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THE PHASE PLANE ANALYSIS OF NONLINEAR EQUATION

ESEN HANAÇ

Abstract. In this paper, I examine the main results concerning the existence and structure of permanent form travelling waves (PTWs) which may occur in the large-time solution to the following initial-boundary value problem

$$u_t + kuu_x = u_{xx} + u(1-u)$$

where $k \neq 0$ is a parameter. To show any solution to above equation with $c > 0$ provides a permanent form travelling wave solution which could develop as the primary large-time structure in the solution of the initial-value problem of the equation.

1. Introduction

The characteristics of many dynamical systems are determined by the propagation of fronts. In particular, scalar and systems of reaction-diffusion equations or reaction-diffusion-convection equations arise in the study of many branches of science for example, genetics [22, 23], nonlinear differential equations in biology [22, 23], combustion [1], chemistry [4] and physics [7, 6]. In these applications the phenomenon propagating wavefronts is of considerable interest. The study of the evolution of travelling wave solutions in scalar and systems of nonlinear partial differential equations is of fundamental importance in a wide variety of applications.In this presentation, I will consider the dynamical system of Burgers Fisher equation's travelling wave style namely,

$$u_t + kuu_x = u_{xx} + u(1-u) \tag{1.1}$$

where $k \neq 0$ is a parameter. This equation has been solved by many authors in different style. When $k = 0$ with a diffusion coefficient (in front of diffusion u_{xx}), which equation reduces to Fisher equation,was solved using spectral analysis method by [9, 18, 21, 24]

$$u_t = \mu u_{xx} + u(1-u). \tag{1.2}$$

and the solutions of equation (1.2) are exponentially small in t were proved. I also note that [25] showed the equation (1.2) displays critical wave form with applying the maximum principle method. By using different methods such as Green's method [21], the spectral method [18], the L^1 weighted energy method together with the Green function method [26] the stability of permanent form travelling waves are

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defined. Later, Bramson [2, 3] showed the solutions of the equation (1.2) displays the formation of a PTWs. I analyze the dynamic system of (1.1) in two different cases. Specifically, there is heteroclinic connection from unstable state to stable state while $k \geqslant 2$ and $c = \frac{2}{k} + \frac{k}{2}$ and a travelling wave solution when $k < 2$ with $c = \frac{2}{k} + \frac{k}{2}$.

2. Travelling Wave Theory

In this section I examine the travelling wave solutions of the Burgers-Fisher equation, namely,

$$u_t + kuu_x = u_{xx} + u(1-u) \tag{2.1}$$

where $k \neq 0$ is a parameter. I begin by looking for a travelling wave solution of the equation (2.1) I obtain

$$z = x - ct, \quad u = U(z), \tag{2.2}$$

where c is the wave speed. Substituting (2.2) into equation (2.1) gives

$$-cU_z + kUU_z = U_{zz} + U(1-U). \tag{2.3}$$

On writing $U_z = W$, I obtain the dynamical system

$$\begin{aligned} U_z &= W, \\ W_z &= -cW + kUW - U(1-U). \end{aligned} \tag{2.4}$$

Dynamical system (2.4) has been examined by a number of authors including Murray [23]. The arguments presented below follow closely those given in [23]. Therefore, I have that

$$\frac{dW}{dU} = -c + kU - \frac{U(1-U)}{W}. \tag{2.5}$$

Dynamical system (2.4) has two equilibrium points at $M : (0,0)$ and $N : (1,0)$. I require a monotone solution in $0 \leqslant U \leqslant 1$ with $U_z(z) \leqslant 0$. I next classify the equilibrium points by linearization. I first consider the equilibrium point $M : (0,0)$. The associated linear system is given by

$$\left.\begin{aligned} U_z &= W \\ W_z &= -cW - U \end{aligned}\right\} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}. \tag{2.6}$$

Eigenvalues of A and associated eigenvectors are given by

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2 - 4}}{2}, \qquad v_{\pm} = \begin{pmatrix} 1 \\ \lambda_{\pm} \end{pmatrix}. \tag{2.7}$$

Since we require $U \geqslant 0$ these eigenvalues must be real and so

$$c \geqslant 2.$$

Now since $0 > \lambda_+ > \lambda_-$ the point $M : (0,0)$ is a stable node. Therefore, the linearization Theorem then indicates that the point $M : (0,0)$ is a stable node for nonlinear system (2.4). Figure 1 displays the (U, W) phase plane in the neigbourhood of the equilibrium point $M : (0,0)$. The stable manifolds $W = \lambda_+ U$ and $W = \lambda_- U$ of the stable node are clearly displayed on the figure. In what follows we label the stable manifold $W = \lambda_- U$ as W_s^-.

I next consider the equilibrium point $N : (1,0)$. On writing $\overline{U} = U - 1$ and $\overline{W} = W$ the associated linear system is given by

$$\left.\begin{aligned} \overline{U}_z &= \overline{W} \\ W_z &= \overline{U} + (k-c)\overline{W}, \end{aligned}\right\} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & k-c \end{pmatrix}. \tag{2.8}$$

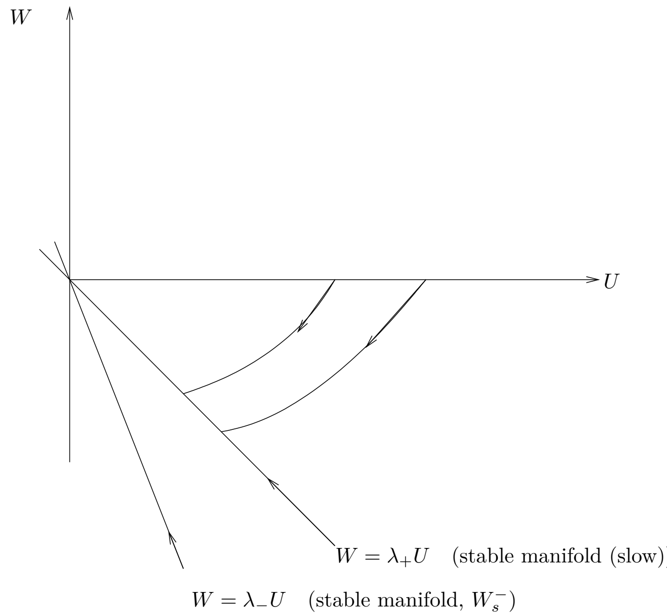

FIGURE 1. (U, W) phase plane in the neigbourhood of the equilibrium point $M : (0, 0)$.

Eigenvalues of A and associated eigenvectors are given by

$$\widehat{\lambda}_{\pm} = \frac{-(c-k) \pm \sqrt{(c-k)^2+4}}{2}, \qquad \widehat{v}_{\pm} = \begin{pmatrix} 1 \\ \widehat{\lambda}_{\pm} \end{pmatrix}. \tag{2.9}$$

Now since $\widehat{\lambda}_+ > 0 > \widehat{\lambda}_-$ the point $N : (1, 0)$ is a saddle point. Therefore, the Linearization Theorem then indicates that point $N : (1, 0)$ is a saddle point for nonlinear system (2.4). Figure 2 displays the (U, W) phase plane in the neigbourhood of the equilibrium point $N : (1, 0)$. The unstable manifold entering the region where $W < 0$ is clearly displayed in the figure. We note that on this unstable manifold

$$U \sim 1 - O(e^{\widehat{\lambda}_+ z}), \quad \text{as} \quad z \to -\infty.$$

In what follows we label this unstable manifold as W^+. We note from (2.5) that since $\frac{d}{dc}\left(\frac{dW}{dU}\right) = -1$ the phase plane rotates clockwise for increasing c. The rotation is counterclockwise for k increasing.

We further note that an exact solution of (2.5) exists and is given by

$$W = -\frac{k}{2}U(1-U) \quad \text{when} \quad c = \frac{k}{2} + \frac{2}{k}. \tag{2.10}$$

On recalling that $U_z = W$ I find that the solution of equation (2.10) is given by

$$U(z) = \frac{Ae^{-\frac{k}{2}z}}{1+Ae^{-\frac{k}{2}z}} \quad \begin{cases} \sim 1 - \frac{1}{A}e^{\frac{k}{2}z} & \text{as} \quad z \to -\infty, \\ \sim Ae^{-\frac{k}{2}z} & \text{as} \quad z \to \infty, \end{cases} \tag{2.11}$$

where A is a constant. Figure 3 displays the (U, W) phase portrait of (2.4) when $k < 2$ and $k \geqslant 2$, respectively. The phase path connecting the equilibrium points $M : (0, 0)$ and $N : (1, 0)$ is given by (2.10). I note that when $k < 2$ this phase path

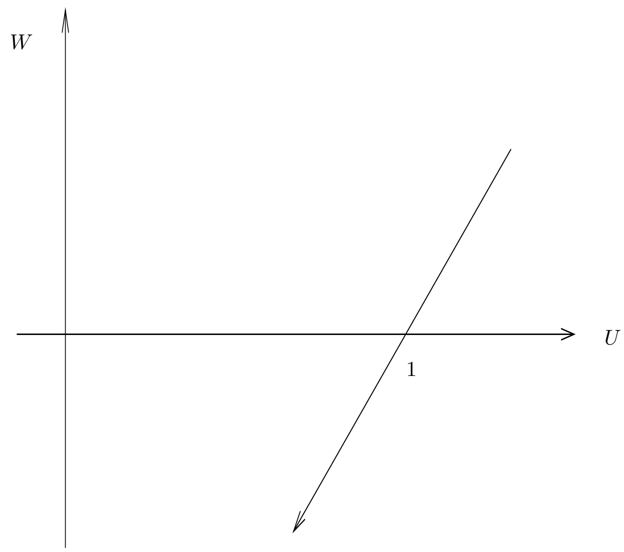

FIGURE 2. (U, W) phase plane in the neigbourhood of the equilibrium point $N : (1, 0)$.

enters $M : (0, 0)$ along the stable manifold $W = \lambda_+ U$, while when $k \geqslant 2$ the phase path enters $M : (0, 0)$ along the stable manifold $W = \lambda_- U$. Consideration of the phase path (2.10) gives that

$$\begin{aligned} \frac{dW}{dU} &= -\frac{k}{2} + kU \\ &\sim -\frac{k}{2} \quad \text{as} \quad U \to 0^+, \end{aligned} \tag{2.12}$$

indicating that when $k \geqslant 2$ the phase path approaches the equilibrium point $M : (0, 0)$ along the stable manifold W_s^-.

Before we proceed further with the phase plane analysis it is instructive to examine the asymptotic behaviour of the stable manifold W_s^- for $c \gg 1$ with fixed k. It is straightforward as

$$W_s^-(U) = -cU + \left[\frac{(k-1)}{2} U^2 + \frac{U^3}{3} \right] + o(1), \quad U \in (0, 1), \tag{2.13}$$

for $c \gg 1$. The stable manifold W_s^- for $c \gg 1$ is sketched in Figure 4. As trajectories cannot cross W_s^- the hashed region in Figure 4 is a positively invariant region for the dynamical system. In what follows I must consider the cases $k \geqslant 2$ and $k < 2$ separetely. I begin with the case when $k \geqslant 2$.

(a) $k \geqslant 2$

In this case the earlier established facts that:

(i) The vector field rotates anticlockwise for decreasing c.

(ii) The stable manifold W_s^- crosses the line $U = 1$ at $W = -c + \frac{k}{2} - \frac{1}{6}$ for $c \gg 1$.

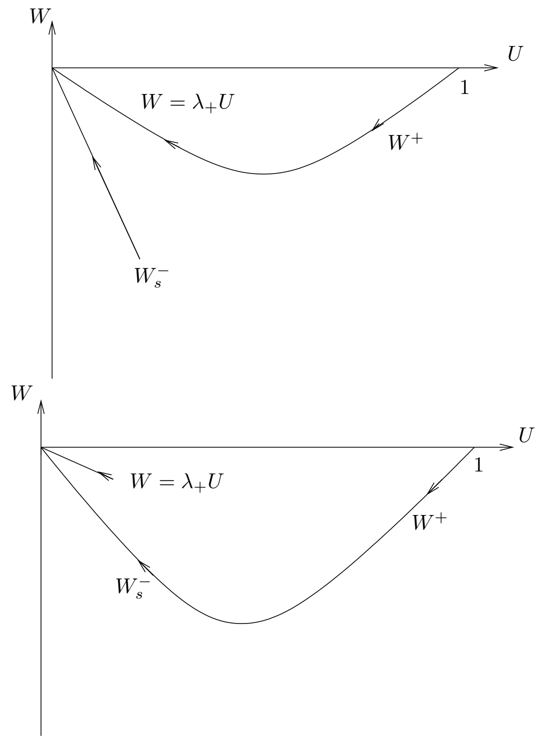

FIGURE 3. The (U, W) phase portrait of dynamical system (2.4) when $k < 2$ and $k \geqslant 2$, respectively.

(iii) The phase path (2.10) forms a heteroclinic connection between the equilibrium points $M : (0, 0)$ and $N : (1, 0)$ when $c = \frac{k}{2} + \frac{2}{k}$.
allow after consideration of the flow on $(0 < U < 1,\ W = 0)$ and $(U = 1,\ W < 0)$ that permanent form travelling wave solutions of (2.1) are only possible when

$$c \geqslant \frac{k}{2} + \frac{2}{k}.$$

(b) $k < 2$
In this case we first show that the phase path $W(U; c = 2, k = 2)$ and the portion of the U-axis $(0 < U < 1,\ W = 0)$ form a positively invariant region for the unstable manifold emanating from $N : (1, 0)$. Since $\frac{dW}{dU}$ increases with increasing k for sufficiently close to $U = 1$, the phase path $W(U; c, k)$ satisfies

$$W(U; c = 2, k) > W(U; c = 2, k = 2).$$

Now suppose there exists a number U^*, where $0 < U^* < 1$ such that

$$W(U^*; c = 2, k = 2) = W(U^*; c = 2, k)$$

where $W(U; c = 2, k = 2) < W(U; c = 2, k)$ for $U^* < U < 1$.

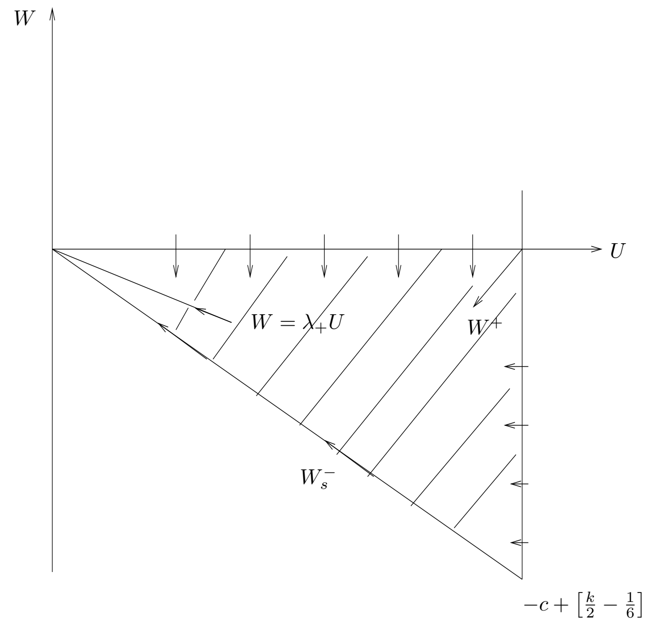

FIGURE 4. The (U, W)phase portrait of the dynamical system (2.13) when $c \gg 1$

If there phase paths cross (touch) then (using (2.5)) $\frac{dW}{dU}(U^*; c = 2, k = 2) \leqslant \frac{dW}{dU}(U^*; c = 2, k)$. However, via (2.5) we obtain

$$-2 + 2U^* - \frac{U^*(1 - U^*)}{W(U^*; c = 2, k = 2)} \leqslant -2 + kU^* - \frac{U^*(1 - U^*)}{W(U^*; c = 2, k)}$$

giving that

$$k \geqslant 2$$

which is a contradiction and no such number U^* exists for $k < 2$. On noting that $W < 0$ on $0 < U < 1$, $W = 0$ I have established that the unstable manifold emanating from $N : (1, 0)$ into $W < 0$ enters a positively invariant region formed by the portion of U-axis given by $(0 < U < 1, W = 0)$ and the phase path $W(U; c = 2, k = 2)$. Therefore, the unstable manifold emanating from $N : (1, 0)$ must approach $M : (0, 0)$ along $W = \lambda_+ U$ and a permanent form travelling wave solution of (2.1) is possible for $c \geqslant 2$.

3. Numerical Solutions

The interested reader is referred to the following texts[12, 13] for details of the numerical method employed here. I present numerical solutions of IVP3 in two different cases when $k = 5$ ($\in (2, \infty)$) and when $k = -5$ ($\in (-\infty, 2]$). We consider each case in turn:

(i) When $k = 5$, in Figure the PTW of wave speed $c = c^*(5) = 2.9$ (the minimum wave speed available in this case) is seen to develop rapidly
(ii) When $k = -5$, in Figure 6 the PTW of wave speed $c = c^*(-5) = 2$(the minimum wave speed available in this case) is seen to develop rapidly

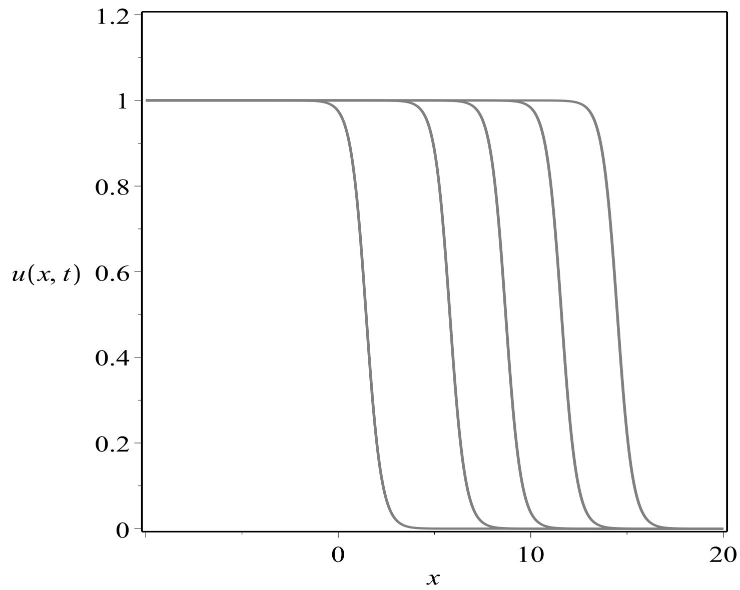

FIGURE 5. Graphs of the solution of IVP3 at times $t = 0.5, 1, 2, 3, 4$.

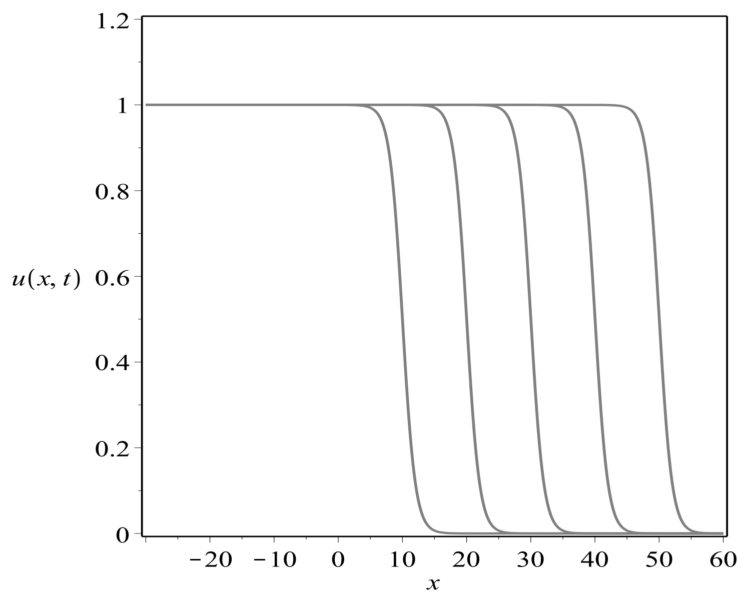

FIGURE 6. Graphs of the solution of IVP3 at times $t = 5, 10, 15, 20, 25$.

4. CONCLUSION

As a conclusion, the orbit $(U(z), W(z))$ is determined for $-\infty < z < \infty$ connecting at equilibrium points (1,0) and (0,0). Therefore, as $z \to -\infty$ $U(z)$ tends to 1 and $z \to \infty$ $U(z)$ tends to 0. Which means U is decreasing and becoming stable at

$\pm\infty$ that gives permanent form travelling wave solution with the minimum possible speed $c = c^*(k)$, where

$$c^*(k) = \begin{cases} 2, & -\infty < k \leqslant 2, \\ \frac{2}{k} + \frac{k}{2}, & 2 < k < \infty. \end{cases}$$

It is also important to note that above results are supported by the numerical simulations.

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Esen HANAÇ
Department of Mathematics, Adiyaman University,Turkey
E-mail address: ehanac@adiyaman.edu.tr