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ANALYTIC AND SEMI ANALYTIC SOLUTION FOR MOTION OF FRACTIONAL SECOND GRADE FLUID IN A CIRCULAR CYLINDER

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Abstract. Unsteady flow of fractional second grade fluid subject to a time dependent shear stress through a circular cylinder is considered. The motion is produced by the boundary of the cylinder which is subject to the longitudinal time dependent shear stress. The governing equation corresponding to second grade fluid given by a new definition of fractional derivatives without singular kernel is used, given by Caputo and Fabrizio. The flow is studied analytically by using finite Hankel and Laplace transforms. The corresponding solutions for ordinary second grade and Newtonian fluids are obtained as limiting cases of our general solutions, where as Stehfest's algorithm is used to developed the numerical solution of our problem. The numerically obtained solutions are in terms of the modified Bessel's equations of first and second kind, satisfying all imposed conditions. A good comparison between existing analytical solution and our solutions are made. Finally, the effect of different parameters and their comparison are explained graphically.

1. Introduction

The study of non-Newtonian fluids has got much attention because of their practical applications. Non-Newtonian characteristics are displayed by a number of industrially important fluids such as clay coatings, greases, polymer melts and many emulsions etc. Due to the great diversity in the physical structure of non-Newtonian fluids it is difficult to suggest a single model which exhibits all properties of non-Newtonian fluids. For this reason many non-Newtonian models such as differential type, rate type and integral type fluids have been proposed in recent years. Amongst these models, the fluids of differential type have received special attention [1, 2, 3, 4]. Further, a subclass of differential type of fluids which are the second grade fluids have been successfully studied in various kinds of flows by different researchers [5, 6, 7, 8, 9, 10, 11, 12, 13]. The development of one dimensional fractional derivative is one of the recent advances in the theoretical studies in rheology. The fact that they can be used to study shear-thinning behavior which have opened the way for the solution to a series of engineering problems.

Movement of the fluid in a translating, oscillating or rotating cylindrical system is very important and interesting. The study of the mechanism of viscoelastic fluids is very critical. It has a lot of applications in many walks of industry, such as chemical industry, bio engineering and oil exploitation.

Viscoelasticity is the property of fluids which has major effect on the motion of the fluids or movement of the bodies through fluids even at micro level. The most important phenomena corresponding to viscoelasticity are blood circulation, high speed vehicles and aeroplanes etc. It is practically verified that differential equations with non-integer order is very reasonable to describe the viscoelasticity and some other properties of the fluids. Thats why fractional calculus approach has been used mostly in flow

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problems for last few years. Some investigations with fractional derivatives can be seen [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]

It is found that most of the researchers show considerable interest in generalizing the flow problems to make more perfect with experimental results. For this purpose, they introduced some definitions of fractional derivatives like Riemann-Liouville, Caputo and Caputo-Fabrizio etc. The definition of fractional derivative in Caputo-Fabrizio sense is very important because it has very smooth and non-singular kernel. We can say Caputo-Fabrizio fractional derivative remove all the hurdles facing other definitions.

The purpose of this communication is to establish solutions for the velocity field and shear stress of an incompressible fractional second grade fluid (IFSGF) in a circular cylinder. Initially the cylinder and fluid both are at rest. At the moment ($t = 0^+$), the cylinder pulled by a longitudinal shear which is time dependent. The well known integral transforms namely Laplace and Hankel are used to solve the above flow model analytically. The final results are expressed in terms of generalized G function. Numerical solution also obtained by using Laplace transform and Stehfest's algorithm with the help of MATHCAD Software. The solutions for ordinary second grade and Newtonian fluids are also obtained as a limiting cases.

2. Governing equations

The flow considered have the velocity field $\mathbf{v}$ and extra stress tensor $\mathbf{S}$ of the form [27]

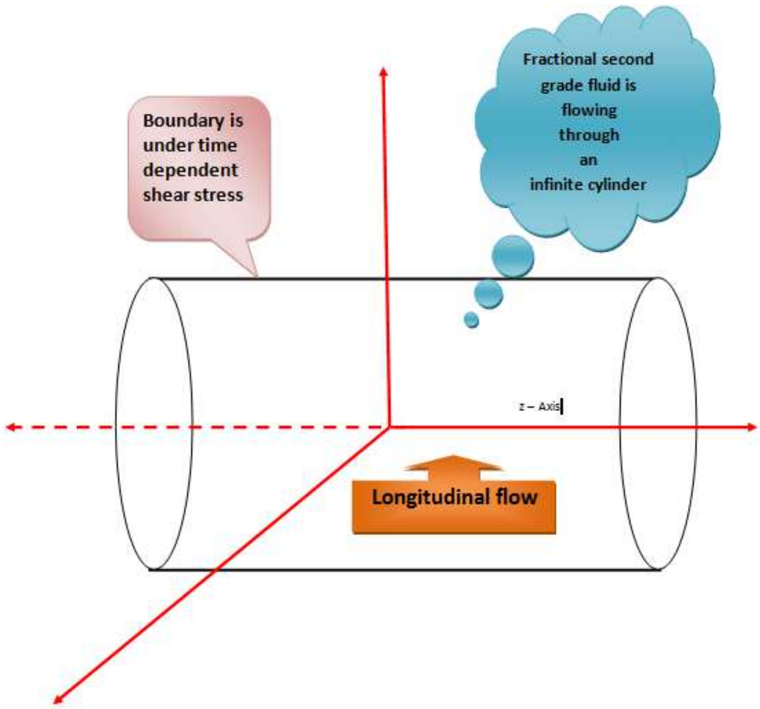

Figure 1. Geometry of the problem.

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{e_z}, \quad \mathbf{S} = \mathbf{S}(r,t), \tag{2.1}$$

where $\mathbf{e_z}$ is the unit vector in the z-direction of the cylindrical coordinate system r, θ and z. For such flows, the constraint of incompressibility is satisfied. Furthermore, if the fluid is at rest up the moment $(t = 0)$, then

$$\mathbf{v}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \mathbf{0}. \tag{2.2}$$

The governing equations, corresponding to such motions for second-grade fluid, are [10, 27].

$$\frac{\partial v(r,t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) v(r,t); \tag{2.3}$$

$$\tau(r,t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial r}, \tag{2.4}$$

where μ is the dynamic viscosity of the fluid, α_1 is a material constant (one of the tow material moduli which define a second-grade fluid), $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid being its constant density, $\alpha = \frac{\alpha_1}{\rho}$ and $\tau(r,t) = \mathbf{S}_{rz}$ is the shear stress which is different to zero. The governing equations corresponding to an incompressible fractional second-grade fluid, performing the same motion, are obtained by replacing the inner time derivative from Eqs. (2.3) and (2.4) by the fractional differential operator [29]

$$D_t^{\beta} g(t) = \begin{cases} \frac{M(\beta)}{\Gamma(1-\beta)} \int_0^t g(\tau) \exp\left[\frac{-\beta(t-\tau)}{1-\beta}\right] d\tau, \\ \\ \frac{d}{dt} g(t), \end{cases} \tag{2.5}$$

where $M(\beta)$ is a normalization function such that $M(0) = 1 = M(1)$.

Consequently, the governing equations for (IFSGF) model are

$$\frac{\partial v(r,t)}{\partial t} = \left(\nu + \alpha D_t^{\beta}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) v(r,t); \tag{2.6}$$

$$\tau(r,t) = \left(\mu + \alpha_1 D_t^{\beta}\right)\frac{\partial v(r,t)}{\partial r}, \tag{2.7}$$

where α and α_1, are the material constants.

3. Axial Couette flow due to a time dependent shear stress

Let us consider an [IFSGF] is at rest in an infinite circular cylinder of radius R. At ($t = 0^+$), the cylinder is suddenly pulled with a time dependent shear stress. Due to the shear, the fluid is gradually moved. The governing equations are given by Eqs. (2.6) and (2.7) and the initial and boundary conditions are

$$v(r,0) = 0 = \tau(r,0), \quad r \in [0,R], \tag{3.1}$$

$$\tau(R,t) = \left(\mu + \alpha_1 D_t^{\beta}\right)\frac{\partial v(r,t)}{\partial r}|_{r=R} = At, \tag{3.2}$$

where A is any constant. For the solution of above flow problem, there exist a class of methods in literature but we use most efficient, systematic and powerful integral transform techniques. In this article, Laplace and finite Hankel transforms are used for temporal and spatial variables respectively.

3.1. **Calculation of the velocity field.**

Keeping in mind initial condition Eq. (3.1), we apply the Laplace transformation to Eqs. (2.6) and (3.2), we get

$$q\overline{v}(r,q) = \left[\nu + \frac{\alpha q}{\beta + (1-\beta)q}\right]\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\overline{v}(r,q), \tag{3.3}$$

$$\frac{\partial \overline{v}(r,q)}{\partial r}|_{r=R} = \frac{A}{q^2\left[\mu + \frac{\alpha_1 q}{\beta+(1-\beta)q}\right]}, \tag{3.4}$$

where, $\overline{v}(r,q)$ represents the Laplace transforms of the function $v(r,t)$ and q is Laplace transform parameter. Eqs. (3.3) and (3.4) can be written as

$$q[\beta+(1-\beta)q]\overline{v}(r,q)=[\nu\beta+\beta_2 q]\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right]\overline{v}(r,q), \tag{3.5}$$

$$\frac{\partial\overline{v}(r,q)}{\partial r}|_{r=R}=\frac{[\beta+(1-\beta)q]A}{q^2[\beta\mu+\beta_1 q]}, \tag{3.6}$$

where $\beta_2=\nu(1-\beta)+\alpha$ and $\beta_1=\mu(1-\beta)+\alpha_1$. The Hankel transform of $\overline{v}(r,q)$ is denoted by $\overline{v}_H(r_n,q)$, and defined as [30]

$$\overline{v}_H(r_n,q)=\int_0^R r\,\overline{v}(r,q)J_0(rr_n)dr, \quad n=1,2,3,... \tag{3.7}$$

where r_n are the positive roots of the transcendental equation $J_0(Rr)=0$. In order to obtain the analytical solution of the problem, the Hankel transform with respect to r is applied on Eqs. (3.5) and (3.6). Multiplying both sides of Eq. (3.5) by $rJ_0(rr_n)$, integrating with respect to r from 0 to R taking into account the condition (3.6) and the equality [31]

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\overline{v}(r,q)J_0(rr_n)dr=-r_n^2\overline{v}_H(r_n,q)+RJ_0(Rr_n)\frac{\partial\overline{v}(R,q)}{\partial r}, \tag{3.8}$$

we obtain

$$\overline{v}_H(r_n,q)=\frac{RA}{\rho}J_0(Rr_n)\frac{\beta+(1-\beta)q}{q^2[(1-\beta)q^2+(r_n^2\beta_2+\beta)q+r_n^2\nu\beta]}. \tag{3.9}$$

To get the more suitable form of the final result, we rewrite above equation as

$$\begin{aligned}\overline{v}_H(r_n,q)=&\frac{RAJ_0(Rr_n)}{{r_n}^2}\frac{[\beta+(1-\beta)q]}{q^2(\beta\mu+\beta_1 q)}-\frac{RAJ_0(Rr_n)}{{r_n}^2}\\&\times\frac{[\beta+(1-\beta)q]^2}{q(\beta\mu+\beta_1 q)[(1-\beta)q^2+(r_n^2\beta_2+\beta)q+\nu r_n^2\beta]}\end{aligned} \tag{3.10}$$

Now we define Inverse Hankel transform as [31]

$$\overline{v}(r,q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{J_0^2(Rr_n)}\overline{v}_H(r_n,q). \tag{3.11}$$

Applying inverse Hankel transform given by Eq. (3.11) to Eq. (3.10), we obtain

$$\begin{aligned}\overline{v}(r,q)=&\frac{r^2A}{2R}\frac{[\beta+(1-\beta)q]}{q^2(\beta\mu+\beta_1 q)}-\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0^2(Rr_n)}\\&\times\frac{[\beta+(1-\beta)q]^2}{q(\beta\mu+\beta_1 q)[(1-\beta)q^2+(r_n^2\beta_2+\beta)q+\nu r_n^2\beta]}.\end{aligned} \tag{3.12}$$

To find the required velocity field, we apply discreet inverse Laplace transform instead of lengthy calculations of residues and contour integrations. By using expansions

$$\frac{1}{q^2(\beta\mu+\beta_1 q)}=\frac{1}{\beta_1}\left[\frac{q^{-2}}{q+\frac{\beta\mu}{\beta_1}}\right], \tag{3.13}$$

$$\begin{aligned}\frac{1}{q[(1-\beta)q^2+(r_n^2\beta_2+\beta)q+\nu r_n^2\beta]}&=\frac{q^{-2}}{[(1-\beta)q+(r_n^2\beta_2+\beta)+\nu r_n^2\beta q^{-1}]}\\&=\sum_{n=0}^{\infty}\frac{(-\nu r_n^2\beta)^k q^{-2-k}}{[(1-\beta)q+(r_n^2\beta_2+\beta)]^{k+1}}=\sum_{n=0}^{\infty}\frac{\beta_3^k q^{-2-k}}{(q+\beta_4)^{k+1}},\end{aligned} \tag{3.14}$$

where $\beta_3^k=\frac{(-\nu r_n^2\beta)^k}{(1-\beta)^{k+1}}$ and $\beta_4=\frac{r_n^2\beta_2+\beta}{1-\beta}$.

By convolution theorem and definition of generalized $G_{\ell,\jmath,\zeta}(\sigma\,,t)$ function which is defined as [27]

$$G_{\ell,\jmath,\zeta}(\sigma\,,t) = L^{-1}\left\{\frac{s^{\jmath}}{(s^{\ell}-\sigma)^{\zeta}}\right\} =$$

$$\sum_{i=0}^{\infty}\frac{\sigma^{i}\,\Gamma(\zeta+i)}{\Gamma(\zeta)\Gamma(i+1)}\frac{t^{(\zeta+i)\ell-\jmath-1}}{\Gamma[(\zeta+i)\ell-\jmath]}\,;\ \mathrm{Re}(\ell\zeta-\jmath)>0,\ \mathrm{Re}(s)>0,\quad \left|\frac{\sigma}{s^{\ell}}\right|<1, \tag{3.15}$$

we find

$$\begin{aligned} v(r,t) &= \frac{Ar^2}{2\beta_1 R}\left[\beta G_{1,-2,1}\left(-\frac{\beta\mu}{\beta_1},t\right)+(1-\beta)G_{1,-1,1}\left(-\frac{\beta\mu}{\beta_1},t\right)\right] \\ &\quad -\frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0^2(Rr_n)}\sum_{k=0}^{\infty}\beta_3^k\int_0^t G_{1,0,1}(-\frac{\beta\mu}{\beta_1},s)\Big[\beta^2G_{1,-2-k,k+1}(-\beta_4,t-s) \\ &\quad +(1-\beta)^2G_{1,-k,k+1}(-\beta_4,t-s)+2\beta(1-\beta)G_{1,-1-k,k+1}(-\beta_4,t-s)\Big]ds. \end{aligned} \tag{3.16}$$

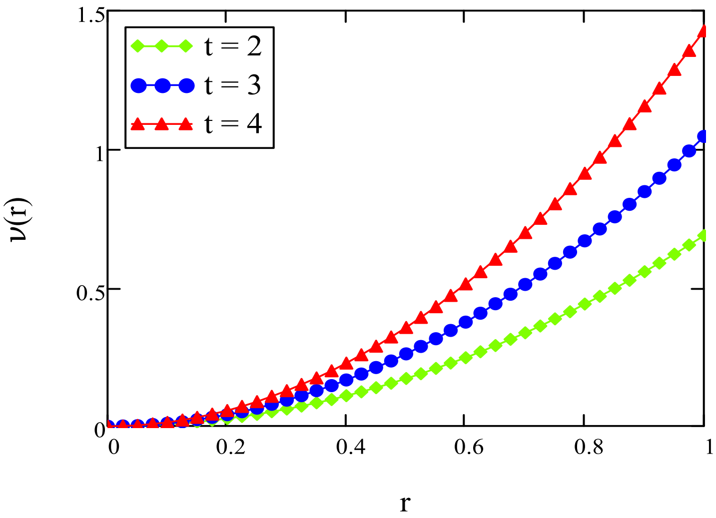

Figure 2. Variation in velocity field $v(r)$ given by Eq. (3.16), for different values of t and $[R=1, A=2, \mu=2.916, \nu=0.00003, \rho=725, \alpha_1=0.002, \beta=0.2]$.

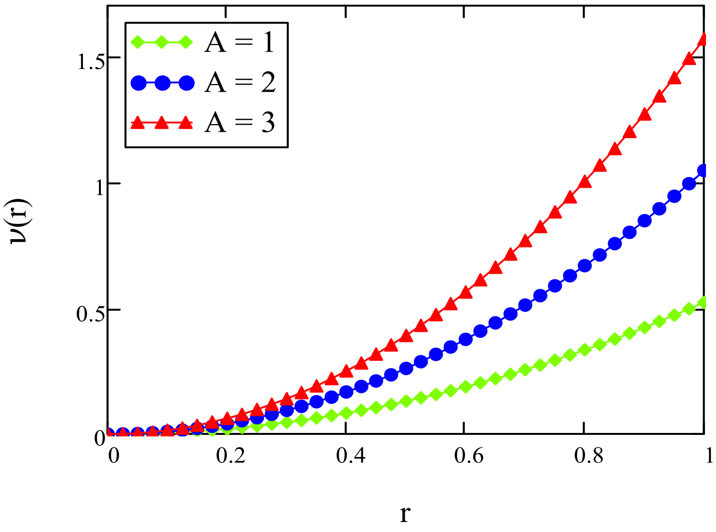

Figure 3. Variation in velocity field $v(r)$ given by Eq. (3.16), for different values of A and $[R = 1, t = 3, \mu = 2.916, \nu = 0.00003, \rho = 725, \alpha_1 = 0.002, \beta = 0.2]$.

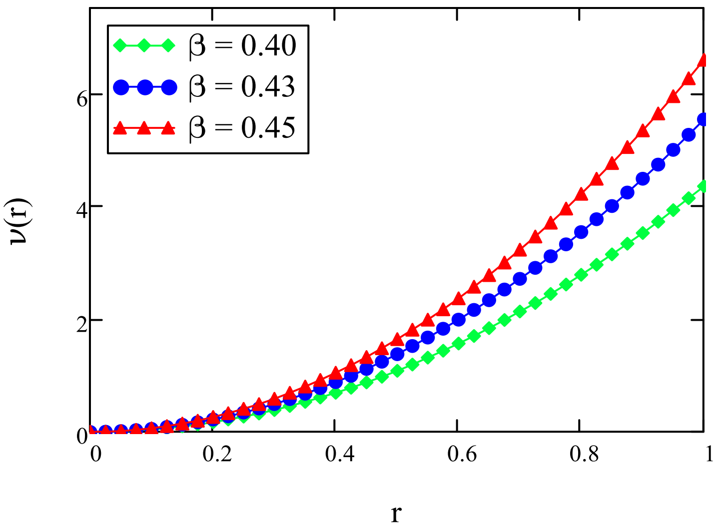

Figure 4. Variation in velocity field $v(r)$ given by Eq. (3.16), for different values of β and $[R = 1, A = 2, \mu = 2.916, \nu = 0.00003, \rho = 725, \alpha_1 = 0.002, t = 3]$.

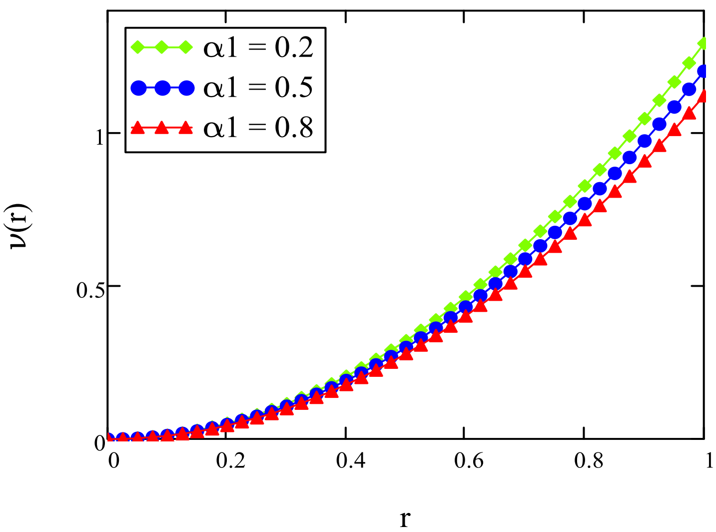

Figure 5. Variation in velocity field $v(r)$ given by Eq. (3.16), for different values of α_1 and $[R = 1, A = 2, \mu = 2.916, \nu = 0.0003, \rho = 725, \beta = 0.2, t = 4]$.

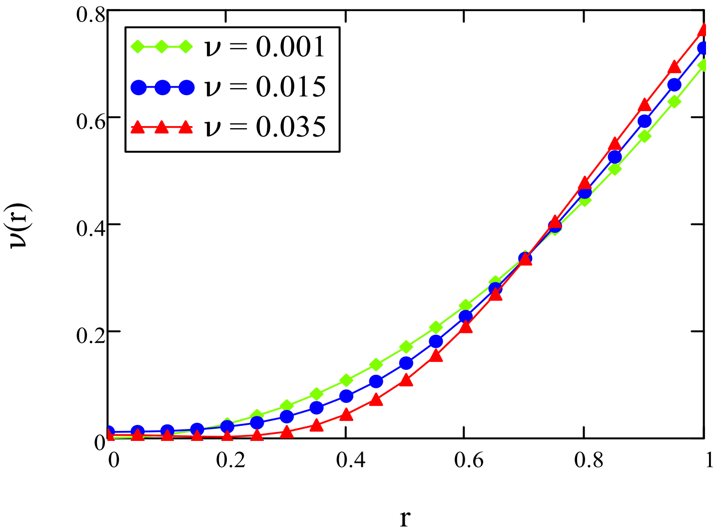

Figure 6. Variation in velocity field $v(r)$ given by Eq. (3.16), for different values of ν and $[R = 1, A = 2, \mu = 2.916, \beta = 0.2, \rho = 725, \alpha_1 = 0.002, t = 3]$.

3.2. **Calculation of the shear stress.** Applying the Laplace transformation to Eq. (2.7), we find that

$$\overline{\tau}(r,q) = \frac{\mu\beta + [\mu(1-\beta)q + \alpha_1 q]}{\beta + (1-\beta)q}\frac{\partial \overline{v}(r,q)}{\partial r}. \tag{3.17}$$

Differentiating Eq. (3.12) with respect to r and using the identity

$$\frac{d}{dr}J_0(rr_n) = -r_n J_1(rr_n), \tag{3.18}$$

we find that

$$\overline{\tau}(r,q) = \frac{Ar}{Rq^2} + \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\frac{[\beta + (1-\beta)q]}{q[(1-\beta)q^2 + (r_n^2\beta_2 + \beta)q + \nu r_n^2\beta]}. \tag{3.19}$$

By applying Discrete inverse Laplace transform to Eq. (3.19), we obtain tangential stress of the form

$$\tau(r,t) = \frac{Art}{R} + \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}$$

$$\times\sum_{k=0}^{\infty}\beta_3^k\Big[\beta G_{1,-2-k,k+1}(-\beta_4,t) + (1-\beta)G_{1,-1-k,k+1}(-\beta_4,t)\Big]. \tag{3.20}$$

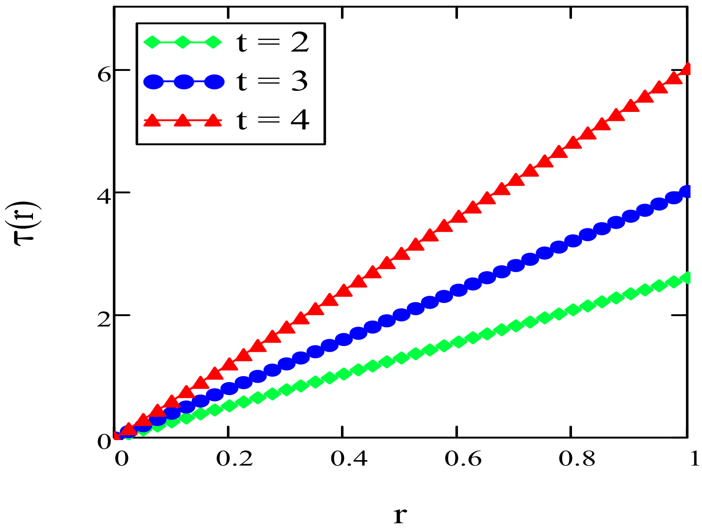

Figure 7. Variation in shear stress $\tau(r)$ given by Eq. (3.20), for different values of t and $[R = 1, A = 2, \mu = 2.916, \nu = 0.00003, \rho = 725, \alpha = 0.002, \beta = 0.2]$.

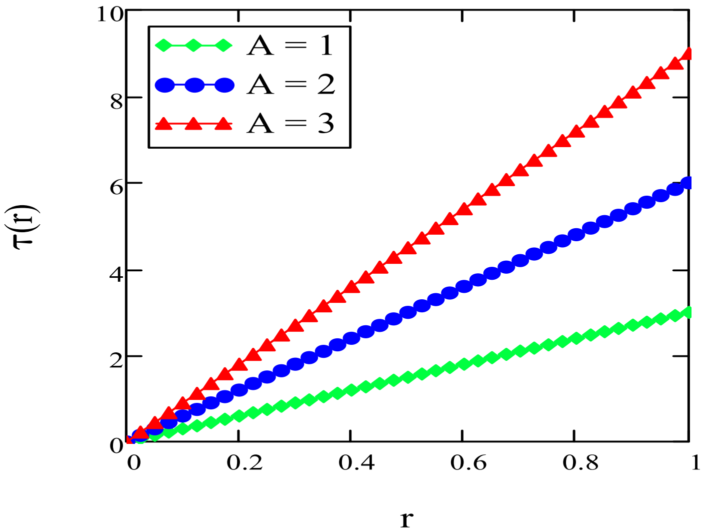

Figure 8. Variation in shear stress $\tau(r)$ given by Eq. (3.20), for different values of A and $[R = 1, t = 3, \mu = 2.916, \nu = 0.00003, \rho = 725, \alpha_1 = 0.002, \beta = 0.2]$.

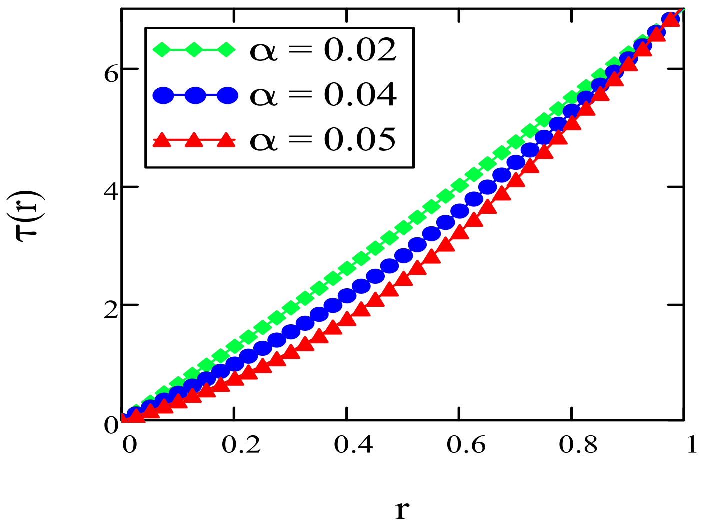

Figure 9. Variation in shear stress $\tau(r)$ given by Eq. (3.20), for different values of α and $[R = 1, A = 2, \mu = 2.916, \nu = 0.0003, \rho = 725, \beta = 0.5, t = 3.5]$.

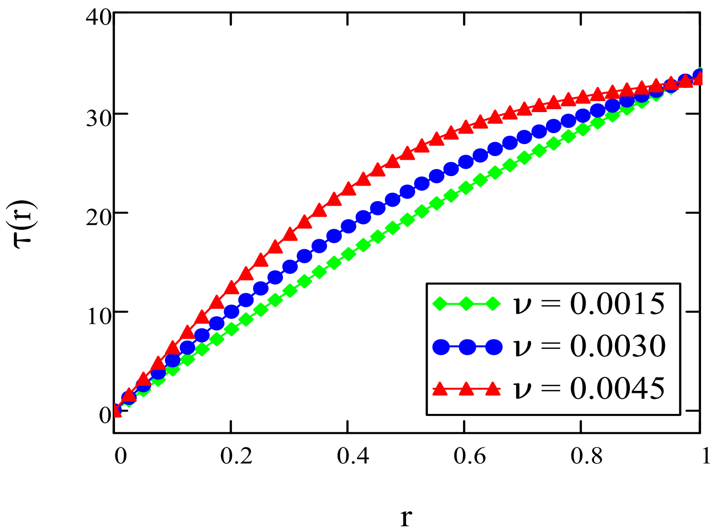

Figure 10. Variation in shear stress $\tau(r)$ given by Eq. (3.20), for different values of ν and $[R = 1, A = 2, \mu = 2.916, \beta = 0.8, \rho = 725, \alpha_1 = 0.002, t = 3.5]$.

4. Limiting cases

4.1. **Ordinary second grade fluid.** If we impose $\beta \to 1$ into Eqs. (3.16) and (3.20), then we get velocity field

$$v(r,t) = \frac{Ar^2}{2\alpha_1 R} G_{1,-2,1}\Big(-\frac{\mu}{\alpha_1}, t\Big) - \frac{2A}{\alpha_1 R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\int_0^t G_{1,0,1}\Big(-\frac{\mu}{\alpha_1}, t\Big)$$

$$\times \sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-k-2,k+1}\Big(-\alpha r_n^2, t-s\Big)ds, \tag{4.1}$$

and associated stress is

$$\tau(r,t) = \frac{Art}{R} + \frac{2A}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}(-\nu r_n^2)^k G_{0,-k-2,k+1}\Big(-\alpha r_n^2, t\Big). \tag{4.2}$$

Using $A1$ from appendix and by simple calculation, Eqs. (4.1) and (4.2) can be written in simplified form as

$$v(r,t) = \frac{Ar^2}{2\mu R}\Big(t - \frac{\alpha_1}{\mu}\Big) - \frac{2A}{\mu\nu R}\sum_{n=1}^{\infty}\Big[1-(1+\alpha r_n^2)\exp(\frac{-\nu r_n^2 t}{1+\alpha r_n^2})\Big]\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}, \tag{4.3}$$

$$\tau(r,t) = \frac{Art}{R} + \frac{2A}{\nu R}\sum_{n=1}^{\infty}\Big[1-\exp(\frac{-\nu r_n^2 t}{1+\alpha r_n^2})\Big]\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}. \tag{4.4}$$

4.2. **Newtonian fluid.** If we impose $\alpha_1 \to 1$ into Eqs. (4.3) and (4.4), then we get velocity field

$$v(r,t) = \frac{Ar^2 t}{2\mu R} - \frac{2A}{\mu\nu R}\sum_{n=1}^{\infty}\Big[1-e^{-\nu r_n^2 t}\Big]\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}, \tag{4.5}$$

and corresponding shear stress is

$$\tau(r,t) = \frac{Art}{R} + \frac{2A}{\nu R}\sum_{n=1}^{\infty}\Big[1-e^{-\nu r_n^2 t}\Big]\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}. \tag{4.6}$$

5. Numerical solution

To solve above problem numerically, we follow the technique used by [32].

5.1. **Calculation of the velocity field.** Eqs. (3.5) and (3.6) can be written as

$$\frac{\partial^2 \overline{v}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}(r,q)}{\partial r} - f(q)\overline{v}(r,q) = 0, \tag{5.1}$$

$$\frac{\partial \overline{v}(r,q)}{\partial r}|_{r=R} = h(q), \tag{5.2}$$

where $f(q) = \frac{q[\beta+(1-\beta)q]}{[\nu\beta+\beta_2 q]}$ and $h(q) = \frac{[\beta+(1-\beta)q]A}{q^2[\beta\mu+\beta_1 q]}$.

By using variable transformation $p = r\sqrt{f(q)}$ in Eq. (5.1), we obtain

$$p^2\frac{d^2\overline{v}}{dp^2} + p\frac{d\overline{v}}{dp} - (p^2 - 0^2) = 0, \tag{5.3}$$

which is the modified form of Bessel equation with general solution of the form

$$\overline{v}(r,q) = C_1 I_0(p) + C_2 K_0(p). \tag{5.4}$$

In above relation C_1, C_2 are constants and $I_0(p), K_0(p)$ represent modified bessel functions of first and second kind respectively. For the finite solution at $p = 0$, C_2 must be zero. Then by using Eq. (5.2) we get

$$C_1 = \frac{h(q)}{\sqrt{f(q)} I_1(R\sqrt{f(q)})}, \tag{5.5}$$

finally we get

$$\overline{v}(r, q) = \frac{h(q) I_0(r\sqrt{f(q)})}{\sqrt{f(q)} I_1(R\sqrt{f(q)})}. \tag{5.6}$$

Inverse laplace obtained by using Stehfest's principle with the help of MATHCAD [32, 33].

5.2. **Calculation of the shear stress.** Eq. (3.17) can be written in the form of

$$\overline{\tau}(r, q) = \frac{\mu\beta + \beta_1 q}{\beta + (1 - \beta)q} \frac{\partial \overline{v}(r, q)}{\partial r}, \tag{5.7}$$

using Eq. (5.6) in Eq. (5.7) we fined

$$\overline{\tau}(r, q) = \frac{\mu\beta + \beta_1 q}{\beta + (1 - \beta)q} h(q) \frac{I_1(r\sqrt{f(q)})}{I_1(R\sqrt{f(q)})}. \tag{5.8}$$

. Inverse laplace obtained by using Stehfest's principle with the help of MATHCAD [32, 33].

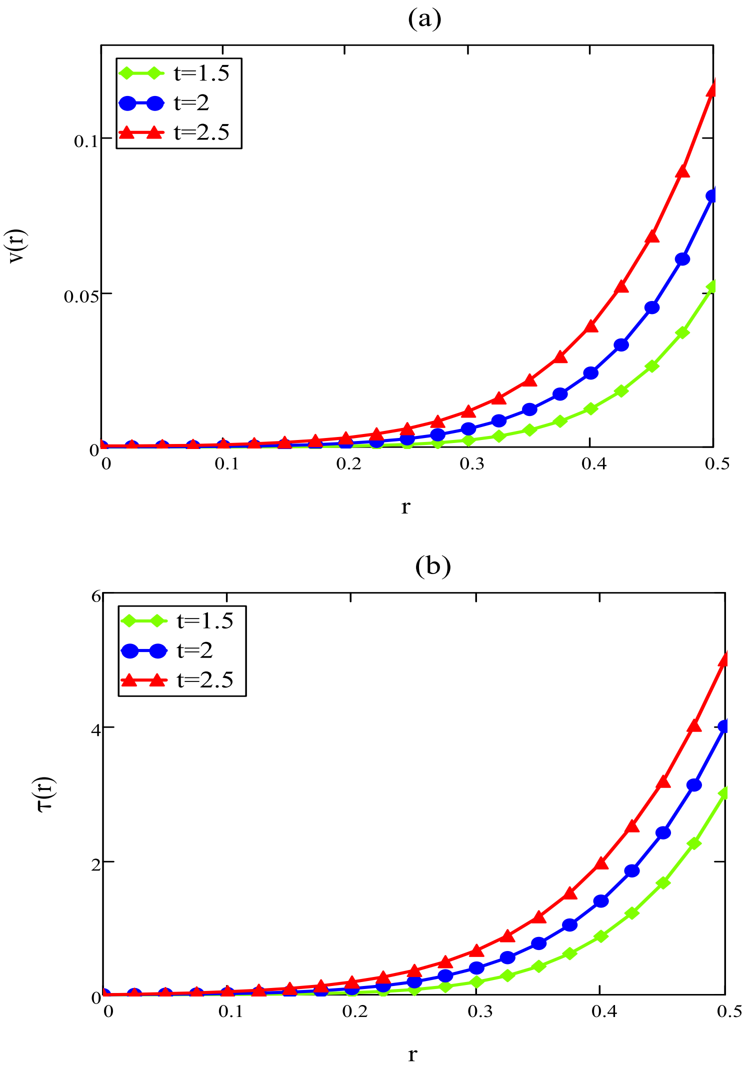

Figure 11. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (5.6) and (5.8), for different values of t and $[R = 0.5, A = 2, \mu = 2.916, \nu = 0.004, \rho = 725, \alpha = 0.002, \beta = 0.2]$.

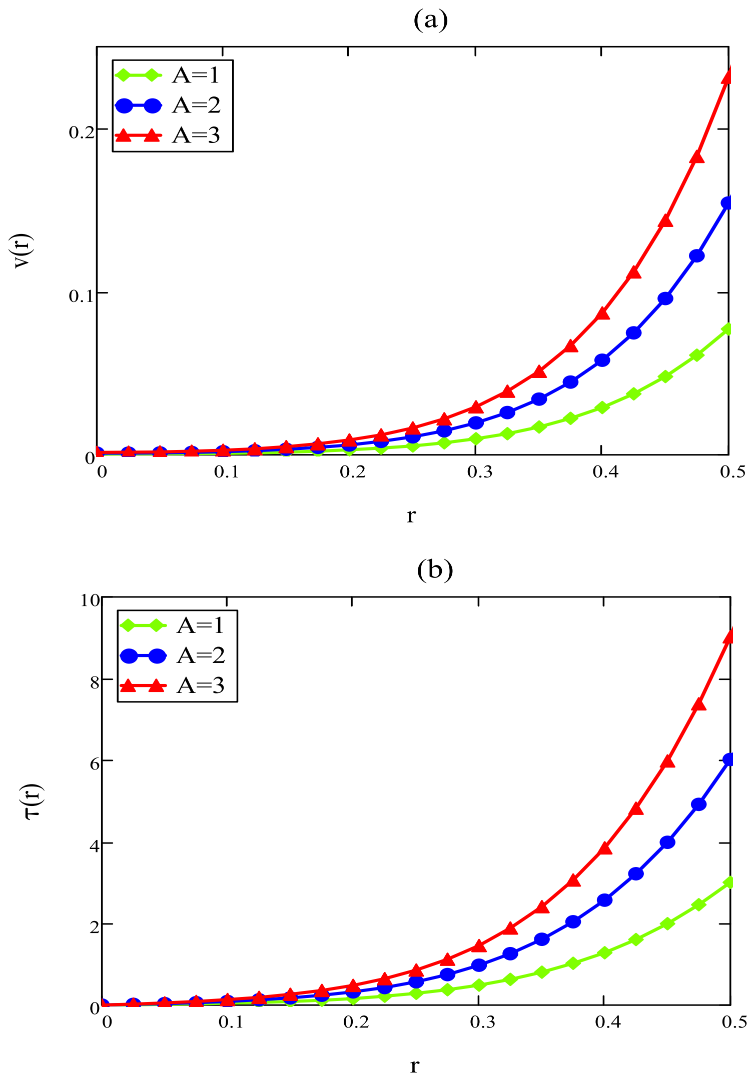

Figure 12. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (5.6) and (5.8), for different values of A and $[R = 0.5, t = 3, \mu = 2.916, \nu = 0.004, \rho = 725, \alpha = 0.002, \beta = 0.2]$.

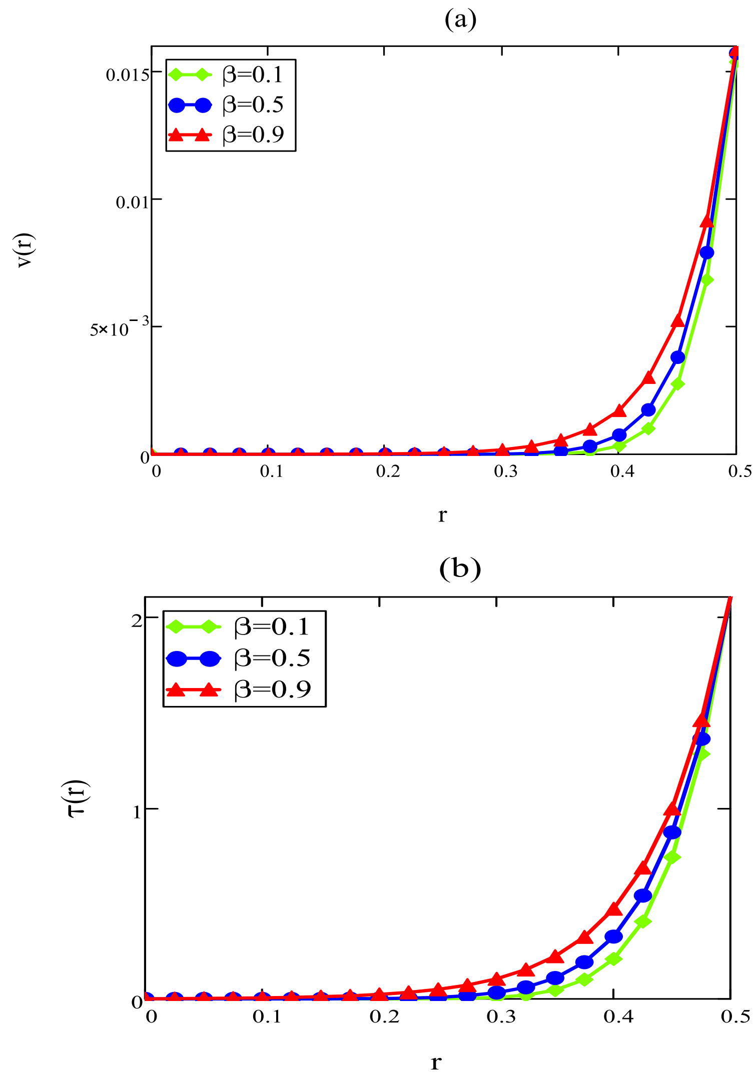

Figure 13. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (5.6) and (5.8), for different values of β and $[R = 0.5, A = 3, \mu = 2.916, \nu = 0.004, t = 0.7, \rho = 725, \alpha = 0.002]$.

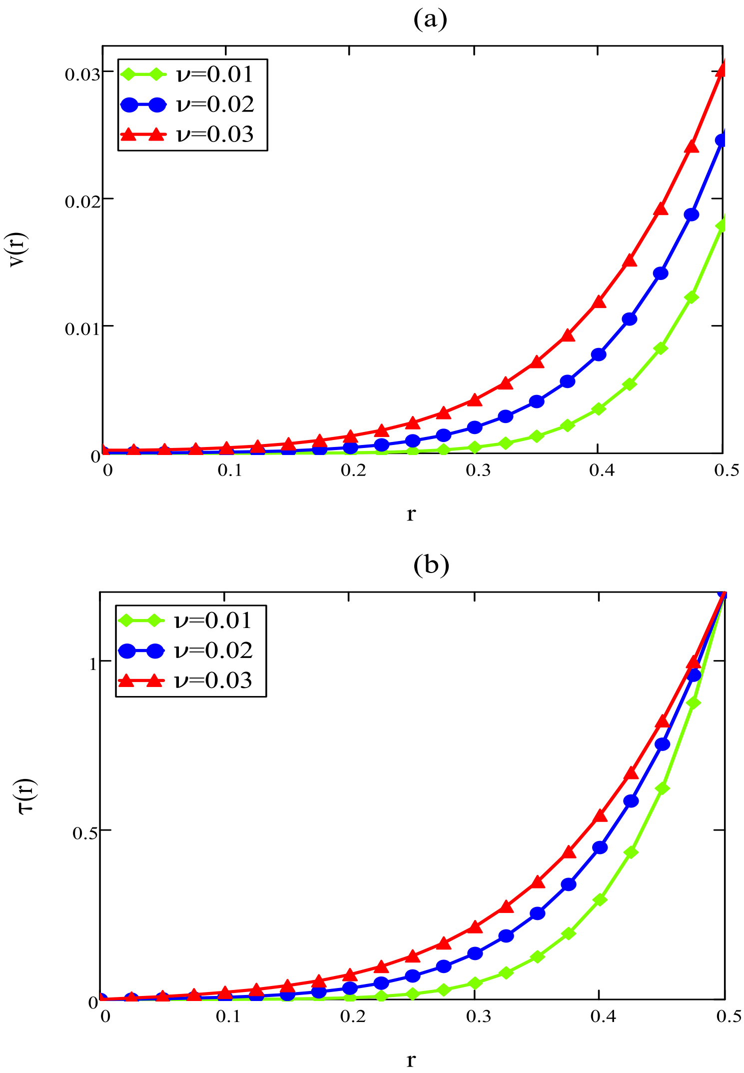

Figure 14. Variation in velocity field $v(r)$ and shear stress $\tau(r)$ given by Eqs. (5.6) and (5.8), for different values of ν and $[R = 0.5, A = 2, \mu = 2.916, t = 0.6, \rho = 725, \alpha = 0.002, \beta = 0.2]$.

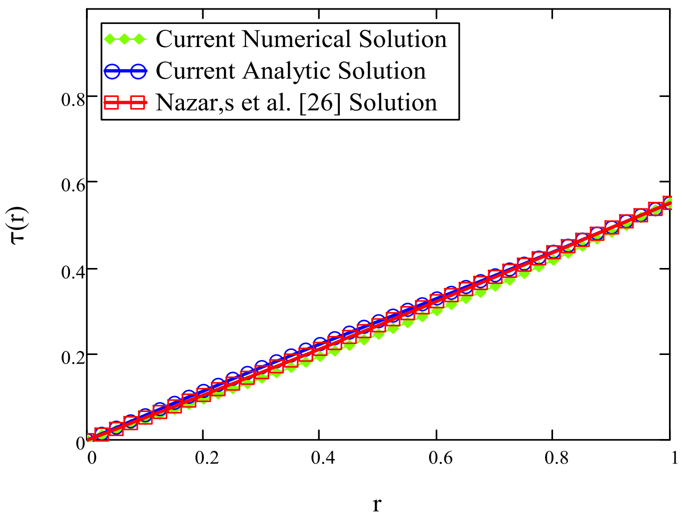

Figure 15. Comparison of shear stress $\tau(r)$ at $[R = 1, A = 0.5, \mu = 3, \nu = 0.0003, \rho = 10000, \alpha_1 = 0.8, \beta = 0.002]$.

6. Numerical results and discussion

This research article is written because we want to know the behavior of circular cylinder with unsteady flow of second grade fluid that is depending upon stress. The new definition of Caputo and Fabrizo for fractional derivative is used to define the governing equations of stress dependent second grade fluid. The finite Hankel and Laplace transform is used to find out the exact solution of the under consideration problem where as Stehfest's algorithm is used to developed the numerical solution of our problem. We are interested to show the behavior of different physical parameters graphically of the results given in Eqs. (3.16) and (3.20) for exact solution of velocity and shear stress. Graphical representation of different physical parameters for numerical solutions are also given in this article. The velocity and shear stress is an increasing function in exact and numerical solutions with respect to time and constant A in Figs. 2, 3, 7 and 8 for exact solutions and in Figs. 11(a,b) and 12(a,b) for numerical solutions. It is noted that exact solution of velocity field is an increasing function of fractional parameter β,which clear from Fig. 4. Figs. 13(a) and 13(b) represent numerical solutions for velocity and stress fields both are also increasing with respect to β. Figs. 5 and 9 indicate that exact solutions for velocity and stress fields are decreasing functions of α_1 and α respectively. Analytical and numerical solutions for velocity field and shear stress both are increasing functions of kinematic viscosity ν with respect to $'r'$, it can be seen from Figs. 6, 10, and 14(a,b). The comparison of solutions that obtained from two different methods analytically [31] and numerically by Stehfest's algorithm with the help of MATHCAD software, is given by table1 where as graphical comparison is available in Fig 15. The units of the material constants in Figs. 2-15 are SI.

Comparison of shear stress with existing result				
Nazar et al. [29] result	Current analytic	Current numerical	Error with analytic	Error with numerical
0.0000	0.0000	0.0000	0.0000	00.0000
0.0261	0.0289	0.0236	-0.0028	0.0025
0.0522	0.0577	0.0473	-0.0055	0.0049
0.0785	0.0862	0.0711	-0.0077	0.0074
0.1048	0.1144	0.0951	-0.0096	0.0097
0.1313	0.1422	0.1193	-0.0109	0.012
0.1579	0.1696	0.1439	-0.0117	0.014
0.1848	0.1965	0.1688	-0.0117	0.016
0.2119	0.2232	0.1939	-0.0113	0.018
0.2392	0.2496	0.2199	-0.0104	0.0193
0.2667	0.276	0.2461	-0.0093	0.0206
0.2944	0.3023	0.2729	-0.0079	0.0215
0.3224	0.3287	0.3006	-0.0063	0.0218
0.3506	0.3554	0.3287	-0.0048	0.0219
0.3789	0.3824	0.3579	-0.0035	0.021
0.4073	0.4097	0.3872	-0.0024	0.0201
0.4359	0.4373	0.4179	-0.0014	0.018
0.4646	0.4652	0.4502	-0.0006	0.0144
0.4932	0.4934	0.4825	-0.0002	0.0107
0.5219	0.5218	0.5164	0.0001	0.0055
0.5505	0.5501	0.5505	0.0004	0.0009

7. Conclusion

In this paper, the unsteady flow of incompressible second grade fluid governed by the fractional differential equations with new definition of Caputo and Febrizio are studied. The domain of the flow is the inner of circular pipe and we are observed the stress force on the cylinder surface, The Laplace and Hankel transformations are used for exact solutions of velocity and adequate shear stress. The Stehfest's method used to find the semi-numerical solution in terms of modified Bessel's functions, MATHCAD software used to calculate the fluid velocity and shear stress numerically. The result categorically have the following aspects.

- The velocity of the fluid increases for this model as fluid becomes more thick.
- The physical parameters t, A, β, ν have increasing behavior for velocity and stress functions.
- Velocity field and shear stress both are decreasing functions of α_1 and α respectively.
- Our obtained solutions from new fractional derivative definition given in [29], derived analytically and numerically by Stehfest's algorithm both are equivalent to [31].

Appendix

$$\sum_{k=0}^{\infty}(-\nu r_n^2)^k\, G_{0,\,-k-p-1,\,k+1}(-\alpha r_n^2, t) = \left(-\frac{1}{\nu}\right)^p \frac{(1+\alpha r_n^2)^{p-1}}{r_n^{2p}} \times \left[\exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right) - \sum_{j=0}^{p-1}\frac{1}{j\,!}\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right)^j\right] \quad p = 1, 2, 3, ..., \tag{A$_1$}$$

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