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APPROXIMATE BEST PROXIMITY FOR SET-VALUED CONTRACTIONS IN METRIC SPACES

FAHIMEH MIRDAMADI, MEHDI ASADI* AND SOMAYEH ABBASI

Abstract. In this paper, we introduce the concept of set-valued cyclic almost contraction mappings. The existence of approximate best proximity points for such mappings on a metric space is established as well. We also obtain the approximate best proximity for two cyclic set-valued nonlinear contraction maps.

1. Introduction and preliminaries

Let X be a metric space and A, B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be cyclic, whenever $T(A) \subset B$ and $T(B) \subset A$. If $T : A \cup B \to A \cup B$ is a cyclic mapping, then a point $x \in A \cup B$ is called a best proximity point for T if $d(x, T(x)) = d(A, B)$, where

$$d(A, B) = \inf\{d(x, y) : (x, y) \in A \times B\}.$$

A best proximity point also evolves as a generalization of the concept of fixed point of mappings. Because if $A \cap B \neq \emptyset$, every best proximity point is a fixed point of T. Recently, many authors studied the existence of a best proximity point under some suitable contraction conditions, for more details; see [1, 2, 7–11, 14, 15] and references therein.

Another important and current branch of fixed point theory is investigating the approximate fixed point property, for more details; see [4, 5, 12, 19] and references therein.

The interest in approximate fixed point results arises naturally in probing into some problems in economics and game theory, see [3,13] and references therein. Recently, Mohsenalhosseni and Mazaheri [17] introduced the notion of approximate best proximity point for single-valued cyclic maps as finding a point $x \in A \cup B$ such that $d(x, T(x)) \leq d(A, B) + \varepsilon$, for some $\varepsilon > 0$ and it is stronger than best proximity point.

Our goal in this paper is to extend the concept of single-valued nonlinear almost contractions to set-valued cyclic maps that was introduced by Berinde [5] and Ciric [6].

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*Corresponding Author.

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We obtain the existence of approximate best proximity point for such maps in metric spaces. Some existence results concerning approximate best proximity coincide point property of the set-valued cyclic I-contractions T is also obtained. We also prove some quantitative theorems regarding the set of approximate best proximity for set-valued almost I-contractions.

Now, we give some notions and definitions.
Let (X, d) be a metric space and $\mathcal{P}(X)$ and $Cl(X)$ denote the families of all nonempty subsets and nonempty closed subsets of X respectively. For any $A, B \subset X$, we consider

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

the Hausdorff metric on $Cl(X)$ induced by the metric d.
Let X and Y be two topological Hausdorff spaces and $T : X \multimap \mathcal{P}(Y)$ be a set-valued mapping with nonempty values. Then T is said to be

- upper semi-continuous (u.s.c.) if, for each closed set $B \subset Y$,
$$T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$$
is closed in X;
- lower semi-continuous (l.s.c.) if, for each open set $B \subset Y$,
$$T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$$
is open in X;
- continuous if it is both u.s.c. and l.s.c.;
- closed if its graph $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed;
- compact if $ClT(X)$ is a compact subset of Y.
We also use from notation $\multimap$ for set-valued maps.

2. Main Results

In this section, first we prove the existence of an approximate best proximity point for set-valued cyclic almost contraction map in metric spaces. Also, some existence results concerning approximate best proximity coincidence point property of the set-valued cyclic I-contractions T is also obtained. We begin with the notion of set-valued cyclic almost contraction map.

Definition 2.1. *Let (X, d) be a metric space, A and B be nonempty subsets of X. Then a set-valued mapping $T : A \cup B \multimap A \cup B$ is called a set-valued cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.*

Note that $T(A) = \cup\{Tx : x \in A\}$.

Definition 2.2. *Let (X, d) be a metric space, A and B be nonempty subsets of X. Then a set-valued cyclic mapping $T : A \cup B \multimap A \cup B$ is called:*
(1) a set-valued cyclic contraction (or set-valued cyclic k-contraction), if there exists a number $0 < k < 1$ such that

$$H(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B), \quad \forall x \in A, y \in B.$$

(2) a set-valued cyclic almost contraction or a set-valued cyclic (θ, L)-almost contraction, if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x,y) + L.d(y,Tx) + (1-\theta)d(A,B), \quad \forall x \in A, y \in B.$$

Definition 2.3. *Let A and B be nonempty subsets of a metric space X. Then a set-valued map $T : A \cup B \multimap A \cup B$ said to have an approximate best proximity point property provided*

$$\inf_{x \in X} d(x, Tx) = d(A,B)$$

or, equivalently, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A \cup B$ such that

$$d(x_\varepsilon, Tx_\varepsilon) \leq d(A,B) + \varepsilon$$

or, equivalently, for any $\varepsilon > 0$, there exists $x_\varepsilon \in A \cup B$ such that

$$T(x_\varepsilon) \cap B(x_\varepsilon, d(A,B) + \varepsilon) \neq \emptyset,$$

where $B(x,r)$ denotes a closed ball of radius r centered at x.

Theorem 2.4. *Let A and B be nonempty subsets of a metric space X. Suppose that $T : A \cup B \multimap A \cup B$ is a cyclic set-valued map. If there exist two sequences (x_n) and (y_n) such that $x_n \in A \cup B$, $y_n \in T(x_n)$ and*

$$\lim_n d(x_n, y_n) = d(A,B).$$

Then T has approximate best proximity point x in $A \cup B$ i.e.

$$d(x, T(x)) \leq d(A,B) + \varepsilon$$

for any $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be given and there exist $x_n \in A \cup B$ and $y_n \in T(x_n)$ such that $\lim_n d(x_n, y_n) = d(A,B)$. So

$$\exists N_0 > 0 \quad \text{such that} \quad \forall n \geq N_0 : d(x_n, y_n) \leq d(A,B) + \varepsilon.$$

If $n = N_0$, then $d(x_{N_0}, y_{N_0}) \leq d(A,B) + \varepsilon$. Thus $d(x_{N_0}, T(x_{N_0})) \leq d(A,B) + \varepsilon$ and so x_{N_0} is an approximate best proximity. □

We first prove that every set-valued cyclic almost contraction has the approximate best proximity property.

Theorem 2.5. *Let (X,d) be a metric space and A and B be nonempty subsets of X. Suppose that $T : A \cup B \multimap A \cup B$ is a closed-valued cyclic almost contraction. Then T has approximate best proximity point property.*

Proof. Choose $x_0 \in A \cup B$ and $x_1 \in T(x_0)$. Then by the definition of H, there exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq H(T(x_0), T(x_1)) + \theta.$$

Similarly, there exists $x_3 \in T(x_2)$ such that

$$d(x_2, x_3) \leq H(T(x_1), T(x_2)) + \theta^2.$$

By following the same way, there exists a sequence $\{x_n\}$ in $A \cup B \cup T(A \cup B)$ such that $x_{n+1} \in T(x_n)$ and

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq H(T(x_{n-1}), T(x_n)) + \theta^n \\
&\leq \theta d(x_{n-1}, x_n) + L.d(x_n, T(x_{n-1})) + (1-\theta)d(A,B) + \theta^n \\
&\leq \theta(\theta d(x_{n-2}, x_{n-1}) + (1-\theta)d(A,B) + \theta^{n-1}) + (1-\theta)d(A,B) + \theta^n \\
&= \theta^2 d(x_{n-2}, x_{n-1}) + (1-\theta^2)d(A,B) + 2\theta^n \\
&\vdots \\
&\leq \theta^n d(x_0, x_1) + (1-\theta^n)d(A,B) + \theta^n + \ldots + \theta^n
\end{aligned}$$

Thus

$$d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1) + (1-\theta n)d(A,B) + n\theta^n,$$

hence, $\lim_n d(x_n, x_{n+1}) \leq d(A,B)$. Also we have $\lim_n d(x_n, x_{n+1}) \geq d(A,B)$, so

$$\lim_n d(x_n, x_{n+1}) = d(A,B).$$

Therefore, by Theorem 2.4, T has approximate best proximity point property. □

In the following example we show that T is a set-valued cyclic almost contraction and T has approximate best proximity but it has not best proximity.

Example 2.6. *Let $A = [\frac{2}{3}, 1]$ and $B = [0, \frac{1}{2}]$ with the Euclidean distance , and let $T(x)$ be defined as follows:*

$$T(x) = \begin{cases} (\frac{2}{3}, \frac{5}{6}) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ (\frac{1}{3}, \frac{3}{8}) & \text{if } \frac{2}{3} \leq x \leq 1, \end{cases}$$

We have T is a set-valued cyclic almost contraction. Indeed, for every $x \in A$ and $y \in B$, we have $H(T(x), T(y)) = \frac{11}{24}$, $d(y, T(x)) \neq 0$ and $d(A,B) = \frac{1}{6}$ and $\theta = \frac{1}{2}$, we see that T is a set-valued cyclic almost contraction provided that $L > 0$ is large enough. Also $x = \frac{1}{2}$ is approximate best proximity of T, while T has not best proximity.

Definition 2.7. *[17] Let (X,d) be a metric space and A and B be nonempty subsets of X. Suppose $T : A \cup B \to A \cup B$ is a single-valued cyclic map. For each $\varepsilon > 0$, we set*

$$P_{T_\varepsilon}^a(A,B) = \{x \in A \cup B : d(x, Tx) \leq d(A,B) + \varepsilon\},$$

of approximate best proximity of single-valued almost contraction T. We define diameter $P_{T_\varepsilon}^a(A,B)$ by

$$\operatorname{diam}(P_{T_\varepsilon}^a(A,B)) = \sup\{d(x,y) : x, y \in P_{T_\varepsilon}^a(A,B)\}.$$

Now, we obtain the following quantitative estimate of the diameter of the set $P_{T_\varepsilon}^a(A,B)$ of approximate best proximity points of single-valued almost contraction.

Theorem 2.8. *Let (X,d) be a metric space. If $T : A \cup B \to A \cup B$ is a single-valued cyclic almost contraction with $\theta + L < 1$, then*

$$\operatorname{diam}(P_{T_\varepsilon}^a(A,B)) \leq \frac{(2+L)\varepsilon + (3-\theta)d(A,B)}{1-(\theta+L)}, \quad \forall \varepsilon > 0.$$

Proof. If $x, y \in P_{T_\varepsilon}^a(A,B)$, then

$$\begin{aligned} d(x,y) &\le d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\ &\le d(A,B) + \varepsilon + \theta d(x,y) + L.d(y,Tx) + (1-\theta)d(A,B) + d(A,B) + \varepsilon \\ &\le 2d(A,B) + 2\varepsilon + \theta d(x,y) + L.(d(x,y) + d(x,Tx)) + (1-\theta)d(A,B) \\ &\le (3-\theta)d(A,B) + (2+L)\varepsilon + (\theta+L)d(x,y). \end{aligned}$$

Therefore, $d(x,y) \le \frac{(3-\theta)d(A,B)+(2+L)\varepsilon}{1-(\theta+L)}$. Hence

$$\operatorname{diam}(P_{T_\varepsilon}^a(A,B)) \le \frac{(2+L)\varepsilon + (3-\theta)d(A,B)}{1-(\theta+L)}.$$

□

The following example indicates that the above argument is not valid for set-valued almost contraction map T.

Example 2.9. *Let $X = \mathbb{R}$ with Euclidean metric, $A = [0,1]$ and $B = [\frac{1}{2}, 2]$. Assume that $T(x) = \{\frac{1}{2}, 1\}$, for each $x \in A \cup B$. Then*

$$H(T(x), T(y)) = 0 < \frac{1}{2} d(x,y)$$

for each $x, y \in A \cup B$. Therefore, T is a continuous set-valued cyclic almost contraction with $\theta + L = \frac{1}{2} < 1$. Moreover, $x = \frac{1}{2}$ and $x = 1$ are best proximity points in A and so $\operatorname{diam}(P_{T_\varepsilon}^a(A,B)) = \frac{1}{2}$. *This shows that Theorem 2.8 is not valid whenever T is set-valued almost contraction.*

Theorem 2.10. *Let (X,d) be a metric space and A and B be nonempty subsets of X. Assume that $T : A \cup B \multimap A \cup B$ is a closed-valued cyclic almost contraction mapping, then T has a best proximity point provided either A, B is compact and the function $f(x) = d(x,Tx)$ is lower semi-continuous or T is closed and compact.*

Proof. By Lemma 2.5, we have $\inf_{x \in X} f(x) = \inf_{x \in X} d(x,Tx) = d(A,B)$. The lower semi-continuity of the function $f(x) = d(x,Tx)$ and the compactness of $A \cup B$ imply that the infimum is attained. Thus there exists an $x_0 \in A \cup B$ such that $d(x_0, Tx_0) = d(A,B)$ and so T has a best proximity point. Suppose that T is closed and compact map. According to Lemma 2.5, T has the approximate best proximity property. Therefore for any $\varepsilon > 0$, there exist $x_\varepsilon \in A$ and $y_\varepsilon \in B$ such that

$$y_\varepsilon \in T(x_\varepsilon) \cap B(x_\varepsilon, d(A,B) + \varepsilon).$$

Now, since $Y := Cl(T)$ is compact, we may assume that y_ε converges to a point $z \in Y$ as $\varepsilon \to 0$. Consequently, x_ε converges to z' as $\varepsilon \to 0$ such that $d(z,z') \le d(A,B) + \varepsilon$. On the other hand, since T is closed, then $z \in T(z')$. So $d(T(z'), z') \le d(A,B) + \varepsilon$. This completes the proof. □

Now, we introduce the notion of set-valued cyclic almost I-contraction. Also, we obtain the existence of approximate best proximity point for such maps in metric spaces.

Definition 2.11. *Let $I : A \cup B \to A \cup B$ be a single-valued cyclic map and $T : A \cup B \multimap Cl(A \cup B)$ be a set-valued cyclic map. Then T is called a set-valued cyclic almost I-contraction if there exist constants $\theta \in (0,1)$ and $L \ge 0$ such that*

$$H(Tx,Ty) \le \theta d(Ix,Iy) + L.d(Iy,Tx) + (1-\theta)d(A,B), \quad \forall x \in A,\ y \in B.$$

Definition 2.12. *The mappings I and T are said to have an approximate best proximity coincidence point property provided that*

$$\inf_{x\in A\cup B} d(Ix,Tx)=d(A,B)$$

or, equivalently, for any $\varepsilon>0$, there exists $z\in A\cup B$ such that

$$d(Iz,Tz)\le d(A,B)+\varepsilon.$$

A point $(x,y)\in A\times B$ is called a coincidence best proximity (common best proximity) point of I and T if $Ix\in Tx$ $(d(x,Ix)=d(A,B))$

Theorem 2.13. *Let (X,d) be a metric space, A and B be nonempty subsets of X.. Suppose that $T: A\cup B \multimap A\cup B$ is a cyclic closed-valued map and $I: A\cup B\to A\cup B$ is a single-valued cyclic map and*

$$\lim_n d(I(x_n),y_n)=d(A,B)$$

for some $x_n\in A\cup B$ and $y_n\in T(x_n)$. Then I and T have a coincidence best proximity point.

Proof. By a similar proof as that of Theorem 2.4, we obtain the conclusion for T and I. □

Theorem 2.14. *Every set-valued cyclic almost I-contraction in a metric space (X,d) has the approximate best proximity coincidence point property provided that each Tx is I-invariant. Further, if A,B is compact and the function $f(x)=d(Ix,Tx)$ is lower semi-continuous, then I and T have a coincidence best proximity point.*

Proof. Choose $x_0\in A\cup B$ and $x_1\in T(x_0)$. Then, by the definition of H, there exists $x_2\in T(x_1)$ such that

$$d(I(x_1),x_2)\le H(I(x_1),T(x_1))+\theta.$$

Since each Tx is I-invariant, i.e., for each $y\in Tx$, $Iy\in Tx$, then $I(x_1)\in T(x_0)$ and so we have

$$d(I(x_1),x_2)\le H(T(x_0),T(x_1))+\theta.$$

Similarly, there exists $x_3\in T(x_2)$ such that

$$d(I(x_2),x_3)\le H(T(x_1),T(x_2))+\theta^2.$$

By following the same way, there exists a sequence $\{x_{n-1}\}$ in $A\cup B\cup T(A\cup B)$ such that $x_n\in T(x_{n-1})$ and

$$\begin{aligned}
d(I(x_{n-1}),x_n) &\le H(I(x_{n-1}),T(x_{n-1}))+\theta^n\\
&\le H(T(x_{n-2}),T(x_{n-1}))+\theta^n\\
&\le \theta d(I(x_{n-2}),I(x_{n-1}))+L.d(I(x_{n-1}),T(x_{n-2}))+(1-\theta)d(A,B)+\theta^n\\
&\le \theta(H(T(x_{n-3}),T(x_{n-2}))+\theta^{n-1})+(1-\theta)d(A,B)+\theta^n\\
&\le \theta(\theta d(I(x_{n-3}),I(x_{n-2}))+(1-\theta)d(A,B)+\theta^{n-1})+(1-\theta)d(A,B)+\theta^n\\
&= \theta^2 d(I(x_{n-3}),I(x_{n-2}))+(1-\theta^2)d(A,B)+2\theta^n\\
&\vdots\\
&\le \theta^n d(I(x_1),I(x_0))+(1-\theta^n)d(A,B)+\theta^n+...+\theta^n.
\end{aligned}$$

Then

$$d(I(x_{n-1}),x_n) \leq \theta^n d(I(x_1),I(x_0)) + (1-\theta^n)d(A,B) + n\theta^n.$$

Thus

$$\lim_n d(I(x_{n-1}),x_n) \leq d(A,B),$$

for every $x_n \in T(x_{n-1})$, Also we have $\lim_n d(I(x_{n-1}),x_n)) \geq d(A,B)$, so

$$\lim_n d(I(x_{n-1}),x_n) = d(A,B).$$

Therefore, by Theorem 2.13, T has approximate best proximity coincidence point property.

Further, the lower semi-continuity of the function $f(x) = d(Ix,Tx)$ and compactness of A,B imply that the infimum is attained. Thus there exists $z \in A \cup B$ such that $f(z) = d(Iz,Tz) = d$. This completes the proof. □

Remark. If I is the identity map on $A \cup B$ in Theorem (2.14), we obtain the conclusion of Theorem 2.5.

Theorem 2.15. *Let (X,d) be a metric space and A and B be nonempty subsets of X. Assume that $T : A \cup B \multimap A \cup B$ is a closed-valued map and suppose that sequences $x_n \in X$ and $y_n \in Tx_n$ satisfying following two conditions:*

$$\lim_{n\to\infty} d(x_n,y_n) = \inf_{x\in X} d(x,Tx) \tag{2.1}$$

and

$$f(y_n) \leq \theta d(x_n,y_n) + (1-\theta)d(A,B), \tag{2.2}$$

where $f(x) = d(x,Tx)$. Then T has the approximate best proximity property. Further, T has a best proximity provided either A,B is compact and the function $f(x)$ is lower semi-continuous or T is closed and compact.

Proof. Let $x_n \in A \cup B$ and $y_n \in Tx_n$ be the sequences that satisfy (2.1) and (2.2).

Then we have

$$\begin{aligned}
\inf_{x\in X} f(x) - d(A,B) &= \inf_{x\in X} d(x,Tx) - d(A,B)\\
&\leq \inf_{x\in X}\inf_{y\in Tx} d(y,Ty) - d(A,B)\\
&\leq \inf_{n\in\mathbb{N}}\inf_{y\in Tx_n} d(y,Ty) - d(A,B)\\
&\leq \inf_{n\in\mathbb{N}} d(y_n,Ty_n) - d(A,B)\\
&\leq \inf_{n\in\mathbb{N}} \theta d(x_n,y_n) + (1-\theta)d(A,B) - d(A,B)\\
&\leq \theta(\lim_{n\to\infty} d(x_n,y_n) - d(A,B))\\
&\leq \theta(\inf_{x\in X} f(x) - d(A,B)).
\end{aligned}$$

Since $\theta < 1$, we get $\inf_{x\in X} f(x) = \inf_{x\in X} d(x,Tx) = d(A,B)$.
Further, the lower semi-continuity of the function $f(x) = d(x,Tx)$ and the compactness of A,B implies that the infimum is attained. Thus there exists a $z_0 \in A\cup B$ such that $f(z_0) = d(z_0,Tz_0) = d(A,B)$.

The second assertion follows as in the proof of Theorem 2.10. This completes the proof. □

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Fahimeh Mirdamadi
Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran
E-mail address: `mirdamadi.f@gmail.com`

Mehdi Asadi
Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran
E-mail address: `masadi@iauz.ac.ir; masadi.azu@gmail.com`

Somayeh Abbasi
Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran
E-mail address: `s.abbasi@khuisf.ac.ir`