the Banach space $(c_0^\beta, \|\cdot\|_{c_0^\beta})$ can be formulated as follows (see [7]):

$$\chi(Q) = \lim_{n\to\infty}\Big[\sup_{z(t_1,\ldots,t_N)\in Q}\Big(\max_{k\geq n}\beta_k|z_k(t_1,\ldots,t_N)|\Big)\Big], \tag{3.1}$$

where $z(t_1,\ldots,t_N) = (z_i(t_1,\ldots,t_N))_{i=1}^\infty \in c_0^\beta$ for each $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$ and $Q\in\mathfrak{M}_{c_0^\beta}$.

In the Banach space $(l_1^\beta, \|\cdot\|_{l_1^\beta})$, the Huasdorff measure of noncompactness χ can be defined as follows :

$$\chi(Q) = \lim_{n\to\infty}\Big[\sup_{z(t_1,\ldots,t_N)\in Q}\Big(\sum_{k=n}^\infty\beta_k|z_k(t_1,\ldots,t_N)|\Big)\Big], \tag{3.2}$$

where $z(t_1,\ldots,t_N) = (z_i(t_1,\ldots,t_N))_{i=1}^\infty \in l_1^\beta$ for each $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$ and $Q\in\mathfrak{M}_{l_1^\beta}$.

Consider the infinite system of integral equations in N-variables

$$z_n(t_1,\ldots,t_N) = f_n\Big(t_1,\ldots,t_N,\int_0^{a_1(t_1)}\cdots\int_0^{a_N(t_N)}g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,z(s_1,\ldots,s_N))ds_1\ldots ds_N, z(t_1,\ldots,t_N)\Big), \tag{3.3}$$

where $z(t_1,\ldots,t_N) = (z_i(t_1,\ldots,t_N))_{i=1}^\infty$, $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$, $n\in\mathbb{N}$ and $z_i\in C(\mathbb{R}_+^N,\mathbb{R})$ for all $i\in\mathbb{N}$.

4. Existence of solutions for infinite systems of integral equations in N-variables in tempered sequence space c_0^β

In this section, we are going to show how the measure χ, defined in (3.1), can be applied to the infinite system of integral equations (3.3) in the sequence space c_0.

Theorem 4.1. *Assume that the following conditions are satisfied.*
(i) $a_1,\ldots,a_N:\mathbb{R}_+\to[0,\infty)$ *are continuous.*
(ii) $f_n:\mathbb{R}_+^N\times\mathbb{R}\times\mathbb{R}^\infty\to\mathbb{R}$ $(n\in\mathbb{N})$ *is continuous with*

$$K_n = \sup_{k\geq n}\Big\{\beta_k|f_k(t_1,\ldots,t_N,0,z^0(t_1,\ldots,t_N))| : t_1,\ldots,t_N\in\mathbb{R}_+\Big\} < \infty,$$

where $z^0(t_1,\ldots,t_N) = \Big(z_i^0(t_1,\ldots,t_N)\Big)_{i=1}^\infty \in\mathbb{R}^\infty$ *and* $z_i^0(t_1,\ldots,t_N) = 0$, $\forall i\in\mathbb{N}$, $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$. *Also, continuous functions* $u_n, m_n:\mathbb{R}_+^N\to\mathbb{R}_+$ $(n\in\mathbb{N})$ *exist such that*

$$\begin{aligned}&|f_n(t_1,\ldots,t_N,p(t_1,\ldots,t_N),z(t_1,\ldots,t_N)) - f_n(t_1,\ldots,t_N,q(t_1,\ldots,t_N),\overline{z}(t_1,\ldots,t_N))|\\ &\quad\leq u_n(t_1,\ldots,t_N)\max_{i\geq n}\beta_i|z_i(t_1,\ldots,t_N)-\overline{z_i}(t_1,\ldots,t_N)|\\ &\quad\quad+m_n(t_1,\ldots,t_N)|p(t_1,\ldots,t_N)-q(t_1,\ldots,t_N)|,\end{aligned}$$

where p *and* q *are mappings from* $\mathbb{R}_+^N$ *into* $\mathbb{R}$, $z(t_1,\ldots,t_N) = \Big(z_i(t_1,\ldots,t_N)\Big)_{i=1}^\infty$, $\overline{z}(t_1,\ldots,t_N) = \Big(\overline{z_i}(t_1,\ldots,t_N)\Big)_{i=1}^\infty\in\mathbb{R}^\infty$.

(iii) $g_n : \mathbb{R}_+^{2N} \times \mathbb{R}^\infty \to \mathbb{R}$ $(n \in \mathbb{N})$ *is continuous and a constant* G_n *exists so that*

$$G_n = \sup_{k\geq n} \Big\{ \beta_k m_k(t_1, \ldots, t_N) \Big| \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_k(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N \Big| \Big\},$$

where $t_1, \ldots, t_N, s_1, \ldots, s_N \in \mathbb{R}_+$ *and* $z(s_1, \ldots, s_N) \in \mathbb{R}^\infty$. *Moreover, for each* n *we have*

$$\lim_{t_1,\ldots,t_N\to\infty} \beta_n \Big| m_n(t_1, \ldots, t_N) \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} \Big[g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N))$$
$$- g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, \overline{z}(s_1, \ldots, s_N)) \Big] ds_1 \ldots ds_N \Big| = 0.$$

(iv) Define an operator $Z : \mathbb{R}_+^N \times c_0^\beta \to c_0^\beta$ *as follows*
$(t_1, \ldots, t_N, z(t_1, \ldots, t_N)) \to (Zz)(t_1, \ldots, t_N)$, *where*

$$(Zz)(t_1, \ldots, t_N) = \Big(\beta_1 f_1(t_1, \ldots, t_N, v_1(z)(t_1, \ldots, t_N), z(t_1, \ldots, t_N)),$$
$$\beta_2 f_2(t_1, \ldots, t_N, v_2(z)(t_1, \ldots, t_N), z(t_1, \ldots, t_N)), \ldots \Big),$$

where $v_n(z)(t_1, \ldots, t_N) = \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N$.

(v) $\lim_{n\to\infty} K_n = 0$ *and* $\lim_{n\to\infty} G_n = 0$. *Also,* $\sup_n K_n = K$, $\sup_n G_n = G$, *and* $\sup_n \{\beta_n u_n(t_1, \ldots, t_N) : t_1, \ldots, t_N \in \mathbb{R}_+\} = U < \infty$ *so that* $0 < U < 1$.

Then the infinite system (3.3) has at least one solution $z(t_1, \ldots, t_N) = \Big(z_i(t_1, \ldots, t_N) \Big)_{i=1}^\infty$ *which belongs to the space* c_0 *for all* $t_1, \ldots, t_N \in \mathbb{R}_+$. *Also,* $z_i \in C(\mathbb{R}_+^N, \mathbb{R})$ *for all* $i \in \mathbb{N}$.

Proof. By applying our assumptions and Eq. (3.3), for all $t_1, \ldots, t_N \in \mathbb{R}_+$ we have

$$\|z(t_1, \ldots, t_N)\|_{c_0^\beta} = \max_{n\geq1} \Big\{ \beta_n |z_n(t_1, \ldots, t_N)| \Big\}$$
$$\begin{aligned}
&= \max_{n\geq1} \Big\{ \beta_n \Big| f_n\Big(t_1, \ldots, t_N, \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N, z(t_1, \ldots, t_N)\Big) \Big| \Big\} \\
&\leq \max_{n\geq1} \Big\{ \beta_n \Big| f_n\Big(t_1, \ldots, t_N, \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N, z(t_1, \ldots, t_N)\Big) \\
&\quad - f_n\Big(t_1, \ldots, t_N, 0, z^0(t_1, \ldots, t_N)\Big) \Big| \Big\} + \max_{n\geq1} \Big\{ \beta_n \Big| f_n\Big(t_1, \ldots, t_N, 0, z^0(t_1, \ldots, t_N)\Big) \Big| \Big\} \\
&\leq \max_{n\geq1} \Big\{ \beta_n u_n(t_1, \ldots, t_N) \max_{i\geq n} \beta_i |z_i(t_1, \ldots, t_N)| \\
&\quad + \beta_n m_n(t_1, \ldots, t_N) \Big| \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N \Big| \Big\} + K_1 \\
&\leq U \|z(t_1, \ldots, t_N)\|_{c_0^\beta} + G_1 + K_1 \\
&\leq U \|z(t_1, \ldots, t_N)\|_{c_0^\beta} + G + K
\end{aligned}$$

i.e. $(1 - U)\|z(t_1, \ldots, t_N)\|_{c_0^\beta} \leq G + K$ and so $\|z(t_1, \ldots, t_N)\|_{c_0^\beta} \leq \frac{G+K}{1-U} = r$(say).

Suppose that $\overline{B} = \overline{B}(z^0(t_1, \ldots, t_N), r)$ is a closed ball with center at $z^0(t_1, \ldots, t_N)$ and radius r, therefore $\overline{B}$ is a nonempty, bounded, closed and convex subset of c_0^β. Now, we define the operator $Z = (Z_i)$ on $C(\mathbb{R}_+^N, \overline{B})$ by the formula

$$(Zz)(t_1, \ldots, t_N) = \Big((Z_i z)(t_1, \ldots, t_N) \Big) = \Big(\beta_i f_i(t_1, \ldots, t_N, v_i(z)(t_1, \ldots, t_N), z(t_1, \ldots, t_N)) \Big),$$

where $z(t_1,\ldots,t_N)=\Big(z_i(t_1,\ldots,t_N)\Big)\in\overline{B}$ and $z_i\in C(\mathbb{R}_+^N,\mathbb{R})$, $\forall i\in\mathbb{N}$. From hypothesis (iv), for each $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$ we have

$$\lim_{i\to\infty}(Z_iz)(t_1,\ldots,t_N)=\lim_{i\to\infty}\beta_if_i(t_1,\ldots,t_N,v_i(z)(t_1,\ldots,t_N),z(t_1,\ldots,t_N))=0.$$

Hence $(Zz)(t_1,\ldots,t_N)\in c_0^\beta$.
Since $\|(Zz)(t_1,\ldots,t_N)-z^0(t_1,\ldots,t_N)\|_{c_0^\beta}\le r$, then Z is a self mapping on $\overline{B}$.
We claim that the operator Z is a continuous function on $C(\mathbb{R}_+^N,\overline{B})$. To establish this claim, let us $\varepsilon>0$ and take arbitrary $x_m(t_1,\ldots,t_N)=\Big(x_{m,i}(t_1,\ldots,t_N)\Big)_{i=1}^\infty$, $x(t_1,\ldots,t_N)=\Big(x_i(t_1,\ldots,t_N)\Big)_{i=1}^\infty\in\overline{B}\subseteq c_0^\beta$ such that $\|x_m(t_1,\ldots,t_N)-x(t_1,\ldots,t_N)\|_{c_0^\beta}<\frac{\varepsilon}{2U}$ for m sufficiently large. We claim that $\|Zx_m(t_1,\ldots,t_N)-Zx(t_1,\ldots,t_N)\|_{c_0^\beta}\to 0$, for m large enough. To this end, we show that $\beta_n|Z_nx_m(t_1,\ldots,t_N)-Z_nx(t_1,\ldots,t_N)|$ tends to 0 as $m\to\infty$. Taking into account condition (ii), for each $(t_1,\ldots,t_N)\in\mathbb{R}_+^N$ we get

$$\begin{aligned}
&\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)| &&(4.1)\\
=\;&\beta_n|f_n(t_1,\ldots,t_N,v_n(x_m)(t_1,\ldots,t_N),x_m(t_1,\ldots,t_N))\\
&-f_n(t_1,\ldots,t_N,v_n(x)(t_1,\ldots,t_N),x(t_1,\ldots,t_N))|\\
\le\;&\beta_nu_n(t_1,\ldots,t_N)\max_{i\ge n}\beta_i|x_{m,i}(t_1,\ldots,t_N)-x_i(t_1,\ldots,t_N)|\\
&+\beta_nm_n(t_1,\ldots,t_N)|v_n(x_m)(t_1,\ldots,t_N)-v_n(x)(t_1,\ldots,t_N)|\\
\le\;&U\|x_m(t_1,\ldots,t_N)-x(t_1,\ldots,t_N)\|_{c_0^\beta}\\
&+\beta_nm_n(t_1,\ldots,t_N)\Big|\int_0^{a_1(t_1)}\ldots\int_0^{a_N(t_N)}\Big[g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))\\
&-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big]ds_1\ldots ds_N\Big|.
\end{aligned}$$

By applying hypothesis (iii), we choose $T>0$ such that $\max(t_1,\ldots,t_N)>T$, and we derive that

$$\beta_n\Big|m_n(t_1,\ldots,t_N)\int_0^{a_1(t_1)}\ldots\int_0^{a_N(t_N)}\Big[g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))$$
$$-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big]ds_1\ldots ds_N\Big|<\frac{\varepsilon}{2}.$$

It follows that $\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|<\varepsilon$.
For $t_1,\ldots,t_N\in[0,T]$, put
$A_1^T=\sup\{a_1(t_1):t_1\in[0,T]\}$,
$A_2^T=\sup\{a_2(t_2):t_2\in[0,T]\}$,
$\vdots$
$A_N^T=\sup\{a_N(t_N):t_N\in[0,T]\}$,
$M_T=\sup_n\{\beta_nm_n(t_1,\ldots,t_N):t_1,\ldots,t_N\in[0,T]\}$,
and

$$g_{x_m,x}=\sup_n\Big\{\Big|g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big|,$$
$$t_1,\ldots,t_N\in[0,T],s_1\in[0,A_N^T],\ldots,s_N\in[0,A_1^T]\Big\}.$$

By (4.1) we find that

$$|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|<\frac{\varepsilon}{2}+M^Tg_{x_m,x}A_N^T\ldots A_1^T.$$

By using the continuity of g_n on the set $[0,T]^N\times[0,A_N^T]\times\ldots\times[0,A_1^T]\times c_0^\beta$, we have $g_{x_m,x}\to 0$ as $\varepsilon\to 0$. It enforces that $\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|\to 0$ as $\|x_m(t_1,\ldots,t_N)-x(t_1,\ldots,t_N)\|_{c_0^\beta}\to 0$ for m large enough.

Therefore, we infer that Z is a continuous function on $\overline{B} \subset c_0^\beta$.
In order to finish the proof, we show that Z is a Meir-Keeler condensing operator on $\overline{B}$. Let Q be any bounded subset of $\overline{B}$ and $\varepsilon > 0$ be arbitrary. We have to find $\delta > 0$ such that $\varepsilon \leq \chi(Q) < \varepsilon + \delta \Rightarrow \chi(Z(Q)) < \varepsilon$.
In view of conditions (ii) and (iv) we observe that

$$\begin{aligned}
\chi(Z(Q)) &= \lim_{n\to\infty}\Big[\sup_{z(t_1,\dots,t_N)\in Q}\Big\{\max_{k\geq n}\beta_k\Big|f_k(t_1,\dots,t_N,v_k(z),z(t_1,\dots,t_N))\Big|\Big\}\Big]\\
&\leq \lim_{n\to\infty}\Big[\sup_{z(t_1,\dots,t_N)\in Q}\Big\{\max_{k\geq n}\beta_k\Big|f_k(t_1,\dots,t_N,v_k(z),z(t_1,\dots,t_N))\\
&\quad -f_k(t_1,\dots,t_N,0,z^0(t_1,\dots,t_N))\Big| + \max_{k\geq n}\beta_k\Big|f_k(t_1,\dots,t_N,0,z^0(t_1,\dots,t_N))\Big|\Big\}\Big]\\
&\leq \lim_{n\to\infty}\Big[\sup_{z(t_1,\dots,t_N)\in Q}\Big\{\max_{k\geq n}\Big(\beta_k u_k(t_1,\dots,t_N)\max_{i\geq k}\beta_i|z_i(t_1,\dots,t_N)| + \beta_k m_k(t_1,\dots,t_N)\\
&\quad \Big|\int_0^{a_1(t_1)}\cdots\int_0^{a_N(t_N)} g_k(t_1,\dots,t_N,s_1,\dots,s_N,z(s_1,\dots,s_N))ds_1\dots ds_N\Big| + K_n\Big)\Big\}\Big]\\
&\leq U\lim_{n\to\infty}\Big[\sup_{z(t_1,\dots,t_N)\in Q}\Big\{\max_{i\geq k}\beta_i|z_i(s_1,\dots,s_N)| + G_n + K_n\Big\}\Big]
\end{aligned}$$

Since, $G_n \to 0$ and $K_n \to 0$ as $n \to \infty$, we deduce

$$\chi(Z(Q)) \leq U\chi(Q). \tag{4.2}$$

Taking $\delta = \frac{\varepsilon(1-U)}{U}$. From (4.2) we obtain

$$\varepsilon \leq \chi(Q) < \varepsilon + \delta \Rightarrow \chi(Z(Q)) < \varepsilon.$$

Therefore Z is a Meir-Keeler condensing operator defined on the set $\overline{B} \subset c_0^\beta$. Now, Theorem 2.9 guarantees that Z has a fixed point in $\overline{B}$ and thus infinite system of integral equations (3.3) has at least one solution in c_0^β. $\square$

Example 4.2. *Consider the following infinite system of integral equations*

$$x_n(t_1,t_2,t_3) = \frac{1}{t_1t_2+t_3^2+n^4} + \frac{1}{n^2}\sum_{i=n}^{\infty}\Big(\frac{x_i(t_1,t_2,t_3)}{3(2i-1)^2}\Big) + \frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}} \tag{4.3}$$

$$\arctan\Big(\int_0^{e^{t_1^2}}\int_0^{e^{t_2^2}}\int_0^{e^{t_3^2}}\frac{\sin(x_n(s_1,s_2,s_3)) + \sin(s_1)\cos(\ln\sum_{i=1}^{\infty}(x_i(s_1,s_2,s_3))^2)}{3+\cos(\sum_{i=1}^{\infty}x_i(s_1,s_2,s_3))}ds_1ds_2ds_3\Big),$$

where $n \in \mathbb{N}$.
Eq. (4.3) is a special case of Eq. (3.3). Here $a_1(t) = a_2(t) = a_3(t) = e^{t^2}$,

$$\begin{aligned}
f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3)) &= \frac{1}{t_1t_2+t_3^2+n^4} + \frac{1}{n^2}\sum_{i=n}^{\infty}\Big(\frac{x_i(t_1,t_2,t_3)}{3(2i-1)^2}\Big)\\
&+ \frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}\arctan\Big(v_n(x)(t_1,t_2,t_3)\Big),
\end{aligned}$$

$$g_n(t_1,t_2,t_3,s_1,s_2,s_3,x(t_1,t_2,t_3)) = \frac{\sin(x_n(s_1,s_2,s_3)) + \sin(s_1)\cos(\ln\sum_{i=1}^{\infty}(x_i(s_1,s_2,s_3))^2)}{3+\cos(\sum_{i=1}^{\infty}x_i(s_1,s_2,s_3))},$$

where $x(t_1,t_2,t_3) = \Big(x_i(t_1,t_2,t_3)\Big)_{i=1}^{\infty}$ and

$$v_n(x)(t_1,t_2,t_3) = \int_0^{e^{t_1^2}}\int_0^{e^{t_2^2}}\int_0^{e^{t_3^2}} g_n(t_1,t_2,t_3,s_1,s_2,s_3,x(t_1,t_2,t_3))ds_1ds_2ds_3.$$

Further, take the tempering sequence of the form $\beta = (\beta_n) = (\frac{1}{n^2})$. *From the definition of* a_1,a_2 *and* a_3, *hypothesis (i) of Theorem 4.1 is obviously satisfied. If* $x(t_1,t_2,t_3) \in c_0^\beta$, *then* $f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3)) \in c_0^\beta$. *Now, if* $y(t_1,t_2,t_3) = \Big(y_i(t_1,t_2,t_3)\Big)_{i=1}^\infty \in c_0^\beta$, *then by taking* $u_n(t_1,t_2,t_3) = \frac{\Pi^2}{24}$ *and* $m_n(t_1,t_2,t_3) = \frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}$. *We can write*

$$\Big|f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3)) - f_n(t_1,t_2,t_3,v_n(y)(t_1,t_2,t_3),y(t_1,t_2,t_3))\Big|$$

$$\begin{aligned}
\leq\ & \sum_{i=n}^{\infty}\frac{1}{3(2i-1)^2}\times\frac{1}{n^2}|x_i(t_1,t_2,t_3)-y_i(t_1,t_2,t_3)| \\
& +\frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}|\arctan v_n(x)(t_1,t_2,t_3)-\arctan v_n(y)(t_1,t_2,t_3)| \\
\leq\ & \frac{\Pi^2}{24}\max_{i\geq n}\frac{1}{i^2}|x_i(t_1,t_2,t_3)-y_i(t_1,t_2,t_3)| + \frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}|v_n(x)(t_1,t_2,t_3)-v_n(y)(t_1,t_2,t_3)|.
\end{aligned}$$

Then the condition (ii) holds. Further,

$$\begin{aligned}
K_n &= \sup\Big\{\frac{1}{n^2}|f_n(t_1,t_2,t_3,0,z^0(t_1,t_2,t_3))| : t_1,t_2,t_3\in\mathbb{R}_+\Big\} \\
&= \sup\Big\{\frac{1}{n^2}\Big|\frac{1}{t_1t_2+t_3^2+n^4}\Big| : t_1,t_2,t_3\in\mathbb{R}_+\Big\}\leq 1,
\end{aligned}$$

and $K_n \to 0$ *as* $n\to\infty$, *and* $0<U<1$. *Evidently,* g_n *is continuous and since*

$$\Big|\int_0^{e^{t_1^2}}\int_0^{e^{t_2^2}}\int_0^{e^{t_3^2}}\frac{\sin(x_n(s_1,s_2,s_3))+\sin(s_1)\cos(\ln\sum_{i=1}^\infty(x_i(s_1,s_2,s_3))^2)}{3+\cos(\sum_{i=1}^\infty x_i(s_1,s_2,s_3))}ds_1ds_2ds_3\Big|$$

$$\leq 2\Big|\int_0^{e^{t_1^2}}\int_0^{e^{t_2^2}}\int_0^{e^{t_3^2}}ds_1ds_2ds_3\Big| \leq 2e^{t_1^2+t_2^2+t_3^2}$$

for all $t_1,t_2,t_3\in\mathbb{R}_+$, *so we deduce that*

$$G_n \leq \sup\Big\{\frac{1}{n^2}(\frac{3e^{t_1^2}e^{t_2^2}e^{t_3^2}}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}) : t_1,t_2,t_3\in\mathbb{R}_+\Big\} = \frac{1}{n^2}(\frac{3}{n^3+2n^2}).$$

As $n\to\infty$ *we obtain* $G_n\to 0$. *These prove condition (v). Moreover, for each* n *we get as* $t_1,t_2,t_3\to\infty$

$$\begin{aligned}
\frac{1}{n^2}\Big|\frac{1}{(n^3+2n^2)e^{t_1^2+t_2^2+t_3^3}}\int_0^{e^{t_1^2}}\int_0^{e^{t_2^2}}\int_0^{e^{t_3^2}}\Big[&g_n(t_1,t_2,t_3,s_1,s_2,s_3,x(t_1,t_2,t_3)) \\
&-g_n(t_1,t_2,t_3,s_1,s_2,s_3,y(t_1,t_2,t_3))\Big]ds_1ds_2ds_3\Big|\to 0,
\end{aligned}$$

which shows that condition (iii) is satisfied. Consequently, all the conditions of Theorem 4.1 are satisfied. Hence the infinite system of integral equations (4.3) has at least one solution, which belongs to the space c_0^β.

5. Existence of solutions for infinite systems of integral equations in N-variables in tempered sequence space l_1^β

To demonstrate the applicability of the Hausdorff measure of noncompactness (3.2) in the space l_1^β, we look for solutions of the Eq. (3.3) in the space l_1^β. Assume that

(a) $a_1, \ldots, a_N : \mathbb{R}_+ \to [0, \infty)$ are continuous.
(b) $f_n : \mathbb{R}_+^N \times \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ $(n \in \mathbb{N})$ is continuous with

$$\sum_{n=1}^{\infty} \beta_n |f_n(t_1, \ldots, t_N, 0, z^0(t_1, \ldots, t_N))|$$

is convergent to zero for all $t_1, \ldots, t_N \in \mathbb{R}_+$ and $z^0(t_1, \ldots, t_N) = \Big(z_i^0(t_1, \ldots, t_N)\Big)_{i=1}^{\infty} \in \mathbb{R}^\infty$ with $z_i^0(t_1, \ldots, t_N)) = 0, \forall i \in \mathbb{N}$. Also, continuous functions $\alpha_n, \gamma_n : \mathbb{R}_+^N \to \mathbb{R}_+$ $(n \in \mathbb{N})$ exist such that

$$|f_n(t_1, \ldots, t_N, p(t_1, \ldots, t_N), z(t_1, \ldots, t_N)) - f_n(t_1, \ldots, t_N, q(t_1, \ldots, t_N), \overline{z}(t_1, \ldots, t_N))|$$

$$\leq \alpha_n(t_1, \ldots, t_N)|z_n(t_1, \ldots, t_N) - \overline{z_n}(t_1, \ldots, t_N)| + \gamma_n(t_1, \ldots, t_N)|p(t_1, \ldots, t_N) - q(t_1, \ldots, t_N)|,$$

where p and q are mappings from $\mathbb{R}_+^N$ into $\mathbb{R}$, $z(t_1, \ldots, t_N) = \Big(z_i(t_1, \ldots, t_N)\Big)_{i=1}^{\infty}$, $\overline{z}(t_1, \ldots, t_N) = \Big(\overline{z_i}(t_1, \ldots, t_N)\Big)_{i=1}^{\infty}$ are in $\mathbb{R}^\infty$.
(c) $g_n : \mathbb{R}_+^{2N} \times \mathbb{R}^\infty \to \mathbb{R}$ $(n \in \mathbb{N})$ is continuous and a constant Q_k exists so that

$$Q_k = \sup\Big\{ \sum_{n \geq k} \beta_n \Big(\gamma_n(t_1, \ldots, t_N)\Big| \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N\Big|\Big)\Big\},$$

where $z(s_1, \ldots, s_N) \in \mathbb{R}^\infty$, $t_1, \ldots, t_N, s_1, \ldots, s_N \in \mathbb{R}_+$. Moreover,

$$\lim_{t_1, \ldots, t_N \to \infty} \sum_{n=1}^{\infty} \beta_n \Big|\gamma_n(t_1, \ldots, t_N) \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} \Big(g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N))$$

$$-g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, \overline{z}(s_1, \ldots, s_N))\Big) ds_1 \ldots ds_N\Big| = 0.$$

(d) Define an operator $Z : \mathbb{R}_+^N \times l_1^\beta \to l_1^\beta$ as follows
$(t_1, \ldots, t_N, z(t_1, \ldots, t_N)) \to (Zz)(t_1, \ldots, t_N)$ where

$$(Zz)(t_1, \ldots, t_N) = \Big(\beta_1 f_1(t_1, \ldots, t_N, v_1(t_1, \ldots, t_N), z(t_1, \ldots, t_N)), \beta_2 f_2(t_1, \ldots, t_N, v_2(t_1, \ldots, t_N), z(t_1, \ldots, t_N)), \ldots\Big)$$

and $v_n(z)(t_1, \ldots, t_N) = \int_0^{a_1(t_1)} \cdots \int_0^{a_N(t_N)} g_n(t_1, \ldots, t_N, s_1, \ldots, s_N, z(s_1, \ldots, s_N)) ds_1 \ldots ds_N$.
(e) $\lim_{k \to \infty} Q_k = 0$, $\sup_k Q_k = Q$, $\eta_1 = \sup\Big\{ \sum_{n=1}^{\infty} \beta_n \gamma_n(t_1, \ldots, t_N) : t_1, \ldots, t_N \in \mathbb{R}_+\Big\}$ and $\sup_n \Big\{\alpha_n(t_1, \ldots, t_N) : t_1, \ldots, t_N \in \mathbb{R}_+\Big\} = \alpha < \infty$ such that $0 < \alpha < 1$.

Theorem 5.1. *Under the hypotheses (a)-(e), Eq. (3.3) has at least one solution* $z(t_1, \ldots, t_N) = \Big(z_i(t_1, \ldots, t_N)\Big)_{i=1}^{\infty}$ *which belongs to the space* l_1^β *for all* $t_1, \ldots, t_N \in \mathbb{R}_+$. *Also,* $z_i \in C(\mathbb{R}_+^N, \mathbb{R})$ *for all* $i \in \mathbb{N}$.

Proof. Applying conditions (a)-(e), and Eq. (3.3), for all $t_1,\dots,t_N\in\mathbb{R}_+$ we obtain

$$\begin{aligned}
\|z(t_1,\dots,t_N)\|_{l_1^\beta} &= \sum_{n=1}^{\infty}\beta_n|z_n(t_1,\dots,t_N)|\\
&= \sum_{n=1}^{\infty}\beta_n\Big|f_n\Big(t_1,\dots,t_N,\int_0^{a_1(t_1)}\dots\int_0^{a_N(t_N)}g_n(t_1,\dots,t_N,s_1,\dots,s_N,z(s_1,\dots,s_N))ds_1\dots ds_N,z(t_1,\dots,t_N)\Big)\Big|\\
&\le \sum_{n=1}^{\infty}\beta_n\Big|f_n\Big(t_1,\dots,t_N,\int_0^{a_1(t_1)}\dots\int_0^{a_N(t_N)}g_n(t_1,\dots,t_N,s_1,\dots,s_N,z(s_1,\dots,s_N))ds_1\dots ds_N,z(t_1,\dots,t_N)\Big)\\
&\quad -f_n\Big(t_1,\dots,t_N,0,z^0(t_1,\dots,t_N)\Big)\Big|+\sum_{n=1}^{\infty}\beta_n\Big|f_n\Big(t_1,\dots,t_N,0,z^0(t_1,\dots,t_N)\Big)\Big|\\
&\le \sum_{n=1}^{\infty}\alpha_n(t_1,\dots,t_N)\beta_n|z_n(t_1,\dots,t_N)|\\
&\quad +\sum_{n=1}^{\infty}\beta_n\gamma_n(t_1,\dots,t_N)\Big|\int_0^{a_1(t_1)}\dots\int_0^{a_N(t_N)}g_n(t_1,\dots,t_N,s_1,\dots,s_N,z(s_1,\dots,s_N))ds_1\dots ds_N\Big|\\
&\le \alpha\sum_{n=1}^{\infty}\beta_n|z_n(t_1,\dots,t_N)|+Q_1\le\alpha\|z(t_1,\dots,t_N)\|_{l_1^\beta}+Q
\end{aligned}$$

i.e. $(1-\alpha)\|z(t_1,\dots,t_N)\|_{l_1^\beta}\le Q\Rightarrow\|z(t_1,\dots,t_N)\|_{l_1^\beta}\le\frac{Q}{1-\alpha}=r(\text{say})$.

Suppose that $\overline{D}=\overline{D}(z^0(t_1,\dots,t_N),r)$ is a closed ball in l_1^β with center at $z^0(t_1,\dots,t_N)$ and radius r, thus $\overline{D}$ is a nonempty, bounded, closed and convex subset of l_1^β.
Now, we define the operator $Z=(Z_i)$ on $C(\mathbb{R}_+^N,\overline{D})$ by formula

$$(Zz)(t_1,\dots,t_N)=\Big(\beta_i(Z_iz)(t_1,\dots,t_N)\Big)_{i=1}^{\infty}=\Big(\beta_if_i(t_1,\dots,t_N,v_i(z)(t_1,\dots,t_N),z(t_1,\dots,t_N))\Big)_{i=1}^{\infty},$$

where $z(t_1,\dots,t_N)=\Big(z_i(t_1,\dots,t_N)\Big)_{i=1}^{\infty}\in\overline{D}$ and $z_i\in C(\mathbb{R}_+^N,\mathbb{R})$, $\forall i\in\mathbb{N}$. Since, by condition (d), for each $(t_1,\dots,t_N)\in\mathbb{R}_+^N$ we have

$$\sum_{i=1}^{\infty}\beta_i|(Z_iz)(t_1,\dots,t_N)|=\sum_{i=1}^{\infty}\beta_i|f_i(t_1,\dots,t_N,v_i(z)(t_1,\dots,t_N),z(t_1,\dots,t_N))|<\infty,$$

hence $(Zz)(t_1,\dots,t_N)\in l_1^\beta$.
Further, $\|(Zz)(t_1,\dots,t_N)-z^0(t_1,\dots,t_N)\|_{l_1^\beta}\le r$ and so Z is a self mapping on $\overline{D}$.
Next, we show that Z is continuous. For this, take $x_m(t_1,\dots,t_N)=\Big(x_{m,i}(t_1,\dots,t_N)\Big)_{i=1}^{\infty}$, $x(t_1,\dots,t_N)=\Big(x_i(t_1,\dots,t_N)\Big)_{i=1}^{\infty}\in l_1^\beta$ and $\varepsilon>0$ arbitrary with $\|x_m(t_1,\dots,t_N)-x(t_1,\dots,t_N)\|_{l_1^\beta}<\frac{\varepsilon}{2\alpha}$ for m sufficiently large. We claim that $\|Zx_m(t_1,\dots,t_N)-Zx(t_1,\dots,t_N)\|_{l_1^\beta}\to0$, for m large enough. We will show that $\beta_n|Z_nx_m(t_1,\dots,t_N)-Z_nx(t_1,\dots,t_N)|\to0$, for m large enough. Then, for each $(t_1,\dots,t_N)\in\mathbb{R}_+^N$, we have

$$\begin{aligned}
&\beta_n|(Z_nx_m)(t_1,\dots,t_N)-(Z_nx)(t_1,\dots,t_N)|\\
&= \beta_n|f_n(t_1,\dots,t_N,v_n(x_m)(t_1,\dots,t_N),x_m(t_1,\dots,t_N))-f_n(t_1,\dots,t_N,v_n(x)(t_1,\dots,t_N),x(t_1,\dots,t_N))|\\
&\le \beta_n\alpha_n(t_1,\dots,t_N)|x_m(t_1,\dots,t_N)-x(t_1,\dots,t_N)|\\
&\quad +\beta_n\gamma_n(t_1,\dots,t_N)|v_n(x_m)(t_1,\dots,t_N)-v_n(x)(t_1,\dots,t_N)|\\
&\le \alpha\beta_n|x_m(t_1,\dots,t_N)-x(t_1,\dots,t_N)|\\
&\quad +\beta_n\gamma_n(t_1,\dots,t_N)\Big|\int_0^{a_1(t_1)}\dots\int_0^{a_N(t_N)}\Big[g_n(t_1,\dots,t_N,s_1,\dots,s_N,x_m(s_1,\dots,s_N))\\
&\quad -g_n(t_1,\dots,t_N,s_1,\dots,s_N,x(s_1,\dots,s_N))\Big]ds_1\dots ds_N\Big|
\end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)| \quad \leq \quad \alpha\|x_m(t_1,\ldots,t_N)-x(t_1,\ldots,t_N)\|_{l_1^{\beta}}+\sum_{n=1}^{\infty}\beta_n\gamma_n(t_1,\ldots,t_N)$$

$$\Big|\int_0^{a_1(t_1)}\ldots\int_0^{a_N(t_N)}\Big[g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big]ds_1\ldots ds_N\Big|.$$

In view of condition (c), $T_1>0$ exists such that if $\max(t_1,\ldots,t_N)>T_1$, then

$$\sum_{n=1}^{\infty}\beta_n\gamma_n(t_1,\ldots,t_N)\Big|\int_0^{a_1(t_1)}\ldots\int_0^{a_N(t_N)}\Big[g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))$$

$$-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big]ds_1\ldots ds_N\Big|<\frac{\varepsilon}{2}.$$

Hence for $\max(t_1,\ldots,t_N)>T_1$, $\sum_{n=1}^{\infty}\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|<\varepsilon$ *i.e.*

$$\|(Zx_m)(t_1,\ldots,t_N)-(Zx)(t_1,\ldots,t_N)\|_{l_1^{\beta}}<\varepsilon.$$

For $t_1,\ldots,t_N\in[0,T]$, let
$A_1^T=\sup\{a_1(t_1):t_1\in[0,T]\}$,
$A_2^T=\sup\{a_2(t_2):t_2\in[0,T]\}$,
$\vdots$
$A_N^T=\sup\{a_N(t_N):t_N\in[0,T]\}$,
and

$$g_{x_m,x}=\sup_n\Big\{\Big|g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x_m(s_1,\ldots,s_N))-g_n(t_1,\ldots,t_N,s_1,\ldots,s_N,x(s_1,\ldots,s_N))\Big|,$$

$$t_1,\ldots,t_N\in[0,T],s_1\in[0,A_N^T],\ldots,s_N\in[0,A_1^T]\Big\}.$$

Then $\sum_{n=1}^{\infty}\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|<\frac{\varepsilon}{2}+g_{x_m,x}A_N^T\ldots A_1^T\eta_1$. By using the continuity of g_n on the set $[0,T]^N\times[0,A_N^T]\times\ldots\times[0,A_1^T]\times l_1^{\beta}$, we obtain $g_{x_m,x}\to 0$ as $m\to\infty$, thus

$$\sum_{n=1}^{\infty}\beta_n|(Z_nx_m)(t_1,\ldots,t_N)-(Z_nx)(t_1,\ldots,t_N)|\to 0$$

as $\|x_m(t_1,\ldots,t_N)-x(t_1,\ldots,t_N)\|_{l_1^{\beta}}\to 0$.
We infer that Z is a continuous function on $\overline{D}\subset l_1^{\beta}$.
In what follows, we verify that Z is a Meir-Keeler condensing operator.
For $\varepsilon>0$, we have to find $\delta>0$ such that $\varepsilon\leq\chi(\mathfrak{D})<\varepsilon+\delta\Rightarrow\chi(Z(\mathfrak{D}))<\varepsilon$ for any nonempty bounded subset $\mathfrak{D}$ of $\overline{D}$.
From (b) and (d) we deduce

$$
\begin{aligned}
&\chi(Z(\mathfrak{D}))\\
=\ & \lim_{n\to\infty}\Big(\sup_{z(t_1,\ldots,t_N)\in\mathfrak{D}}\Big\{\sum_{k\geq n}\beta_k\Big|f_k(t_1,\ldots,t_N,v_k(z)(t_1,\ldots,t_N),z(t_1,\ldots,t_N))\Big|\Big\}\Big)\\
\leq\ & \lim_{n\to\infty}\Big(\sup_{z(t_1,\ldots,t_N)\in\mathfrak{D}}\Big\{\sum_{k\geq n}\beta_k\Big|f_k(t_1,\ldots,t_N,v_k(z)(t_1,\ldots,t_N),z(t_1,\ldots,t_N))\\
& -f_k(t_1,\ldots,t_N,0,z^0(t_1,\ldots,t_N))\Big|+\sum_{k\geq n}\beta_k\Big|f_k(t_1,\ldots,t_N,0,z^0(t_1,\ldots,t_N))\Big|\Big\}\Big)\\
\leq\ & \lim_{n\to\infty}\Big(\sup_{z(t_1,\ldots,t_N)\in\mathfrak{D}}\Big\{\sum_{k\geq n}\Big(\beta_k\alpha_k(t_1,\ldots,t_N)|z_k(t_1,\ldots,t_N)|\\
& +\beta_k\gamma_k(t_1,\ldots,t_N)\Big|\int_0^{a_1(t_1)}\cdots\int_0^{a_N(t_N)}g_k(t_1,\ldots,t_N,s_1,\ldots,s_N,z(s_1,\ldots,s_N))ds_1\ldots ds_N\Big|\Big)\Big\}\Big)\\
\leq\ & \lim_{n\to\infty}\Big(\sup_{z(t_1,\ldots,t_N)\in\mathfrak{D}}\Big\{\alpha\sum_{k\geq n}\beta_k|z_k(t_1,\ldots,t_N)|+Q_n\Big\}\Big).
\end{aligned}
$$

Since, $Q_n\to 0$, as $n\to\infty$, we derive that

$$\chi(Z(\mathfrak{D}))\leq\alpha\chi(\mathfrak{D}). \tag{5.1}$$

Let us choose $\delta=\frac{\varepsilon(1-\alpha)}{\alpha}$. From (5.1), it ts easy to see that Z is a Meir-Keeler condensing operator defined on the set $\overline{D}\subset l_1^{\beta}$. Now, by Theorem 2.9 we find that Z has a fixed point in $\overline{D}$ and thus the infinite system of integral equations (3.3) has at least one solution in l_1^{β}. □

Example 5.2. *Consider the following infinite system of integral equations*

$$x_n(t_1,t_2,t_3)=\sum_{i=n}^{\infty}\Big(\frac{\sin(\frac{\Pi}{2i})\cos(e^{t_1t_2t_3})x_i(t_1,t_2,t_3)}{3i}\Big) \tag{5.2}$$

$$+\frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}\sin\Big(\int_0^{e^{t_1}}\int_0^{e^{t_2}}\int_0^{e^{t_3}}\frac{\tanh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}{5+\cosh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}ds_1ds_2ds_3\Big),$$

where $n\in\mathbb{N}$. *Eq. (5.2) is a special case of Eq. (3.3). Here* $a_1(t)=a_1(t)=a_3(t)=e^t$,

$$f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3))=\sum_{i=n}^{\infty}\Big(\frac{\sin(\frac{\Pi}{2i})\cos(e^{t_1t_2t_3})x_i(t_1,t_2,t_3)}{3i}\Big)$$

$$+\frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}\sin(v_n(x)(t_1,t_2,t_3)),$$

where $v_n(x)(t_1,t_2,t_3)=\int_0^{e^{t_1}}\int_0^{e^{t_2}}\int_0^{e^{t_3}}g_n(t_1,t_2,t_3,s_1,s_2,s_3,x(t_1,t_2,t_3))ds_1ds_2ds_3$, *and*

$$g_n(t_1,t_2,t_3,s_1,s_2,s_3,x(t_1,t_2,t_3))=\frac{\tanh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}{5+\cosh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}.$$

Furthermore, take $\beta = (\beta_n) = (\frac{1}{n^3})$. *If* $x(t_1,t_2,t_3) \in l_1^\beta$ *is arbitrary, then we have*

$$\sum_{n=1}^{\infty} \beta_n |f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3))|$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n^3}\Big|\sum_{i=n}^{\infty}\big(\frac{\sin(\frac{\Pi}{2i})\cos(e^{t_1t_2t_3})x_i(t_1,t_2,t_3)}{3i}\big) \\
&\quad + \frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}\sin(v_n(x)(t_1,t_2,t_3))\Big| \\
&\leq \sum_{n=1}^{\infty}\sum_{i=n}^{\infty}\frac{1}{n^3}\Big|\frac{\sin(\frac{\Pi}{2i})x_i(t_1,t_2,t_3)}{3i}\Big| + \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} \\
&\leq \sum_{n=1}^{\infty}\sum_{i=n}^{\infty}|\frac{\Pi}{6i^2}||\frac{1}{n^3}x_i(t_1,t_2,t_3)| + \frac{1}{4} \\
&\leq \frac{\pi^3}{36}\|x(t_1,t_2,t_3)\|_{l_1^\beta} + \frac{1}{4} \\
&\leq \|x(t_1,t_2,t_3)\|_{l_1^\beta} + \frac{1}{4} < \infty.
\end{aligned}$$

Therefore $f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3),x(t_1,t_2,t_3)) \in l_1^\beta$. *Now, if* $y(t_1,t_2,t_3) = \Big(y_i(t_1,t_2,t_3)\Big)_{i=1}^{\infty} \in l_1^\beta$, *then by taking* $\alpha_n(t_1,t_2,t_3) = \frac{\Pi^3}{36}$ *and* $\gamma_n(t_1,t_2,t_3) = \frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}$ *we get*

$$\Big|f_n(t_1,t_2,t_3,v_n(x)(t_1,t_2,t_3) - f_n(t_1,t_2,t_3,v_n(y)(t_1,t_2,t_3)\Big|$$

$$\begin{aligned}
&\leq \Big|\sum_{i=n}^{\infty}\frac{\sin(\frac{\Pi}{2i})\cos(e^{t_1t_2t_3})}{3i}(x_i(t_1,t_2,t_3) - y_i(t_1,t_2,t_3))\Big| \\
&\quad + \frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}\Big|\sin v_n(x)(t_1,t_2,t_3) - \sin v_n(y)(t_1,t_2,t_3)\Big| \\
&\leq \frac{\Pi^3}{36}|x_i(t_1,t_2,t_3) - y_i(t_1,t_2,t_3)| + \frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}}|v_n(x)(t_1,t_2,t_3) - v_n(y)(t_1,t_2,t_3)|.
\end{aligned}$$

Evidently $0 < \alpha < 1$, $\sum_{n=1}^{\infty}\beta_n|f_n(t_1,t_2,t_3,0,z^0(t_1,t_2,t_3))|$ *is convergent to zero for all* $t_1,t_2,t_3 \in \mathbb{R}_+$, $\eta_1 = \sup\Big\{\sum_{n=1}^{\infty}\beta_n\gamma_n(t_1,t_2,t_3) : t_1,t_2,t_3 \in \mathbb{R}_+\Big\} \leq \frac{1}{4}$ *and* f_n *and* g_n *are continuous functions.*
On the other hand, we have

$$\begin{aligned}
\sum_{n=k}^{\infty}\beta_n\gamma_n(t_1,t_2,t_3)|v_n(x)(t_1,t_2,t_3)| &\leq \sum_{n=k}^{\infty}\frac{1}{n(n+1)(n+2)e^{t_1+t_2+t_3^2}} \\
&\quad \times\Big|\int_0^{e^{t_1}}\int_0^{e^{t_2}}\int_0^{e^{t_3}}\frac{\tanh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}{5+\cosh(\sum_{i=1}^{\infty}x_i(t_1,t_2,t_3))}ds_1ds_2ds_3\Big| \\
&\leq \frac{e^{t_1+t_2+t_3}}{e^{t_1+t_2+t_3^2}}\sum_{n=k}^{\infty}\frac{1}{n(n+1)(n+2)}.
\end{aligned}$$

It in turn implies that

$$Q_k \leq \sup\Big\{\frac{e^{t_1+t_2+t_3}}{e^{t_1+t_2+t_3^2}}\sum_{n=k}^{\infty}\frac{1}{n(n+1)(n+2)}; t_1,t_2,t_3,s_1,s_2,s_3 \in \mathbb{R}_+\Big\}.$$

As $k\to\infty$ *we obtain* $\displaystyle\sum_{n\geq k}\frac{1}{n(n+1)(n+2)}\to 0$. *Thus, we infer that* $Q_k\to 0$ *as* $k\to\infty$ *and* $Q\leq\frac{1}{4}$.

Also, we observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\beta_n\Big|\gamma_n(t_1,t_2,t_3)\int_0^{e^{t_1}}\int_0^{e^{t_2}}\int_0^{e^{t_3}}&\big[g_n(t_1,t_2,t_3,s_1,s_2,s_3,z(t_1,t_2,t_3))\\ &- \; g_n(t_1,t_2,t_3,s_1,s_2,s_3,\overline{z}(t_1,t_2,t_3))\big]ds_1ds_2ds_3\Big|\\ &\leq \; \sum_{n=1}^{\infty}\frac{2e^{t_1+t_2+t_3}}{e^{t_1+t_2+t_3^2}}\frac{1}{n(n+1)(n+2)}=\frac{1}{2e^{t_3}}.\end{aligned}$$

It enforces that

$$\begin{aligned}\lim_{t_1,t_2,t_3\to\infty}\sum_{n=1}^{\infty}\beta_n\Big|\gamma_n(t_1,t_2,t_3)\int_0^{e^{t_1}}\int_0^{e^{t_2}}\int_0^{e^{t_3}}&\big[g_n(t_1,t_2,t_3,s_1,s_2,s_3,z(t_1,t_2,t_3))\\ -g_n(t_1,t_2,t_3,s_1,s_2,s_3,\overline{z}(t_1,t_2,t_3))\big]ds_1ds_2ds_3\Big|&=0.\end{aligned}$$

Consequently, all the conditions of Theorem 5.1 are satisfied. Hence the infinite system of integral equations (5.2) has at least one solution in l_1^{β}.

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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