

**THE MODIFIED PICARD-FB ITERATIVE ALGORITHM FOR  
APPROXIMATING THE FIXED POINTS OF CONDITIONAL  
QUASI-CONTRACTIVE MAPPINGS IN CONVEX METRIC  
SPACES AND ITS RATE OF CONVERGENCE**

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**ABSTRACT.** In this paper, the new modified Picard-FB iterative algorithm based on the idea of the iterative algorithm of Fukhar-ud-din and Berinde in [H. Fukhar-ud-dina, V. Berinde, Iterative methods for the class of quasi-contractive type operators and comparison of their rate of convergence in convex metric spaces, Filomat 30 (2016), 223–230.] is defined. We also introduce the class of conditional quasi-contractive mappings in convex metric spaces and prove the convergence result of the purposed iterative algorithm for approximating the fixed points for conditional quasi-contractive mappings in convex metric spaces. Furthermore, a comparison result of the rate of convergence of the new purposed iterative algorithm, the iterative algorithm of Fukhar-ud-din and Berinde and many iterative algorithms in convex metric spaces is investigated.

1. INTRODUCTION AND THE OBJECTIVE

In 2004, Berinde [2] first introduced the following concept of the comparison of the rate of convergence for two iterative algorithms.

**Definition 1.1** ([2]). *Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of real numbers that converge to  $a$  and  $b$ , respectively, and assume that there exists*

$$l := \lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|}.$$

( $\mathcal{C}_1$ ): *If  $l = 0$ , then it can be said that  $\{a_n\}$  converges faster to  $a$  than  $\{b_n\}$  to  $b$ .*

( $\mathcal{C}_2$ ): *If  $0 < l < \infty$ , then it can be said that  $\{a_n\}$  and  $\{b_n\}$  have the same rate of convergence.*

Suppose that two iterative sequences  $\{x_n\}$  and  $\{y_n\}$  in a metric space  $(X, d)$  converge to the same fixed point  $z \in X$  of a mapping  $T : X \rightarrow X$  such that

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$$d(x_n, z) \leq a_n \text{ and } d(y_n, z) \leq b_n \text{ for all } n \in \mathbb{N},$$

where  $\{a_n\}$  and  $\{b_n\}$  are two sequences of nonnegative real numbers converging to zero. From the Definition 1.1, the following concept appears to be very natural.

- We say that the iterative sequence  $\{x_n\}$  converges faster than the iterative sequence  $\{y_n\}$  to  $z$  whenever  $\{a_n\}$  converges faster than  $\{b_n\}$ .

Note that the comparison of the rate of convergence in the above sense depends on the choice of two sequences  $\{a_n\}$  and  $\{b_n\}$  which are error bounds of  $\{x_n\}$  and  $\{y_n\}$ , respectively.

In 2013, Phuengrattana and Suantai [12] point out that this method seems to be unclear and hence they redefined the comparison of the rate of convergence for two fixed point iterative sequences as follows.

**Definition 1.2** ([12]). *Let  $\{x_n\}$  and  $\{y_n\}$  be two sequences in a metric space  $(X, d)$  converging to the same fixed point  $z$  of a mapping  $T : X \rightarrow X$ . We say that  $\{x_n\}$  converges faster than  $\{y_n\}$  to  $z$  if and only if  $\lim_{n \rightarrow \infty} \frac{d(x_n, z)}{d(y_n, z)} = 0$ .*

In 2016, Fukhar-ud-din and Berinde [7] defined the new iterative method for finding the fixed point of a quasi-contractive mapping in metric spaces and studied the comparison of their rate of convergence in the sense of Phuengrattana and Suantai [12] in Definition 1.2.

The results of Fukhar-ud-din and Berinde [7] improve and unify a number of many results of the well-known researchers in this area such as Suantai [13], Noor [11], Glowinski and Le Tallec [8], Xu and Noor [15], Ishikawa [9] and Mann [10]. Based on the impact of this iterative algorithm and the fact that the class of quasi-contractive mappings is wider than many classes of contractive mappings, the results of Fukhar-ud-din and Berinde [7] are generalizations and refinements of many recent results such as Theorems 2.1, 2.2, 2.4 in [12], Theorem 3.1 in [6], Theorem 2 in [3] and many corresponded results in  $CAT(0)$  spaces, simultaneously.

In this paper, we define the new modified Picard-FB iterative algorithm which is motivated from the iterative algorithm of Fukhar-ud-din and Berinde [7] and introduce the class of conditional quasi-contractive mappings in convex metric spaces. The convergence result of the suggested iterative algorithm for approximating the fixed points for conditional quasi-contractive mappings in convex metric spaces is investigated. Furthermore, a comparison result of the rate of convergence of the new purposed iterative algorithm, the iterative algorithm of Fukhar-ud-din and Berinde [7] and several iterative algorithms in convex metric spaces is given.

## 2. PRELIMINARIES

In 2005, Suantai [13] introduced the modified Noor iterative algorithm  $\{x_n\}$  in a nonempty convex subset  $C$  of a normed space  $E$  as follows:

$$\left. \begin{aligned} x_1 &\in C, \\ z_n &= a_n T x_n + (1 - a_n) x_n, \\ y_n &= b_n T z_n + c_n T x_n + (1 - b_n - c_n) x_n, \\ x_{n+1} &= \alpha_n T y_n + \beta_n T z_n + (1 - \alpha_n - \beta_n) x_n \end{aligned} \right\} \quad (2.1)$$

for all  $n \in \mathbb{N}$ , where  $T : C \rightarrow C$  is a given mapping and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are real control sequences in the interval  $[0, 1]$ . This iterative algorithm generalizes several types of the iterative algorithm in many works (see in works of

Noor [11], Glowinski and Le Tallec [8], Xu and Noor [15], Ishikawa [9] and Mann [10] and references therein).

It is well known that many new iterative algorithms can be defined by using the idea of the convexity in normed spaces. However, we can not use this idea for defining several types of the iterative algorithms in the arbitrary metric space. From this point, we need the convex structure in a metric space for defining the new iterative algorithm. Next, we give the concept of a convex structure in a metric space.

**Definition 2.1** ([14]). *Let  $(X, d)$  be a metric space. A mapping  $W : X \times X \times [0, 1] \rightarrow X$  is called a convex structure in  $X$  if*

$$d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y) \tag{2.2}$$

for all  $x, y, u \in X$  and for all  $\alpha \in [0, 1]$ . Also, the triplet  $(X, d, W)$  is called a convex metric space.

It is well known that the normed space  $(X, \|\cdot\|)$  has a convex structure  $W : X \times X \times [0, 1] \rightarrow X$  defined by  $W(x, y, \alpha) = \alpha x + (1 - \alpha)y$  for all  $x, y \in X$  and for all  $\alpha \in [0, 1]$ . An another important and interesting example of a convex metric space is a  $CAT(0)$  space. We refer the reader to [5] for getting details of  $CAT(0)$  spaces.

Next, we give the important lemma for proving the second main result of this paper.

**Lemma 2.2.** *Let  $(X, d, W)$  be a convex metric space. Then*

$$d(u, W(x, y, \alpha)) \geq (1 - \alpha)d(u, y) - \alpha d(u, x) \tag{2.3}$$

for all  $x, y, u \in X$  and for all  $\alpha \in [0, 1]$ .

Now, we give the definition of a convex subset in a convex metric space as follows.

**Definition 2.3** ([14]). *A nonempty subset  $C$  of a convex metric space  $X$  is convex if  $W(x, y, \alpha) \in C$  for all  $x, y \in C$  and for all  $\alpha \in [0, 1]$ .*

Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a mapping. In 2006, Fukhar-ud-din and Berinde [7] introduced the following iterative algorithm  $\{x_n\}$  in  $X$ :

$$\left. \begin{aligned} x_1 &\in C, \\ z_n &= W(Tx_n, x_n, a_n), \\ y_n &= W\left(Tz_n, W\left(Tx_n, x_n, \frac{c_n}{1 - b_n}\right), b_n\right), \\ x_{n+1} &= W\left(Ty_n, W\left(Tz_n, x_n, \frac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right) \end{aligned} \right\} \tag{2.4}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are real control sequences in the interval  $[0, 1]$ . Using the fact that  $W(x, y, 0) = y$  for all  $x, y \in X$ , the iterative algorithm (2.4) provides, in convex metric spaces, analogues of the following famous iterative algorithms:

- Noor iterative algorithm  $\{x_n\}$  in  $X$  defined by

$$\left. \begin{aligned} x_1 &\in C, \\ z_n &= W(Tx_n, x_n, a_n), \\ y_n &= W(Tz_n, x_n, b_n), \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \end{aligned} \right\} \tag{2.5}$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{a_n\}$  and  $\{b_n\}$  are real control sequences in the interval  $[0, 1]$ ;

- Ishikawa iterative algorithm  $\{x_n\}$  in  $X$  defined by

$$\left. \begin{aligned} x_1 &\in C, \\ y_n &= W(Tx_n, x_n, b_n), \\ x_{n+1} &= W(Ty_n, x_n, \alpha_n) \end{aligned} \right\} \quad (2.6)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  and  $\{b_n\}$  are real control sequences in the interval  $[0, 1]$ ;

- Mann iterative algorithm  $\{x_n\}$  in  $X$  defined by

$$\left. \begin{aligned} x_1 &\in C, \\ x_{n+1} &= W(Tx_n, x_n, \alpha_n) \end{aligned} \right\} \quad (2.7)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$  is a real control sequence in the interval  $[0, 1]$ ;

- Picard iterative algorithm  $\{x_n\}$  in  $X$  defined by

$$\left. \begin{aligned} x_1 &\in C, \\ x_{n+1} &= Tx_n \end{aligned} \right\} \quad (2.8)$$

for all  $n \in \mathbb{N}$ .

Fukhar-ud-din and Berinde [7] also proved a convergence theorem in a convex metric space for the iterative algorithm (2.4) of a quasi-contractive self mapping  $T$  on a nonempty, closed and convex subset  $C$  of a convex metric space  $(X, d, W)$ , that is,  $T$  satisfies the following condition:

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx) \quad (2.9)$$

for all  $x, y \in C$ , where  $\delta \in [0, 1)$  and  $L \geq 0$  (the original idea of a quasi-contractive mapping stating in metric spaces was first introduced by Berinde [4]). They also compared the rate of convergence of the iterative algorithm (2.4) with Noor iterative algorithm, Ishikawa iterative algorithm and Mann iterative algorithm in the setting of convex metric spaces.

### 3. THE CONVERGENCE AND THE RATE OF CONVERGENCE THEOREMS

We first introduce the generalized class of a class of quasi-contractive mappings as follows.

**Definition 3.1.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$ . A mapping  $T : C \rightarrow C$  is called a conditional quasi-contractive mapping if there are constants  $\lambda, \delta \in [0, 1)$  and  $L \geq 0$  such that*

$$x, y \in C \text{ with } \lambda d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx). \quad (3.1)$$

It is easy to see that the class of conditional quasi-contractive mappings is wider than the class of quasi-contractive mappings.

Throughout this section, unless otherwise specified,  $C$  is a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$ . Next, we introduce a modified

Picard-FB iterative algorithm to approximate fixed point of a conditional quasi-contractive mapping  $T : C \rightarrow C$  as follows:

$$\left. \begin{aligned} x_1 &\in C, \\ z_n &= W(T^n x_n, x_n, a_n), \\ y_n &= W\left(T^n z_n, W\left(T^n x_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(T^n y_n, W\left(T^n z_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \end{aligned} \right\} \quad (3.2)$$

for all  $n \in \mathbb{N}$ , where  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$  and  $\{c_n\}$  are real control sequences in the interval  $[0, 1]$ .

We now prove the first main theorem of this paper concerning the convergence analysis of the modified Picard-FB iterative algorithm (3.2) for approximating fixed points of conditional quasi-contractive mappings.

**Theorem 3.2.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a conditional quasi-contractive mapping, that is, there are constants  $\lambda, \delta \in [0, 1)$  and  $L \geq 0$  such that*

$$x, y \in C \text{ with } \lambda d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx). \quad (3.3)$$

Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{b_n + c_n\}$  and  $\{\alpha_n + \beta_n\}$  are sequences in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$ . Then the sequence  $\{x_n\}$  in (3.2) converges to a unique fixed point of  $T$ .

*Proof.* From the facts that  $F(T) \neq \emptyset$  and  $T$  satisfies (3.3), we obtain  $F(T)$  is the singleton set. Assume that  $p \in F(T)$ . Our goal of this proof is to claim that  $\{x_n\}$  in (3.2) converges to a unique fixed point  $p$  of  $T$ . From the convex structure  $W$  and the definitions of  $\{z_n\}$  and  $\{y_n\}$ , we obtain

$$\begin{aligned} d(z_n, p) &= d(W(T^n x_n, x_n, a_n), p) \\ &\leq a_n d(T^n x_n, p) + (1 - a_n) d(x_n, p) \\ &\leq a_n \delta^n d(x_n, p) + (1 - a_n) d(x_n, p) \\ &\leq a_n \delta d(x_n, p) + (1 - a_n) d(x_n, p) \\ &= (1 - a_n(1 - \delta)) d(x_n, p) \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} d(y_n, p) &= d\left(W\left(T^n z_n, W\left(T^n x_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), p\right) \\ &\leq b_n d(T^n z_n, p) + (1 - b_n) d\left(W\left(T^n x_n, x_n, \frac{c_n}{1-b_n}\right), p\right) \\ &\leq b_n d(T^n z_n, p) + (1 - b_n) \left(\frac{c_n}{1-b_n} d(T^n x_n, p) + \left(1 - \frac{c_n}{1-b_n}\right) d(x_n, p)\right) \\ &= b_n d(T^n z_n, p) + c_n d(T^n x_n, p) + (1 - b_n - c_n) d(x_n, p) \\ &\leq b_n \delta^n d(z_n, p) + c_n \delta^n d(x_n, p) + (1 - b_n - c_n) d(x_n, p) \\ &\leq b_n \delta d(z_n, p) + c_n \delta d(x_n, p) + (1 - b_n - c_n) d(x_n, p) \end{aligned} \quad (3.5)$$

for all  $n \in \mathbb{N}$ . From (3.4) and (3.5), we have

$$d(y_n, p) \leq [b_n \delta(1 - a_n(1 - \delta)) + c_n \delta + (1 - b_n - c_n)] d(x_n, p) \quad (3.6)$$

for all  $n \in \mathbb{N}$ . The convex structure  $W$  and the definition of  $\{x_n\}$  yield that

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(W\left(T^n y_n, W\left(T^n z_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\leq \alpha_n d(T^n y_n, p) + (1-\alpha_n) d\left(W\left(T^n z_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\leq \alpha_n d(T^n y_n, p) + (1-\alpha_n) \left(\frac{\beta_n}{1-\alpha_n} d(T^n z_n, p) + (1-\beta_n) d(x_n, p)\right) \\
&= \alpha_n d(T^n y_n, p) + \beta_n d(T^n z_n, p) + (1-\alpha_n-\beta_n) d(x_n, p) \\
&\leq \alpha_n \delta^n d(y_n, p) + \beta_n \delta^n d(z_n, p) + (1-\alpha_n-\beta_n) d(x_n, p) \\
&\leq \alpha_n \delta d(y_n, p) + \beta_n \delta d(z_n, p) + (1-\alpha_n-\beta_n) d(x_n, p)
\end{aligned} \tag{3.7}$$

for all  $n \in \mathbb{N}$ . By using (3.4), (3.6) and (3.7), we get

$$\begin{aligned}
d(x_{n+1}, p) &\leq \{\alpha_n \delta [b_n \delta (1-a_n(1-\delta)) + c_n \delta + (1-b_n-c_n)] + \beta_n \delta (1-a_n(1-\delta)) + (1-\alpha_n-\beta_n)\} d(x_n, p) \\
&\leq \{1 - (1-\delta)\alpha_n [1 + a_n b_n + (b_n + c_n)\delta] - (1-\delta)\beta_n [1 + a_n \delta]\} d(x_n, p) \\
&\leq [1 - (1-\delta)\alpha_n - (1-\delta)\beta_n] d(x_n, p) \\
&= [1 - (1-\delta)(\alpha_n + \beta_n)] d(x_n, p) \\
&\vdots \\
&= \prod_{k=1}^n [1 - (1-\delta)(\alpha_k + \beta_k)] d(x_1, p)
\end{aligned} \tag{3.8}$$

for all  $n \in \mathbb{N}$ . This implies that

$$d(x_{n+1}, p) \leq \prod_{k=1}^n [1 - (1-\delta)(\alpha_k + \beta_k)] d(x_1, p) \tag{3.9}$$

for all  $n \in \mathbb{N}$ . Since  $1 + x \leq e^x$  for all  $x \geq 0$ , the above inequality implies that

$$\begin{aligned}
d(x_{n+1}, p) &\leq \prod_{k=1}^n e^{-(1-\delta)(\alpha_k + \beta_k)} d(x_1, p) \\
&= e^{-(1-\delta) \sum_{k=1}^n (\alpha_k + \beta_k)} d(x_1, p)
\end{aligned} \tag{3.10}$$

for all  $n \in \mathbb{N}$ . Taking limit as  $n \rightarrow \infty$  in the above inequality and using the fact that  $\sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$ , we can conclude that  $d(x_{n+1}, p) \rightarrow 0$ , that is, the sequence  $\{x_n\}$  in (3.2) converges to a unique fixed point of  $T$ . This completes the proof.  $\square$

By using the same technique in the proof of Theorem 3.2, we can prove the following result which is a generalization of the main result of Fukhar-ud-din and Berinde [7].

**Theorem 3.3.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a conditional quasi-contractive mapping. Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{a_n\}$ ,  $\{b_n\}$ ,  $\{c_n\}$ ,  $\{b_n + c_n\}$  and  $\{\alpha_n + \beta_n\}$  are sequences in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$ . Then the Fukhar-ud-din and*

Berinde iterative algorithm  $\{x_n^{(4)}\}$  in  $X$  defined by

$$\left. \begin{aligned} x_1^{(4)} &\in C, \\ z_n^{(4)} &= W(Tx_n^{(4)}, x_n^{(4)}, a_n), \\ y_n^{(4)} &= W\left(Tz_n^{(4)}, W\left(Tx_n^{(4)}, x_n^{(4)}, \frac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1}^{(4)} &= W\left(Ty_n^{(4)}, W\left(Tz_n^{(4)}, x_n^{(4)}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \end{aligned} \right\} \quad (3.11)$$

for all  $n \in \mathbb{N}$  converges to a unique fixed point of  $T$ .

From Theorem 3.2, we get the following immediate consequences.

**Corollary 3.4.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a conditional quasi-contractive mapping. Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}$ ,  $\{a_n\}$  and  $\{b_n\}$  are sequences in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then the modified Picard-Noor iterative algorithm  $\{x_n^{(3)}\}$  in  $X$  defined by*

$$\left. \begin{aligned} x_1^{(3)} &\in C, \\ z_n^{(3)} &= W(T^n x_n^{(3)}, x_n^{(3)}, a_n), \\ y_n^{(3)} &= W\left(T^n z_n^{(3)}, x_n^{(3)}, b_n\right), \\ x_{n+1}^{(3)} &= W\left(T^n y_n^{(3)}, x_n^{(3)}, \alpha_n\right) \end{aligned} \right\} \quad (3.12)$$

for all  $n \in \mathbb{N}$  converges to a unique fixed point of  $T$ .

*Proof.* Taking  $c_n = \beta_n = 0$  for all  $n \in \mathbb{N}$  in Theorem 3.2, we get this result.  $\square$

**Corollary 3.5.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a conditional quasi-contractive mapping. Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}$  and  $\{b_n\}$  are sequences in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ .*

*Then the modified Picard-Ishikawa iterative algorithm  $\{x_n^{(2)}\}$  in  $X$  defined by*

$$\left. \begin{aligned} x_1^{(2)} &\in C, \\ y_n^{(2)} &= W\left(T^n x_n^{(2)}, x_n^{(2)}, b_n\right), \\ x_{n+1}^{(2)} &= W\left(T^n y_n^{(2)}, x_n^{(2)}, \alpha_n\right) \end{aligned} \right\} \quad (3.13)$$

for all  $n \in \mathbb{N}$  converges to a unique fixed point of  $T$ .

*Proof.* Taking  $a_n = c_n = \beta_n = 0$  for all  $n \in \mathbb{N}$  in Theorem 3.2, we get this result.  $\square$

**Corollary 3.6.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a conditional quasi-contractive mapping. Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}$  is a sequence in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ . Then the modified Picard-Mann iterative algorithm  $\{x_n^{(1)}\}$  in  $X$  defined by*

$$\left. \begin{aligned} x_1^{(1)} &\in C, \\ x_{n+1}^{(1)} &= W\left(T^n x_n^{(1)}, x_n^{(1)}, \alpha_n\right) \end{aligned} \right\} \quad (3.14)$$

for all  $n \in \mathbb{N}$  converges to a unique fixed point of  $T$ .

*Proof.* Taking  $a_n = b_n = c_n = \beta_n = 0$  for all  $n \in \mathbb{N}$  in Theorem 3.2, we get this result.  $\square$

Next, we will investigate the rate of convergence of the modified Picard-FB iterative algorithm (3.2) with the iterative algorithm (3.11) of Fukhar-ud-din and Berinde [7], the modified Picard-Noor (3.12), the modified Picard-Ishikawa (3.13) and the modified Picard-Mann (3.14) for approximating the fixed points of conditional quasi-contractive mappings in convex metric spaces.

**Theorem 3.7.** *Let  $C$  be a nonempty, closed and convex subset of a convex metric space  $(X, d, W)$  and  $T : C \rightarrow C$  be a quasi-contractive mapping, that is, there are constants  $\lambda, \delta \in [0, 1)$  and  $L \geq 0$  such that*

$$x, y \in C \text{ with } \lambda d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx). \quad (3.15)$$

Suppose that  $F(T) \neq \emptyset$  and  $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n + \beta_n\}$  and  $\{b_n + c_n\}$  are sequences in  $[0, 1]$  such that  $\sum_{n=1}^{\infty} (\alpha_n + \beta_n) = \infty$ . If the following conditions hold:

- (R<sub>1</sub>) there is a real number  $\eta$  such that  $0 < \eta < \alpha_n + \beta_n < 1$  for all  $n \in \mathbb{N}$ ;
- (R<sub>2</sub>)  $0 < \alpha_n \leq \alpha_n + \beta_n < \frac{1}{1 + \delta}$  for all  $n \in \mathbb{N}$ ;
- (R<sub>3</sub>)  $\lim_{n \rightarrow \infty} \alpha_n = 0, \lim_{n \rightarrow \infty} \beta_n = 0,$

then the sequence  $\{x_n\}$  in (3.2) converges faster than the Fukhar-ud-din and Berinde iterative method (3.11), the modified Picard-Noor iterative method (3.12), the modified Picard-Ishikawa iterative method (3.13) and the modified Picard-Mann iterative method (3.14) to a unique fixed point of  $T$  provided that all the methods have the same initial guess  $x_1 \in C$ .

*Proof.* Since  $F(T) \neq \emptyset$  and  $T$  satisfies (3.15), we get  $F(T)$  is the singleton set. Assume that  $p \in F(T)$ . By Theorem 3.2, Theorem 3.3 and Corollaries 3.4, 3.5, 3.6, the sequences  $\{x_n\}, \{x_n^{(4)}\}, \{x_n^{(3)}\}, \{x_n^{(2)}\}$  and  $\{x_n^{(1)}\}$  defined by (3.2), (3.11), (3.12), (3.13), (3.14), respectively, converges to  $p$ . First, we will show that the sequence  $\{x_n\}$  in (3.2) converges faster than the sequence  $\{x_n^{(4)}\}$  generated by Fukhar-ud-din and Berinde iterative method (2.5), to  $p$ . From Lemma 2.2 and the definition of a

convex structure  $W$ , we obtain

$$\begin{aligned}
d\left(x_{n+1}^{(4)}, p\right) &= d\left(W\left(Ty_n^{(4)}, W\left(Tz_n^{(4)}, x_n^{(4)}, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\geq (1-\alpha_n)d\left(W\left(Tz_n^{(4)}, x_n^{(4)}, \frac{\beta_n}{1-\alpha_n}\right), p\right) - \alpha_n\delta d\left(y_n^{(4)}, p\right) \\
&= (1-\alpha_n)d\left(W\left(Tz_n^{(4)}, x_n^{(4)}, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\quad - \alpha_n\delta d\left(W\left(Tz_n^{(4)}, W\left(Tx_n^{(4)}, x_n^{(4)}, \frac{c_n}{1-b_n}\right), b_n\right), p\right) \\
&\geq (1-\alpha_n)\left[\left(1-\frac{\beta_n}{1-\alpha_n}\right)d\left(x_n^{(4)}, p\right) - \left(\frac{\beta_n}{1-\alpha_n}\right)\delta d\left(z_n^{(4)}, p\right)\right] \\
&\quad - \alpha_n\delta\left[b_n\delta d\left(z_n^{(4)}, p\right) + (1-b_n)d\left(W\left(Tx_n^{(4)}, x_n^{(4)}, \frac{c_n}{1-b_n}\right)\right)\right] \\
&= (1-\alpha_n-\beta_n)d\left(x_n^{(4)}, p\right) - (\beta_n\delta + \alpha_n\delta^2b_n)d\left(z_n^{(4)}, p\right) \\
&\quad - \alpha_n\delta\left[(1-b_n)d\left(W\left(Tx_n^{(4)}, x_n^{(4)}, \frac{c_n}{1-b_n}\right)\right)\right] \\
&= (1-\alpha_n-\beta_n)d\left(x_n^{(4)}, p\right) - (\beta_n\delta + \alpha_n\delta^2b_n)d\left(W\left(Tx_n^{(4)}, x_n^{(4)}, a_n\right), p\right) \\
&\quad - \alpha_n\delta\left[(1-b_n)d\left(W\left(Tx_n^{(4)}, x_n^{(4)}, \frac{c_n}{1-b_n}\right)\right)\right] \\
&\geq (1-\alpha_n-\beta_n)d\left(x_n^{(4)}, p\right) - (\beta_n\delta + \alpha_n\delta^2b_n)\left[a_n\delta d\left(x_n^{(4)}, p\right) + (1-a_n)d\left(x_n^{(4)}, p\right)\right] \\
&\quad - \alpha_n\delta(1-b_n)\left[\left(\frac{c_n}{1-b_n}\right)\delta d\left(x_n^{(4)}, p\right) + \left(1-\frac{c_n}{1-b_n}\right)d\left(x_n^{(4)}, p\right)\right] \\
&= (1-\alpha_n-\beta_n)d\left(x_n^{(4)}, p\right) - (\beta_n\delta + \alpha_n\delta^2b_n)(1-a_n(1+\delta))d\left(x_n^{(4)}, p\right) \\
&\quad - \alpha_n\delta(1-b_n-c_n(1+\delta))d\left(x_n^{(4)}, p\right) \\
&\geq (1-\alpha_n-\beta_n)d\left(x_n^{(4)}, p\right) - (\beta_n\delta + \alpha_n\delta^2b_n)d\left(x_n^{(4)}, p\right) - \alpha_n\delta(1-b_n)d\left(x_n^{(4)}, p\right) \\
&\geq (1-\alpha_n-\beta_n-\alpha_n\delta-\beta_n\delta)d\left(x_n^{(4)}, p\right) \\
&= [1-(\alpha_n+\beta_n)(1+\delta)]d\left(x_n^{(4)}, p\right) \\
&\quad \vdots \\
&\geq \prod_{k=1}^n [1-(\alpha_k+\beta_k)(1+\delta)]d\left(x_1^{(4)}, p\right) \tag{3.16}
\end{aligned}$$

for all  $n \in \mathbb{N}$ . Similarly, we can prove that

$$d\left(x_{n+1}^{(i)}, p\right) \geq \prod_{k=1}^n [1-\alpha_k(1+\delta)]d\left(x_1^{(i)}, p\right) \tag{3.17}$$

for all  $n \in \mathbb{N}$ , where  $i = 1, 2, 3$ . From (3.8) and  $(R_1)$ , we obtain

$$\begin{aligned} d(x_{n+1}, p) &\leq \prod_{k=1}^n [1 - (1 - \delta)(\alpha_k + \beta_k)] d(x_1, p) \\ &\leq (1 - \eta(1 - \delta))^n d(x_1, p) \end{aligned} \quad (3.18)$$

for all  $n \in \mathbb{N}$ .

Next, we will claim that the sequence  $\{x_n\}$  in (3.2) converges faster than the Fukhar-ud-din and Berinde iterative method (3.11) to a unique fixed point of  $T$  provided that all the methods have the same initial guess  $x_1 \in C$ . From (3.16) and (3.18), we obtain

$$\frac{d(x_{n+1}, p)}{d(x_{n+1}^{(4)}, p)} \leq \frac{(1 - \eta(1 - \delta))^n}{\prod_{k=1}^n [1 - (\alpha_k + \beta_k)(1 + \delta)]} \quad (3.19)$$

for all  $n \in \mathbb{N}$ . Let  $\theta_n := \frac{(1 - \eta(1 - \delta))^n}{\prod_{k=1}^n [1 - (\alpha_k + \beta_k)(1 + \delta)]}$  for all  $n \in \mathbb{N}$ . It follows from

$(R_3)$  that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\theta_{n+1}}{\theta_n} &= \lim_{n \rightarrow \infty} \frac{(1 - \eta(1 - \delta))^{n+1}}{\prod_{k=1}^{n+1} [1 - (\alpha_k + \beta_k)(1 + \delta)]} \cdot \frac{\prod_{k=1}^n [1 - (\alpha_k + \beta_k)(1 + \delta)]}{(1 - \eta(1 - \delta))^n} \\ &= \lim_{n \rightarrow \infty} \frac{1 - \eta(1 - \delta)}{1 - (\alpha_{n+1} + \beta_{n+1})(1 + \delta)} \\ &= 1 - \eta(1 - \delta) \\ &< 1. \end{aligned}$$

From the ratio test of real sequences, we have  $\sum_{n=1}^{\infty} \theta_n < \infty$ . It yields that  $\lim_{n \rightarrow \infty} \theta_n = 0$ .

From Definition 1.2, we conclude that the sequence  $\{x_n\}$  in (3.2) converges faster than the Fukhar-ud-din and Berinde iterative method (3.11) to a unique fixed point of  $T$  provided that all the methods have the same initial guess  $x_1 \in C$ .

Finally, we will show that the sequence  $\{x_n\}$  in (3.2) converges faster than the iterative methods (3.12), (3.13) and (3.14) to a unique fixed point of  $T$  provided that all the methods have the same initial guess  $x_1 \in C$ . From (3.17) and (3.18), we obtain

$$\frac{d(x_{n+1}, p)}{d(x_{n+1}^{(i)}, p)} \leq \frac{(1 - \eta(1 - \delta))^n}{\prod_{k=1}^n [1 - \alpha_k(1 + \delta)]} \quad (3.20)$$

for all  $n \in \mathbb{N}$ , where  $i = 1, 2, 3$ .

Let  $\theta'_n := \frac{(1 - \eta(1 - \delta))^n}{\prod_{k=1}^n [1 - \alpha_k(1 + \delta)]}$  for all  $n \in \mathbb{N}$ . It follows from  $(R_3)$  that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\theta'_{n+1}}{\theta'_n} &= \lim_{n \rightarrow \infty} \frac{(1 - \eta(1 - \delta))^{n+1}}{\prod_{k=1}^{n+1} [1 - \alpha_k(1 + \delta)]} \cdot \frac{\prod_{k=1}^n [1 - \alpha_k(1 + \delta)]}{(1 - \eta(1 - \delta))^n} \\ &= \lim_{n \rightarrow \infty} \frac{1 - \eta(1 - \delta)}{1 - \alpha_{n+1}(1 + \delta)} \\ &= 1 - \eta(1 - \delta) \\ &< 1. \end{aligned}$$

From the ratio test, we have  $\sum_{n=1}^{\infty} \theta_n < \infty$ . It means that  $\lim_{n \rightarrow \infty} \theta_n = 0$ . Definition 1.2 yields that the sequence  $\{x_n\}$  in (3.2) converges faster than the iterative methods (3.12), (3.13) and (3.14) to a unique fixed point of  $T$ .  $\square$

#### 4. CONCLUSIONS AND OPEN PROBLEMS

We introduced the definition of a conditional quasi-contractive mapping in convex metric spaces and defined the new modified Picard-FB iterative algorithm in the framework of convex metric spaces which covers various iterative algorithms such as the modified Picard-Noor iterative algorithm, the modified Picard-Ishikawa iterative algorithm and the modified Picard-Mann iterative algorithm. The strong convergence theorem of the new purposed iterative algorithm for approximating fixed points for conditional quasi-contractive mappings in convex metric spaces is established and the comparison of the rate of convergence of the new iterative algorithm with several iterative algorithms is investigated. In view of the fact that the convex metric space is an important extension of a normed space and a  $CAT(0)$  space, our technique and results in this work can be viewed as a generalization of many results in normed spaces and  $CAT(0)$  spaces.

There have been many results on the iterative algorithm for finding the fixed points of nonlinear mappings in various spaces such as the Mann type iterative algorithm of Afshari and Aydi [1]. Inspired by these works, we have the following question.

- Does the main result in this paper holds for the Mann type iterative algorithm of Afshari and Aydi [1] in convex metric spaces?
- Does the main result in this paper holds for another iterative algorithm in convex metric spaces?

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