

SOME NEW FIXED POINT THEOREMS WITH APPLICATIONS

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ABSTRACT. The aim of this paper is to introduce α -generalized multi-valued contractions in cone metric spaces over Banach Algebras with solid cones and show the existence of fixed points of these contractions in this context. Our results generalize and extend several known results in metric and cone metric spaces given by some authors. Also, we give one example to illustrate significance of the main result in this paper.

1. INTRODUCTION

In 1922, Banach [8] proved Banach's contraction principle (or Banach's fixed point theorem), which is one of the pivotal results of nonlinear analysis and establishes that, if a mapping S on a complete metric space (X, d) into itself is such that, for some $\lambda \in [0, 1)$,

$$d(Sx, Sy) \leq \lambda d(x, y) \quad (1.1)$$

for all $x, y \in X$, then S has a unique fixed point in X .

Due to its importance and simplicity, many authors have obtained a lots of interesting improvements, extensions and generalizations of Banach's contraction principle (see [1-25] and references therein).

Especially, in 2007, Huang et al. [10] introduced cone metric spaces with the normal cone as a generalization of metric spaces. Many authors have obtained some fixed point theorems for generalized contractions in cone metric spaces (see [5, 7, 17]). In [22], Liu and Xu used cones over a Banach algebra and proved some fixed point theorems for generalized Lipschitz mappings in cone metric spaces. They improved the contractive conditions on self-mappings in cone metric spaces by replacing the contractive constant by a vector of cone.

Suppose that \mathbb{A} is a real Banach algebra, i.e., \mathbb{A} is a *real Banach space* in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in \mathbb{A}$ and $a \in \mathbb{R}$,

- (1) $x(yz) = (xy)z$;
- (2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
- (3) $a(xy) = (ax)y = x(ay)$;

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$$(4) \|xy\| \leq \|x\|\|y\|.$$

We assume that the Banach algebra \mathbb{A} has the unit, i.e., the multiplicative identity e such that $ex = xe = x$ for all $x \in \mathbb{A}$. An element $x \in \mathbb{A}$ is said to be *invertible* if there is an inverse element $y \in \mathbb{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1} .

The properties are well known ([33]), that is, let \mathbb{A} be a real Banach algebra with the unit e and let $x \in \mathbb{A}$. If the *spectral radius* $\rho(x)$ of x is less than one, that is,

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $-x$ is invertible. Actually, we have

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x_i.$$

A subset P of \mathbb{A} is called a *cone* if

- (1) $\{e, \theta\} \subset P$;
- (2) $P^2 = PP \subset P$, $P \cap (-P) = \{\theta\}$;
- (3) for all $a, b \in \mathbb{R}$ with $a, b \geq 0$, $aP + bP \subset P$.

For any cone $P \subset \mathbb{A}$, we define a *partial ordering* \preceq with respect to P by $x \preceq y$ if and only if $y - x \in P$, where $x \prec y$ stands for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . If $\text{int}P \neq \emptyset$, then P is called a *solid cone*. We write $\|\cdot\|$ as the norm of \mathbb{A} . A cone P is said to be *normal* if there is a number $M > 0$ such that, for all $x, y \in \mathbb{A}$,

$$(1.2) \quad \theta \preceq x \preceq y \implies \|x\| \leq M\|y\|.$$

The least positive number satisfying (1.2) is called the *normal constant* of P . Note that, for any normal cone P , we have $M \geq 1$.

In the following, we always suppose that \mathbb{A} is a real Banach algebra with the unit e , P is a solid cone and \preceq is the partial ordering with respect to P .

The following lemmas and remarks are useful in the sequel:

Lemma 1.1. [18] *If \mathbb{E} is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Lemma 1.2. [28] *If \mathbb{E} is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 1.3. [28] *If \mathbb{E} is a real Banach space with a solid cone P and if $\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.*

Remark 1.1.[32] If $\rho(x) < 1$, then $\|x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.1. [22] Let X be a nonempty set and \mathbb{A} be a Banach algebra. A function $d : X \times X \rightarrow \mathbb{A}$ is called a *cone metric* if the following conditions hold:

- (C1) $\theta \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$;
- (C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (C3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The pair (X, d) is called a *cone metric space* over Banach algebra \mathbb{A} .

Lemma 1.4. [22] Let (X, d) be a cone metric space over Banach algebra \mathbb{A} , $x \in X$ and let $\{x_n\}$ be a sequence in X . Then we have the following:

(1) The sequence $\{x_n\}$ converges to a point $x \in X$ whenever, for any $c \in \mathbb{A}$ with $\theta \ll c$, there exists a natural number n_0 such that

$$d(x_n, x) \ll c$$

for all $n \geq n_0$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$;

(2) The sequence $\{x_n\}$ is a Cauchy sequence whenever, for any $c \in \mathbb{A}$ with $\theta \ll c$, there exists a natural number n_0 such that

$$d(x_n, x_m) \ll c$$

for all $n, m \geq n_0$;

(3) (X, d) is a complete cone metric if every Cauchy sequence in X is convergent.

In [15], Cho et al. introduced the Hausdorff distance function on cone metric spaces and proved some fixed point theorems for multi-valued mappings.

Following the concept of Hausdorff distance function on cone metric spaces, we extend it to cone metric spaces with a Banach Algebra \mathbb{A} as follows:

For a cone metric space (X, d) over Banach algebra \mathbb{A} , denote

$$N(X) = \{A : A \text{ is non empty subset of } X\},$$

$$s(p) = \{q \in \mathbb{E} : p \preceq q\}$$

for all $q \in \mathbb{A}$ and

$$s(a, B) = \bigcup_{b \in B} s(d(a, b)) = \bigcup_{b \in B} \{x \in \mathbb{A} : d(a, b) \preceq x\}$$

for all $a \in X$ and $B \in N(X)$. For all $A, B \in N(X)$, also denote

$$s(A, B) = \left(\bigcap_{a \in A} s(a, B) \right) \cap \left(\bigcap_{b \in B} s(b, A) \right).$$

The following lemma is much needed in the proof of our main result.

Lemma 1.5. Let (X, d) be a cone metric space over Banach algebra \mathbb{A} and let P be a solid cone. Then we have the following:

(1) If, for any $p, q \in \mathbb{A}$, $p \preceq q$, then $s(q) \subset s(p)$.

(2) If, for any $x \in X$ and $A \in N(X)$, $\theta \in s(x, A)$, then $x \in A$.

(3) Let $q \in P$ and let $A, B \in CB(X)$ and $a \in A$. If $q \in s(A, B)$, then $q \in s(a, B)$ for all $a \in A$ or $q \in s(A, b)$ for all $b \in B$.

(4) If $q \in P$ and $\lambda \geq 0$, then $\lambda s(q) \subseteq s(\lambda q)$.

Proof. The proof is same like the proof of lemma of [15] with cone metric space over Banach algebra. \square

Remark 1.2. [15] Let (X, d) be a cone metric space over Banach algebra \mathbb{A} . If $\mathbb{A} = \mathbb{R}$ and $P = [0, +\infty)$, then (X, d) is a metric space. Moreover, for any $A, B \in CB(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by d .

In 2012, Samet et al. [29] introduced the concepts of α -admissible mappings and established some fixed point theorems for self-mappings defined on complete metric spaces.

Definition 1.2. [29] Let S be a self-mapping on X and $\alpha : X \times X \rightarrow [0, +\infty)$ be a function. We say that S is an α -admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \implies \quad \alpha(Sx, Sy) \geq 1.$$

In this paper, we first define the notion of an α -generalized multi-valued contraction on cone metric spaces over Banach algebra \mathbb{A} and then we use it to study of graphs. Also, we furnish a non-trivial example to illustrate our main result.

2. MAIN RESULTS

First, we introduce an α -generalized multi-valued contraction on cone metric spaces.

Definition 2.1. Let (X, d) be a cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone. Then a multi-valued mapping $S : X \rightarrow C(X)$ is called an α -generalized multi-valued contraction if there exist $\alpha : X \times X \rightarrow [0, +\infty)$ and $k \in P$ such that $\rho(k) < 1$ and

$$kd(x, y) \in \alpha(x, y)s(Sx, Sy) \tag{2.1}$$

for all $x, y \in X$.

Theorem 2.1. Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone. Suppose that a multi-valued mapping S is α -admissible, α -generalized multi-valued contraction and there exist $x_0 \in X$, $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x^*$ as $n \rightarrow +\infty$, then $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. Then S has a fixed point in X .

Proof. Let x_0 be an arbitrary point in X . Then $Sx_0 \in C(X)$ and so $Sx_0 \neq \emptyset$. Let $x_1 \in Sx_0$. From (2.1), we have

$$kd(x_0, x_1) \in \alpha(x_0, x_1)s(Sx_0, Sx_1).$$

By Lemma 1.5 (3), we have

$$kd(x_0, x_1) \in \alpha(x_0, x_1) s(x_1, Sx_1).$$

By the definition, we can take $x_2 \in Sx_1$ such that

$$kd(x_0, x_1) \in \alpha(x_0, x_1) s(d(x_1, x_2)).$$

By Lemma 1.5 (4), we have

$$kd(x_0, x_1) \in \alpha(x_0, x_1) s(d(x_1, x_2)) \subseteq s(\alpha(x_0, x_1)d(x_1, x_2))$$

and so

$$\alpha(x_0, x_1)d(x_1, x_2) \preceq kd(x_0, x_1). \tag{2.2}$$

Since $\alpha(x_0, x_1) \geq 1$, it follows from (2.2) that

$$0 \prec d(x_1, x_2) \preceq \alpha(x_0, x_1)d(x_1, x_2) \preceq kd(x_0, x_1). \tag{2.3}$$

If $x_1 = x_2$, then x_1 is the required fixed point and so we have nothing to prove.

If $x_1 \neq x_2$, then $x_2 \notin Sx_1$ and so it follows from (2.1) that

$$kd(x_1, x_2) \in \alpha(x_1, x_2)s(Sx_1, Sx_2).$$

By Lemma 1.5 (3), we have

$$kd(x_1, x_2) \in \alpha(x_1, x_2) s(x_2, Sx_2).$$

By the definition, we can take $x_3 \in Sx_2$ such that

$$kd(x_1, x_2) \in \alpha(x_1, x_2) s(d(x_2, x_3)).$$

By Lemma 1.5 (4), we have

$$kd(x_1, x_2) \in \alpha(x_1, x_2) s(d(x_2, x_3)) \subseteq s(\alpha(x_1, x_2) d(x_2, x_3))$$

and so

$$\alpha(x_1, x_2) d(x_2, x_3) \preceq kd(x_1, x_2). \quad (2.4)$$

Since S is α -admissible, we have $\alpha(Sx_0, Sx_1) \geq 1$, that is, $\alpha(x_1, x_2) \geq 1$. Thus, it follows from (2.4) that

$$0 \prec d(x_2, x_3) \preceq \alpha(x_1, x_2) d(x_2, x_3) \preceq kd(x_1, x_2). \quad (2.5)$$

If $x_2 = x_3$, then x_2 is the required fixed point and so we have nothing to prove.

If $x_2 \neq x_3$, then $x_3 \notin Sx_3$ and so it follows from (2.1) that

$$kd(x_2, x_3) \in \alpha(x_2, x_3) s(Sx_2, Sx_3).$$

By Lemma 1.5 (3), we have

$$kd(x_2, x_3) \in \alpha(x_2, x_3) s(x_3, Sx_3).$$

By the definition, we can take $x_4 \in Sx_3$ such that

$$kd(x_2, x_3) \in \alpha(x_2, x_3) s(d(x_3, x_4)).$$

By Lemma 1.5 (4), we have

$$kd(x_2, x_3) \in \alpha(x_2, x_3) s(d(x_3, x_4)) \subseteq s(\alpha(x_2, x_3) d(x_3, x_4))$$

and so

$$\alpha(x_2, x_3) d(x_3, x_4) \preceq kd(x_2, x_3). \quad (2.6)$$

Since S is α -admissible on X , we have $\alpha(Sx_1, Sx_2) \geq 1$, that is, $\alpha(x_2, x_3) \geq 1$ and so it follows from (2.6) that

$$0 \prec d(x_3, x_4) \preceq \alpha(x_2, x_3) d(x_3, x_4) \preceq kd(x_2, x_3). \quad (2.7)$$

By continuing this process, we can obtain a sequence $\{x_n\}$ in X such that

$$x_n \in Sx_{n-1}, \quad x_n \neq x_{n-1}, \quad \alpha(x_n, x_{n+1}) \geq 1$$

and

$$d(x_n, x_{n+1}) \preceq kd(x_{n-1}, x_n) \preceq \cdots \preceq k^n d(x_0, x_1) \quad (2.8)$$

for all n . Then, for all $m, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &\preceq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + k^{n+2} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\ &\preceq k^n (e + k^1 + k^2 + \cdots + k^{m-n-1}) d(x_0, x_1) \\ &\preceq k^n (e - k)^{-1} d(x_0, x_1). \end{aligned}$$

Since $\rho(k) < 1$, we have $\|k^n\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, for any $c \in \mathbb{A}$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) \preceq k^n (e - k)^{-1} d(x_0, x_1) \ll c$$

for all $n > n_0$. This is sufficient to conclude that $\{x_n\}$ is a Cauchy sequence and so there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Then, by the assumption, we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. Thus it follows from (2.1) that

$$kd(x_n, x^*) \in \alpha(x_n, x^*)s(Sx_n, Sx^*)$$

for all $n \in \mathbb{N}$. By Lemma 1.5 (3), we have

$$kd(x_n, x^*) \in \alpha(x_n, x^*)s(x_{n+1}, Sx^*).$$

By the definition, we can take $v_n \in Sx^*$ such that

$$kd(x_n, x^*) \in \alpha(x_n, x^*)s(d(x_{n+1}, v_n)).$$

By Lemma 1.5 (4), we have

$$kd(x_n, x^*) \in \alpha(x_n, x^*)s(d(x_{n+1}, v_n)) \subseteq s(\alpha(x_n, x^*)d(x_{n+1}, v_n))$$

and so

$$\alpha(x_n, x^*)d(x_{n+1}, v_n) \preceq kd(x_n, x^*).$$

Since $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$, we have

$$0 \prec d(x_{n+1}, v_n) \preceq \alpha(x_n, x^*)d(x_{n+1}, v_n) \preceq kd(x_n, x^*) \prec d(x_n, x^*).$$

Since $x_n \rightarrow x^*$ as $n \rightarrow +\infty$, for any $c \in \text{Int}P$, there exists $k \in \mathbb{N}$ such that

$$d(x_n, x^*) \ll \frac{c}{2}, \quad d(x_{n+1}, x^*) \ll \frac{c}{2}$$

for each $n \geq k = k(c)$. Thus, from the triangular property, it follows that

$$\begin{aligned} d(x^*, v_n) &\preceq d(x^*, x_{n+1}) + d(x_{n+1}, v_n) \\ &\preceq d(x^*, x_{n+1}) + \alpha(x_n, x^*)d(x_{n+1}, v_n) \\ &\preceq d(x^*, x_{n+1}) + kd(x_n, x^*) \\ &\preceq d(x^*, x_{n+1}) + d(x_n, x^*) \\ &\ll \frac{c}{2} + \frac{c}{2} = c \end{aligned}$$

for each $n \geq k = k(c)$. Hence $\lim_{n \rightarrow \infty} v_n = x^*$. Since Sx^* is closed, $x^* \in Sx^*$. This implies that x^* is a fixed point of S . This completes the proof.

Theorem 2.2. *Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone. Suppose $S : X \rightarrow C(X)$ be a multi-valued mapping on X satisfies*

$$kd(x, y) \in s(Sx, Sy)$$

for all $x, y \in X$. Then, there exists a point x^* in X such that $x^* \in Sx^*$.

Proof. Taking $\alpha(x, y) = 1$ in the Theorem 2.1, we have the conclusion.

By Remark 1.2, we have the following:

Corollary 2.1. *Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, +\infty)$ be a function and a multi-valued mapping $S : X \rightarrow CB(X)$ is α -admissible. If there exists a constant $k \in [0, 1)$ such that*

$$\alpha(x, y)H(Sx, Sy) \leq kd(x, y)$$

for all $x, y \in X$. Suppose that there exist $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Assume that, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow u$ as $n \rightarrow +\infty$, then $\alpha(x_n, u) \geq 1$ for all $n \in \mathbb{N}$. Then there exists a point x^* in X such that $x^* \in Sx^*$.

Corollary 2.2. *Let (X, d) be a complete metric space and $S : X \rightarrow CB(X)$ be a multi-valued mapping. If there exists a constant $k \in [0, 1)$ such that*

$$H(Sx, Sy) \leq kd(x, y)$$

for all $x, y \in X$. Then there exists a point x^* in X such that $x^* \in Sx^*$.

Proof. Taking $\alpha(x, y) = 1$ in the Corollary 2.1, we have the conclusion.

Next, we give a lemma which is useful to establish the second main result. Moreover, this lemma is also a tool for analyzing fixed point results in a directed graph in next section.

Lemma 2.1. *Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} , P be the underlying solid cone and $\alpha : X \times X \rightarrow [0, +\infty)$ be a given mapping. If α has the transitive property (that is, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$, then $\alpha(x, z) \geq 1$), then the following conditions are equivalent:*

(1) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x^*$ as $n \rightarrow +\infty$, then $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$.*

(2) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x^*$ as $n \rightarrow +\infty$, then there exists a subsequence $\{x_{n_k}\}$ with $\alpha(x_{n_k}, x^*) \geq 1$ for all $n \in \mathbb{N}$.*

Proof. It easy to see that (1) implies (2). Now, we assume that (2) holds and $\{x_n\}$ is as in (1). By the transitivity, $\alpha(x_n, x_m) \geq 1$ if $n \leq m$. From (2), there exists a subsequence $\{x_{n_k}\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $n \in \mathbb{N}$. Since $n \leq n_k$, we have $\alpha(x_n, x_{n_k}) \geq 1$. Thus, by the transitivity, we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$. This completes the proof.

Using Lemma 2.1, we have the following result.

Corollary 2.3. *Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} , P be the underlying solid cone and $\alpha : X \times X \rightarrow [0, +\infty)$ be a given mapping satisfying*

transitive property. Suppose that a multi-valued mapping S is α -admissible, α -generalized multi-valued contraction and there exist $x_0 \in X$, $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x^$ as $n \rightarrow +\infty$, then there exists a subsequence $\{x_{n_k}\}$ with $\alpha(x_{n_k}, x^*) \geq 1$ for all $n \in \mathbb{N}$. Then S has a fixed point in X .*

Now, we give one example to illustrate Theorem 2.1 as follows:

Example 2.1. Let $X = [0, 1]$, $\mathbb{A} = C_{\mathbb{R}}^1[0, 1] \times C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|u\| = \|u\|_{\infty} + \|u'\|_{\infty}.$$

Define the multiplication on \mathbb{A} as just pointwise multiplication. Then \mathbb{A} is a real Banach algebra with the unit $e = 1$, that is, $e(t) = 1$ for all $t \in X$.

Let $P = \{u(t) \in \mathbb{A} : u(t) \geq 0 : t \in [0, 1]\}$. Then P is a non-normal solid cone. Define the cone metric $d : X \times X \rightarrow \mathbb{A}$ by

$$d(x, y) = (|x - y|)e^t.$$

Then (X, d) is a complete cone metric space over Banach algebra \mathbb{A} . Define a multi-valued mapping $S : X \rightarrow 2^X$ by $Sx = [0, \frac{x}{14}]$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} \frac{1}{|x-y|}, & \text{if } x \neq y, \\ 1, & \text{if } x = y. \end{cases}$$

Then clearly S is α -admissible. Take $k = \frac{1}{2} + \frac{1}{4}t$, where $t \in [0, 1]$. Then $\rho(k) = \frac{2}{3} < 1$. Now, for all $x, y \in X$ with $x \leq y$, we have

$$kd(x, y) = \left(\frac{1}{2} + \frac{1}{4}t\right)|x - y|e^t \succeq \frac{1}{14}|x - y|e^t = \alpha(x, y)s(Sx, Sy),$$

which implies that

$$kd(x, y) \in \alpha(x, y)s(Sx, Sy)$$

and so S is α -generalized multi-valued contraction on X . Thus all the conditions of Theorem 2.1 are satisfied and 0 is a fixed point of S .

3. APPLICATIONS WITH GRAPHS

In 2008, Jachymski [16] proved some fixed point results in metric spaces endowed with a graph and generalized simultaneously Banach's contraction principle from metric and partially ordered metric spaces.

Consistent with Jachymski, let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. Also, assume that the graph G has no parallel edges and thus, one can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [16]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G , then a *path* in G from x to y of length N ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for each $i = 1, \dots, N$.

In [16], Jachymski gave the following definition of G -contraction:

Definition 3.1. An operator $S : X \rightarrow X$ is called a *Banach G -contraction* or, simply, *G -contraction* if

(a) S preserves edges of G ; for each $x, y \in X$ with $(x, y) \in E(G)$, we have

$$(S(x), S(y)) \in E(G);$$

(b) S decreases weights of edges of G ; there exists $\alpha \in [0, 1)$ such that for all $x, y \in X$ with $(x, y) \in E(G)$, we have

$$d(S(x), S(y)) \leq \alpha d(x, y).$$

Notice that a graph G is *connected* if there is a directed path between any two vertices and it is *weakly connected* if \bar{G} is connected, where \bar{G} denotes the undirected graph obtained from G by ignoring the direction of edges. Denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus we have

$$V(G^{-1}) = V(G), \quad E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat \tilde{G} as a directed graph for which the set of its edges is symmetric under this convention; we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

By a *subgraph* of G we mean a graph H satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that $V(H)$ contains the vertices of all edges of $E(H)$. If $E(G)$ is symmetric, then, for any $x \in V(G)$, the subgraph G_x consisting of all edges and vertices that are contained in some path in G beginning at x is called the component of G containing x . In this case, $V(G_x) = [x]_G$, where $[x]_G$ denotes the equivalence class of the relation R defined on $V(G)$ by the rule:

$$yRz \text{ if there exists a path in } G \text{ from } y \text{ to } z.$$

Clearly, G_x is connected for all $x \in G$. We denote by

$$\Psi = \{G : G \text{ is a directed graph with } V(G) = X \text{ and } \Delta \subseteq E(G)\}.$$

In this section, we give the existence of fixed point theorems for multi-valued operator in a cone metric space over a Banach algebra A endowed with a graph under the generalized Hausdorff distance.

Before presenting our results, we introduce new definitions in a cone metric space over a Banach algebra \mathbb{A} endowed with a graph.

Definition 3.2. Let (X, d) be a cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone endowed with a graph G . Then we say that the multi-valued mapping $S : X \rightarrow C(X)$ *weakly preserves* edges of G if, for each $x \in X$ and $y \in S(x)$ with $(x, y) \in E(G)$, $(y, z) \in E(G)$ for all $z \in S(y)$.

Definition 3.3. Let (X, d) be a cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone. Then the multi-valued mapping $S : X \rightarrow C(X)$ is called a *generalized multi-valued G -contraction* if there exists $k \in P$ such that $\rho(k) < 1$ and, for all $x, y \in X$,

$$kd(x, y) \in s(Sx, Sy). \quad (3.1)$$

Example 3.1. Any mapping $S : X \rightarrow C(X)$ defined by $S(x) = \{c\}$, where $c \in X$, is a generalized multi-valued G -contraction for any graph G with $V(G) = X$.

Example 3.1. Any mapping $S : X \rightarrow C(X)$ is trivially a Banach G -contraction, where $G = (V(G), e(G)) = (X, \Delta)$.

Definition 3.4. Let (X, d) be a cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone endowed with a graph G . We say that X has the *G -regular property* if, for any $x \in X$ and any sequence $\{x_n\}$ in X such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, $(x_n, x^*) \in E(G)$ for all $n \in \mathbb{N}$.

Definition 3.5. Let (X, d) be a cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone endowed with a graph G . We say that $E(G)$ is *quasi-ordered* (or *transitive*) if $(x, y) \in E(G)$ and $(y, z) \in E(G)$ imply $(x, z) \in E(G)$ for all $x, y, z \in X$.

Now, we establish two fixed point theorems for multi-valued mappings in a cone metric space over a Banach algebra \mathbb{A} endowed with a graph G .

Theorem 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone endowed with a graph G . Suppose that $S : X \rightarrow C(X)$ is a generalized multi-valued G -contraction and satisfy the following conditions:*

- (a) S weakly preserves edges of G ;
- (b) there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $(x_0, x_1) \in E(G)$;
- (c) X has the G -regular property.

Then there exists a point $x^ \in X$ such that $x^* \in Sx^*$, that is, S has a fixed point in X .*

Proof. Consider a mapping $\alpha : X \times X \rightarrow [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

Since S is generalized multivalued G -contraction, it follows that, for all $x, y \in X$,

$$kd(x, y) \in \alpha(x, y)s(Sx, Sy).$$

This implies that S satisfies (2.1). By construction of α and hypothesis (a), we see that S is an α -admissible operator. From the condition (b) and the definition of α , we have $\alpha(x_0, x_1) \geq 1$. Using the G -regular property of X , we find that the condition (c) in Theorem 2.1 holds. Now, all the hypotheses of Theorem 2.1 are satisfied and so the existence of the fixed point of S follows from Theorem 2.1. This completes the proof.

By Remark 1.2, we have the following multi-valued version of Jachymski's result in [15]:

Corollary 3.1. *Let (X, d) be a complete metric space endowed with a graph G . Suppose the multi-valued mapping $S : X \rightarrow CB(X)$ satisfy the following conditions:*

- (a) $H(Sx, Sy) \leq kd(x, y)$ for all $x, y \in X$;
- (b) S weakly preserves edges of G ;
- (c) there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $(x_0, x_1) \in E(G)$;
- (d) X has the G -regular property.

Then there exists a point $x^ \in X$ such that $x^* \in Sx^*$, that is, S has a fixed point in X .*

Theorem 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra \mathbb{A} and P be the underlying solid cone endowed with a graph G . Suppose that $S : X \rightarrow C(X)$ is a generalized multi-valued G -contraction and satisfy the following conditions:*

- (a) S weakly preserves edges of G ;
- (b) there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $(x_0, x_1) \in E(G)$;
- (c) if $x^* \in X$ and a sequence $\{x_n\}$ in X is such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x^*) \in E(G)$ for all $n \in \mathbb{N}$.

Then there exists a point $x^ \in X$ such that $x^* \in Sx^*$, that is, S has a fixed point in X .*

Proof. Since G is a connected graph, $E(G)$ is a quasi-ordered. Consider the mapping $\alpha : X \times X \rightarrow [0, \infty)$ defined as in the proof of Theorem 3.1. It follows

from $E(G)$ being quasi ordered that α has the transitive property. By Lemma 2.1, we note that the condition (c) is equivalent to the condition (c) in Theorem 3.1. Thus we can prove this theorem similarly to the proof of Theorem 3.1 and, consequently, S has a fixed point in X . This completes the proof.

4. CONCLUSIONS

Recently, Liu and Xu [22] used the cones over a Banach algebra and proved some fixed point theorems for some contractions on cone metric spaces. They improved the contractive condition on self-mappings of cone metric spaces by replacing the contractive constant by a vector of cone. In this paper, we studied cone metric spaces over Banach algebra and proved some fixed point theorems for generalized multi-valued contractions. Also, we give an example to illustrate the significance of the main result. As applications of the main results, we derived some fixed point results on graphs.

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