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ORDER-THEORETIC METRICAL COINCIDENCE THEOREMS INVOLVING (ϕ, ψ)-CONTRACTIONS

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Abstract. In this paper, we prove some results on existence and uniqueness of a coincidence point for (ϕ, ψ)-contractive mappings using order-theoretic analogues of involved metric notions. Our results generalize and extend some well-known results existing in literature.

1. Introduction

The first important result in metric fixed point theory is Banach Contraction Principle which was given by Banach in 1922 [9]. This principle has been generalized by many authors, either by changing the contractive condition, or by changing the underlying space, see [1, 5, 6, 7, 8, 14, 15, 25, 24, 27, 28, 29, 34, 35, 37]. Khan et al. [23] used the concept of altering distance functions to generalize Banach contraction principle. Afterward, Dutta et al. [13] generalized Banach Contraction Principle by using a pair of altering distance functions. In this direction, the concept of (ϕ, ψ)-contractions was further refined by Choudhury et al. [11] by deleting the condition of monotonicity of ϕ.
On the other hand, Ran and Reurings [33] obtained an important generalization of Banach Contraction Principle in the setting of ordered metric spaces, which is further refined by Nieto and Lopez [30]. Recently, Alam et al. [2, 3] introduced notions of O-completeness, O-continuity, (g, O)-continuity. Utilizing these notions, they proved some coincidence theorems by deleting completeness condition of whole space, instead used completeness of subspace in setting of ordered metric spaces. The aim of this paper is to prove some coincidence point theorems under (ϕ, ψ)-contractions in ordered metric spaces. This paper generalizes and extends the results of Harjani and Sadarangani [17].

2. Preliminaries

In this section, to make our exposition self-contained, we recall some basic definitions, relevant notions and auxiliary results. Throughout this paper, $\mathbb{N}$ stands for set of natural numbers and $\mathbb{N}_0$ for the set of whole numbers ($i.e., \mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

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Definition 2.1. [26]. *Let $(X, \preceq))$ be an ordered set. $\succeq$ denotes the dual order of $\preceq$ (i.e., $x \preceq y \Rightarrow y \succeq x$).*

Definition 2.2. [26]. *Two elements x and y of an ordered set $(X, \preceq)$ are said to be comparable if either $x \preceq y$ or $x \succeq y$. Two comparable elements x and y are denoted by $x \prec\succ y$.*

Definition 2.3. [26]. *A subset E of an ordered set $(X, \preceq)$ is called totally (or linearly ordered) if every pair of elements of E is comparable, i.e.,*

$$x \prec\succ y \quad \textit{for all } x, y \in E.$$

Definition 2.4. [38]. *A sequence $\{x_n\}$ in an ordered set $(X, \preceq)$ is said to be*
(i) *increasing or ascending if for any $m, n \in \mathbb{N}_0$ with $m \leq n \Rightarrow x_m \preceq x_n$;*
(ii) *decreasing or descending if for any $m, n \in \mathbb{N}_0$ with $m \leq n \Rightarrow x_n \preceq x_m$;*
(iii) *monotone if either it is increasing or decreasing;*
(iv) *bounded above if there exists an element $u \in X$ such that $x_n \preceq u$ for all $n \in \mathbb{N}_0$. Here, u is called an upper bound of $\{x_n\}$;*
(v) *bounded below if there exists an element $u \in X$ such that $x_n \succeq u$ for all $n \in \mathbb{N}_0$. Here u is called an lower bound of $\{x_n\}$.*

Definition 2.5. [12]. *Let f and g be two self-mappings on an ordered set $(X, \preceq)$. Then f is said to be g-increasing if for any $x, y \in X$,*

$$g(x) \preceq g(y) \Rightarrow f(x) \preceq f(y).$$

Definition 2.6. [16],[19],[20]. *Let f and g be two self-mappings on a non-empty set X. An element $x \in X$ is called a coincidence point of f and g if $f(x) = g(x)$.*

Definition 2.7. *(*[36],[22]*). Let f and g be two self-mappings on a metric space (X, d). Then*
(i) *f and g are said to be compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = z$ implies $\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$.*
(ii) *f is said to be g-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,*

$$g(x_n) \to g(x) \Rightarrow f(x_n) \to f(x).$$

Moreover, f is called g-continuous if it is g-continuous at each point.

Definition 2.8. [31]. *A triplet $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, if the metric space (X, d) is complete, then $(X, d, \preceq)$ is called an ordered complete metric space.*

Notation 1. [3]. *Let $(X, d, \preceq)$ be an ordered metric space and $\{x_n\}$ be a sequence in X. (i) If $\{x_n\}$ is increasing and $x_n \xrightarrow{d} x$, then it is denoted by $x_n \uparrow x$;*
(ii) if $\{x_n\}$ is decreasing and $x_n \xrightarrow{d} x$, then it is denoted by $x_n \downarrow x$;
(iii) if $\{x_n\}$ is monotone and $x_n \xrightarrow{d} x$, then it is denoted by $x_n \uparrow\downarrow x$.

Definition 2.9. [2]. *An ordered metric space $(X, d, \preceq)$ is called*
(i) $\overline{O}$-complete if every increasing cauchy sequence in X converges;
(ii) $\underline{O}$-complete if every decreasing cauchy sequence in X converges;
(iii) O-complete if every monotone cauchy sequence in X converges.

Definition 2.10. [3]. *Let $(X,d,\preceq)$ be an ordered metric space. A subset E of X is called*
(i) $\overline{O}$-closed if for any sequence $\{x_n\}\subset E$,

$$x_n\uparrow x\Rightarrow x\in E.$$

(ii) $\underline{O}$-closed if for any sequence $\{x_n\}\subset E$,

$$x_n\downarrow x\Rightarrow x\in E.$$

(iii) O-closed if for any sequence $\{x_n\}\subset E$,

$$x_n\uparrow\downarrow x\Rightarrow x\in E.$$

Proposition 2.11. [3]. *(i) An $\overline{O}$-complete subspace of an ordered metric space is $\overline{O}$-closed.*
(ii) An $\underline{O}$-complete subspace of an ordered metric space is $\underline{O}$-closed.
(iii) An O-complete subspace of an ordered metric space is O-closed.

Proposition 2.12. [3]. *(i) An $\overline{O}$-closed subspace of an $\overline{O}$-complete ordered metric space is $\overline{O}$-complete.*
(ii) An $\underline{O}$-closed subspace of an $\underline{O}$-complete ordered metric space is $\underline{O}$-complete.
(iii) An O-closed subspace of an O-complete ordered metric space is O-complete.

Definition 2.13. [2]. *Let $(X,d,\preceq)$ be an ordered metric space, $f:X\to X$ be a mapping and $x\in X$. Then f is called*
(i) $\overline{O}$-continuous at x if for any sequence $\{x_n\}\subset X$,

$$x_n\uparrow x\Rightarrow f(x_n)\xrightarrow{d}f(x).$$

(ii) $\underline{O}$-continuous at x if for any sequence $\{x_n\}\subset X$,

$$x_n\downarrow x\Rightarrow f(x_n)\xrightarrow{d}f(x).$$

(iii) O–continuous at x if for any sequence $\{x_n\}\subset X$,

$$x_n\uparrow\downarrow x\Rightarrow f(x_n)\xrightarrow{d}f(x).$$

Definition 2.14. [2]. *Let $(X,d,\preceq)$ be an ordered metric space, f and g be two self-mappings on X and $x\in X$. Then f is called*
(i) (g,$\overline{O}$)-continuous at x if for any sequence $\{x_n\}\subset X$,

$$g(x_n)\uparrow g(x)\Rightarrow f(x_n)\xrightarrow{d}f(x);$$

(ii) (g,$\underline{O}$)-continuous at x if for any sequence $\{x_n\}\subset X$;

$$g(x_n)\downarrow g(x)\Rightarrow f(x_n)\xrightarrow{d}f(x).$$

(iii) (g,O)-continuous at x if for any sequence $\{x_n\}\subset X$,

$$g(x_n)\uparrow\downarrow g(x)\Rightarrow f(x_n)\xrightarrow{d}f(x).$$

Definition 2.15. [2]. *Let $(X,d,\preceq)$ be an ordered metric space and f and g be two self-mappings on X. One says that f and g are*
(i) $\overline{O}$-compatible if for any sequence $\{x_n\}\subset X$ and for any $z\in X$,

$$g(x_n)\uparrow z, f(x_n)\uparrow z\Rightarrow\lim_{n\to\infty}d(gfx_n,fgx_n)=0.$$

(ii) $\underline{O}$-compatible if for any sequence $\{x_n\}\subset X$ and for any $z\in X$,

$$g(x_n)\downarrow z, f(x_n)\downarrow z\Rightarrow\lim_{n\to\infty}d(gfx_n,fgx_n)=0.$$

(iii) O-compatible if for any sequence $\{x_n\} \subset X$ *and for any* $z \in X$,

$$g(x_n) \uparrow\downarrow z, f(x_n) \uparrow\downarrow z \Rightarrow \lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Definition 2.16. [2]. *Let* $(X, d, \preceq)$ *be an ordered space. One says that*
(i) $(X, d, \preceq)$ *has ICC (increasing-convergence-comparable) property if every increasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$*; that is*

$$x_n \uparrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall\ k \in \mathbb{N}_0.$$

(ii) $(X, d, \preceq)$ *has DCC (decreasing-convergence-comparable) property if every decreasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$*; that is*

$$x_n \downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall\ k \in \mathbb{N}_0.$$

(iii) $(X, d, \preceq)$ *has MCC (monotone-convergence-comparable) property if every monotone convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$*, that is,*

$$x_n \uparrow\downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall\ k \in \mathbb{N}_0.$$

Definition 2.17. [2]. *Let* $(X, d, \preceq)$ *be an ordered metric space and* g *a self-mapping on* X*. One says that*
(i) $(X, d, \preceq)$ *has g-ICC property if every increasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{gx_{n_k}\}$ *is comparable with g-image of the limit of* $\{x_n\}$*, i.e.,*

$$x_n \uparrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\text{with } g(x_{n_k}) \prec\succ g(x) \ \forall\ k \in \mathbb{N}_0.$$

(ii) $(X, d, \preceq)$ *has g-DCC property if every decreasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{gx_{n_k}\}$ *is comparable with g-image of the limit of* $\{x_n\}$*, i.e.,*

$$x_n \downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\text{with } g(x_{n_k}) \prec\succ g(x) \ \forall\ k \in \mathbb{N}_0.$$

(iii) $(X, d, \preceq)$ *has g-MCC property if every monotone convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that every term of* $\{gx_{n_k}\}$ *is comparable with g-image of the limit of* $\{x_n\}$*, i.e.,*

$$x_n \uparrow\downarrow x \Rightarrow \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\text{with } g(x_{n_k}) \prec\succ g(x) \ \forall\ k \in \mathbb{N}_0.$$

Observe that under the restriction $g = I$, the identity mapping on X, Definition 2.17 is reduced to Definition 2.16.

Lemma 2.18. [21],[32],[10]. *Let* (X, d) *be a metric space and* $\{x_n\}$ *be a sequence in* X*. If* $\{x_n\}$ *is not Cauchy, then there exist* $\varepsilon > 0$ *and two subsequences* $\{x_{m_k}\}$ *and* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that*
(i) $k \leq m_k < n_k \forall\ k \in N$;
(ii) $d(x_{m_k}, x_{n_k}) \geq \varepsilon$;
(iii) $d(x_{m_k}, x_{p_k}) < \varepsilon \ \forall\ p_k \in \{m_k + 1, m_k + 2, ..., n_k - 2, n_k - 1\}$.
Moreover, suppose that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*. Then*
(iv) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$;
(v) $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon$.

Lemma 2.19. [4]. *Let f and g be two self-mappings defined on an ordered set $(X, \preceq)$. If f is g-increasing and $g(x) = g(y)$, then $f(x) = f(y)$.*

Lemma 2.20. [18]. *Let X be a non-empty set and g be a self-mapping on X. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Definition 2.21. [23]. *A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:*
(i) ψ is monotone increasing and continuous;
(ii) $\psi(t) = 0$ iff $t = 0$.

Note. In the main result, we denote by $\mathbb{N}^0$ the set $\mathbb{N} \cup \{0\}$.

3. Main results

3.1. Existence of a coincidence point.

Definition 3.1. *A function $\phi : [0, \infty) \to [0, \infty)$ is called an ultra-altering distance function, if the following properties are satisfied:*
(i) ϕ is continuous;
(ii) $\phi(t) = 0$ iff $t = 0$.

Example 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x + 2 \sin x$. Clearly, f is continuous and $f(t) = 0$ iff $t = 0$. We show that f is not monotonic increasing. Take $x_1 = \frac{3\pi}{4}$ and $x_2 = \frac{7\pi}{8}$. Note that that $x_1 < x_2$, but $f(x_1) > f(x_2)$. That is, f is an ultra-altering distance function.*

Theorem 3.3. *Let $(X, d, \preceq)$ be an ordered metric space, Y be an $\overline{O}$-complete subspace of X and f, $g : X \to X$ be self-mappings on X. Suppose there exist an ultra-altering distance function ϕ and an altering distance function ψ such that*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)) \ \forall \ x, y \in X \text{ with } gx \prec\succ gy. \tag{3.1}$$

Suppose also that the following conditions hold:
(a) $f(X) \subseteq g(X) \cap Y$;
(b) f is g- increasing;
(c) there exists exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$;
In addition to above, if either $(d_1) - (d_3)$ or $(d'_1) - (d'_2)$ holds, where
(d_1) f and g are $\overline{O}$-compatible;
(d_2) g is $\overline{O}$-continuous;
(d_3) either f is $\overline{O}$-continuous or $(Y, d, \preceq)$ has g-ICC property
and
(d'_1) $Y \subseteq g(X)$;
(d'_2) either f is $(g,\overline{O})$-continuous, or f and g are continuous, or $(Y, d, \preceq)$ has ICC-property.
Then f and g have a coincidence point.

Proof. By assumption (c), there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$. If $g(x_0) = f(x_0)$, then x_0 is a coincidence point of f and g, and so the proof is completed. Otherwise, by using assumption (a), $f(X) \subset g(X)$. We may construct a sequence $\{x_n\} \subset X$ of joint iterations of f and g based at point x_0, *i.e.*,

$$g(x_{n+1}) = f(x_n) \quad \text{for all } n \in \mathbb{N}_0. \tag{3.2}$$

Now, we show that $\{g(x_n)\}$ is an increasing sequence, *i.e.*,

$$g(x_n) \preceq g(x_{n+1}) \quad \text{for all } n \in \mathbb{N}_0. \tag{3.3}$$

We will prove it by mathematical induction. Using (3.2) with $n = 0$, by assumption (c), we have

$$g(x_0) \preceq f(x_0) = g(x_1),$$

that is, (3.3) holds for $n = 0$. We assume that (3.3) holds for $n = r > 0$, *i.e.*, $g(x_r) \preceq g(x_{r+1})$. Now, using (3.2) and assumption (b), we get

$$g(x_{r+1}) = f(x_r) \preceq f(x_{r+1}) = g(x_{r+2}),$$

which shows that (3.3) is also true for $n = r + 1$. By principle of mathematical induction, (3.3) holds for all n, *i.e.*, $g(x_n) \preceq g(x_{n+1})$ for all $n \in \mathbb{N}_0$. In view of (3.2) and (3.3), $\{fx_n\}$ is also an increasing sequence, *i.e.*,

$$f(x_n) \preceq f(x_{n+1}) \quad \text{for all } n \in \mathbb{N}_0. \tag{3.4}$$

If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}_0$, then using (3.2), we have $g(x_{n_0}) = f(x_{n_0})$, so x_{n_0} is a coincidence point of f and g.

On the other hand, $g(x_n) \neq g(x_{n+1})$ for all $n \in \mathbb{N}_0$. Now, we define a sequence $\{R_n\}_{n=0}^{\infty} \subset (0, \infty)$, where

$$R_n = d(gx_n, gx_{n+1}).$$

By (3.3), we have $g(x_{n-1}) \preceq g(x_n)$. From (3.1) and (3.2), we have

$$\begin{array}{rcl} \psi(d(gx_{n+1}, gx_n)) & = & \psi(d(fx_n, fx_{n-1})) \\ . & \leq & \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})), \end{array}$$

that is,

$$\psi(R_n) \leq \psi(R_{n-1}) - \phi(R_{n-1}). \tag{3.5}$$

Using properties of ϕ, we have for all $n \geq 1$

$$\psi(R_n) \leq \psi(R_{n-1}).$$

Since ψ is monotone increasing, we get $R_n \leq R_{n-1}$ for all n. Therefore, $\{R_n\}$ is a monotonic decreasing sequence of non-negative real numbers. Thus

$$\exists\ r \geq 0\ s.t\ R_n \to r\ as\ n \to \infty. \tag{3.6}$$

Taking the limit as $n \to \infty$ in (3.5) and using (3.6) together with the continuities of ψ and ϕ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence

$$R_n \to 0\ as\ n \to \infty, \tag{3.7}$$

i.e., $\lim\limits_{n\to\infty} d(gx_n, gx_{n+1}) = 0$. Next, we show that $\{gx_n\}$ is a Cauchy sequence.

Suppose $\{gx_n\}$ is not a Cauchy sequence. By Lemma 2.18, there exist $\varepsilon > 0$ and subsequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that for $n_k \geq m_k \geq k$,

$$D_k = d(gx_{m_k}, gx_{n_k}) \geq \varepsilon, \tag{3.8}$$

and

$$d(gx_{m_k}, gx_{n_k-1}) < \varepsilon. \tag{3.9}$$

Now, using (3.8),

$$\begin{array}{rcl} \varepsilon \leq D_k & = & d(gx_{m_k}, gx_{n_k}) \\ & \leq & d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}), \end{array}$$

In view of (3.9) and definition of R_n, we get

$$\begin{aligned}\varepsilon \leq D_k &= d(gx_{m_k}, gx_{n_k}) \\ &\leq \varepsilon + R_{n_k-1}.\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.7), we have

$$\lim_{k\to\infty} D_k = d(gx_{m_k}, gx_{n_k}) = \varepsilon. \tag{3.10}$$

Again,

$$\begin{aligned}D_{k+1} &= d(gx_{m_k+1}, gx_{n_k+1}) \\ &\leq d(gx_{m_k+1}, gx_{m_k}) + d(gx_{m_k}, gx_{n_k}) + d(gx_{n_k}, gx_{n_k+1}) \\ &= R_{m_k} + D_k + R_{n_k}.\end{aligned}$$

Letting $k \to \infty$ and using (3.7) and (3.10), we have

$$\lim_{k\to\infty} D_{k+1} = d(gx_{m_k+1}, gx_{n_k+1}) = \varepsilon. \tag{3.11}$$

Since $n_k \geq m_k$, $g(x_{n_k}) \succeq g(x_{m_k})$. Then from (3.1) and (3.2), we have

$$\begin{aligned}\psi(d(gx_{n_k+1}, gx_{m_k+1})) &= \psi(d(fx_{n_k}, fx_{m_k})) \\ &\leq \psi(d(gx_{n_k}, gx_{m_k})) - \phi(d(gx_{n_k}, gx_{m_k})),\end{aligned}$$

that is,

$$\psi(D_{k+1}) \leq \psi(D_k) - \phi(D_k). \tag{3.12}$$

Letting $k \to \infty$ in (3.12) and using (3.10), (3.11) and continuities of ϕ and ψ, we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon).$$

Thus, $\phi(\varepsilon) = 0$. So $\varepsilon = 0$, which is a contradiction. Therefore, $\{gx_n\}$ is a Cauchy sequence.

By (3.2) and $f(X) \subseteq Y$, we can say that $\{gx_n\}$ ia an increasing Cauchy sequence in Y, which is $\overline{O}$-complete, so there exists $z \in Y$ such that $\lim_{n\to\infty} g(x_n) = z$. Due to (3.3),

$$g(x_n) \uparrow z. \tag{3.13}$$

On using (3.2), (3.4) and (3.13), we obtain

$$f(x_n) \uparrow z. \tag{3.14}$$

Now, using assumptions $(d_1) - (d_3)$ and $(d_1') - (d_2')$, we accomplish the rest of the proof. Assume that $(d_1) - (d_3)$ holds. Using assumption (d_2) in (3.13) and (3.14), we have

$$\lim_{n\to\infty} g(gx_n) = g(z), \tag{3.15}$$

$$\lim_{n\to\infty} g(fx_n) = g(z). \tag{3.16}$$

On using (3.13), (3.14) and assumption (d_1), we have

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0. \tag{3.17}$$

Now, we show that z is a coincidence point of f and g. To accomplish this, we use assumption (d_3). Suppose that f is $\overline{O}$-continuous. Using (3.13) and $\overline{O}$-continuity of f,

$$\lim_{n\to\infty} f(gx_n) = f(z). \tag{3.18}$$

By (3.16), (3.17) and (3.18), we obtain

$$\begin{aligned} d(gz, fz) &= d(\lim_{n\to\infty} gfx_n, \lim_{n\to\infty} fgx_n) \\ &= \lim_{n\to\infty} d(gfx_n, fgx_n) \\ &= 0. \end{aligned}$$

This implies that $g(z) = f(z)$. Alternatively, suppose that $(Y, d, \preceq)$ has g-ICC property. Due to (3.13), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k}) \prec\succ g(z) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.19}$$

We have to show that

$$d(fgx_{n_k}, fz) \leq d(ggx_{n_k}, gz) \quad \text{for all } k \in \mathbb{N}.$$

Using (3.19) and (3.1),

$$\psi(d(fgx_{n_k}, fz)) \leq \psi(d(ggx_{n_k}, gz)) - \phi(d(ggx_{n_k}, gz)). \tag{3.20}$$

As $\phi(t) \geq 0$, we have from (3.20),

$$\psi(d(fgx_{n_k}, fz)) \leq \psi(d(ggx_{n_k}, gz)).$$

Again, by using (3.19) and monotonicity of ψ, we get

$$d(fgx_{nk}, fz) \leq d(ggx_{nk}, gz) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.21}$$

Now, by triangular inequality, (3.15), (3.16), (3.17) and (3.21), we get

$$\begin{aligned} d(gz, fz) &= d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(fgx_{n_k}, fz) \\ &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(ggx_{n_k}, gz) \\ &\to 0 \; as \; k \to \infty. \end{aligned}$$

So that $g(z) = f(z)$. Thus $z \in X$ is a coincidence point of f and g.
Now, assume that (d_1') and (d_2') holds. By assumption (d_1'), *i.e.*, $Y \subseteq g(X)$, we can find some $u \in X$ such that $z = g(u)$. Hence (3.13) and (3.14) are reduced to

$$g(x_n) \uparrow g(u) \tag{3.22}$$

and

$$f(x_n) \uparrow g(u), \tag{3.23}$$

respectively. Now, we show that u is a coincidence point of f and g. To accomplish this, we use assumption (d_2'). Firstly, suppose that f is $(g,\overline{O})$-continuous. By (3.22),

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{3.24}$$

Using (3.23) and (3.24), we get $g(u) = f(u)$.
Secondly, suppose that f and g are continuous. By Lemma 2.20, there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one. Without loss of generality, we consider $E \subseteq X$ such that $u \in E$. Now, we define $T : g(E) \to g(X)$ by

$$T(ge) = f(e) \; \forall \; g(e) \in g(E) \quad \text{where } e \in E. \tag{3.25}$$

As $g : E \to X$ is one-to-one and $f(X) \subseteq g(X)$, so T is well defined. Also, as f and g are continuous, it follows that T is continuous. Since $\{x_n\} \subset X$ and $g(X) = g(E)$, there exists $\{e_n\} \subset E$ such that $g(x_n) = g(e_n)$ for all $n \in \mathbb{N}_0$. Using Lemma 2.20, $f(x_n) = f(e_n)$ for all $n \in \mathbb{N}_0$. By using (3.22) and (3.23),

$$\lim_{n\to\infty} g(e_n) = \lim_{n\to\infty} f(e_n) = g(u). \tag{3.26}$$

Making use of (3.25) and (3.26) and continuity of T, we get

$$\begin{aligned} f(u) &= T(gu) \\ &= T(\lim_{n\to\infty} ge_n) \\ &= \lim_{n\to\infty} T(ge_n) \\ &= \lim_{n\to\infty} f(e_n) \\ &= g(u). \end{aligned}$$

Thus $u \in X$ is a coincidence point of f and g. Finally, suppose that $(Y, d, \preceq)$ has ICC-property, then by using (3.22), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(x_{n_k}) \prec\succ g(u) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.27}$$

On using (3.1), (3.2) and (3.27), we obtain

$$\begin{aligned} \psi(d(gx_{n_k+1}, fu)) &= \psi(d(fx_{n_k}, fu)) \\ &\leq \psi(d(gx_{n_k}, gu)) - \phi(d(gx_{n_k}, gu)). \end{aligned}$$

Using property of ϕ and monotonicity of ψ, we get

$$d(gx_{nk+1}, fu) < d(gx_{nk}, gu) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.28}$$

We claim that

$$d(gx_{n_k+1}, fu) \leq d(gx_{n_k}, gu) \quad \text{for all } k \in \mathbb{N}. \tag{3.29}$$

On account of two different possibilities arising here, we consider again the partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, i.e., $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ and $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$, verifying that

$$(i)\ d(gx_{n_k}, gu) = 0 \quad \text{for all } k \in \mathbb{N}^0,$$
$$(ii)\ d(gx_{n_k}, gu) > 0 \quad \text{for all } k \in \mathbb{N}^+.$$

Case (i): By using Lemma 2.20, we get $d(fx_{n_k}, fu) = 0$ for all $k \in \mathbb{N}^0$. In view of (3.2), we get $d(gx_{n_k+1}, fu) = 0$ for all $k \in \mathbb{N}^0$ and so (3.29) holds for all $k \in \mathbb{N}^0$.
Case (ii): On using (3.28), we have $d(gx_{n_k+1}, fu) < d(gx_{n_k}, gu)$ for all $k \in \mathbb{N}^+$. So (3.29) holds for all $k \in \mathbb{N}^+$. Thus (3.29) holds for $k \in \mathbb{N}$. Now, by using (3.22) and (3.29),

$$\begin{aligned} d(gu, fu) &= d(\lim_{n\to\infty} gx_{n+1}, fu) \\ &= \lim_{k\to\infty} d(gx_{n_k+1}, fu) \\ &\leq \lim_{k\to\infty} d(gx_{n_k}, gu) \\ &= 0. \end{aligned}$$

So that $g(u) = f(u)$. Hence u is the coincidence point of f and g. □

We present the following examples.

Example 3.4. *Let $X = [0, \infty)$ be endowed with the usual metric and be equipped with the natural ordering $\leq$. Take $Y = [0, 3]$. Clearly, Y is an $\overline{O}$-complete subspace of ordered metric space $(X, d, \preceq)$. Define the self mappings $f, g : X \to X$ by $f(x) = 1 + \frac{x}{x+1}$ and $g(x) = 4x$. So that $f(X) = [1, 2)$ and $g(X) = [0, \infty)$. It is easy to verify the conditions (a), (b) and (c) of Theorem 3.3. We can see that $Y \subseteq g(X)$ and f and g are continuous. It remains to prove (3.1). For this, we* *define* $\psi, \phi : [0, \infty) \to [0, \infty)$ *by* $\psi(x) = 3x$ *and* $\phi(x) = \frac{x}{2}$*. Now, for any* $x, y \in X$ *with* $gx \prec\succ gy$*, we have*

$$\begin{aligned} \psi[d(gx, gy)] - \phi[d(gx, gy)] &= \psi[d(4x, 4y)] - \phi[d(4x, 4y)] \\ &= \psi(4 \mid x - y \mid) - \phi(4 \mid x - y \mid) \\ &= 12 \mid x - y \mid -2 \mid x - y \mid \\ &= 10 \mid x - y \mid . \end{aligned} \tag{3.30}$$

Again, we have

$$\begin{aligned} \psi[d(fx, fy)] &= \psi[d(1 + \frac{x}{x+1}, 1 + \frac{y}{y+1})] \\ &= \psi(\frac{\mid x - y \mid}{\mid x + 1 \mid\mid y + 1 \mid}) \\ &= \frac{3 \mid x - y \mid}{\mid x + 1 \mid\mid y + 1 \mid} \\ &\leq 3 \mid x - y \mid . \end{aligned} \tag{3.31}$$

From (3.30) and (3.31), we get

$$\psi[d(fx, fy)] \leq \psi[d(gx, gy)] - \phi[d(gx, gy)].$$

Hence all the required conditions of Theorem 3.3 are satisfied. Thus, f *and* g *have a coincidence point in* X*.*

Example 3.5. *Let* $X = [0, \infty)$ *be endowed with the usual metric and the natural ordering* $\leq$*. Take* $Y = [0, 2]$*. Again,* Y *is an* $\overline{O}$*-complete subspace of ordered metric space* $(X, d, \preceq)$*. Define the self mappings* $f, g : X \to X$ *by* $f(x) = \frac{2x}{x+1}$ *and* $g(x) = \frac{5}{3}x$*. So that* $f(X) = [0, 2)$ *and* $g(X) = [0, \infty)$*. It is easy to verify the conditions (a), (b) and (c) of Theorem 3.3. Note that* $Y \subseteq g(X)$ *and* f *and* g *are continuous. We need to prove (3.1). Define* $\psi, \phi : [0, \infty) \to [0, \infty)$ *by* $\psi(x) = \frac{x}{4}$ *and* $\phi(x) = \frac{x}{8}$*. Now, for any* $x, y \in X$ *with* $gx \prec\succ gy$*, we have*

$$\begin{aligned} \psi[d(gx, gy)] - \phi[d(gx, gy)] &= \psi[d(\frac{5}{3}x, \frac{5}{3}y)] - \phi[d(\frac{5}{3}x, \frac{5}{3}y)] \\ &= \psi(\frac{5}{3} \mid x - y \mid) - \phi(\frac{5}{3} \mid x - y \mid) \\ &= \frac{5}{12} \mid x - y \mid -\frac{5}{24} \mid x - y \mid \\ &\leq \frac{1}{2} \mid x - y \mid . \end{aligned} \tag{3.32}$$

Also,

$$\begin{aligned} \psi[d(fx, fy)] &= \psi[d(\frac{2x}{x+1}, \frac{2y}{y+1})] \\ &= \psi(\frac{2 \mid x - y \mid}{\mid x + 1 \mid\mid y + 1 \mid}) \\ &= \frac{\mid x - y \mid}{2 \mid x + 1 \mid\mid y + 1 \mid} \\ &\leq \frac{1}{2} \mid x - y \mid . \end{aligned} \tag{3.33}$$

From (3.32) and (3.33), we get

$$\psi[d(fx, fy)] \leq \psi[d(gx, gy)] - \phi[d(gx, gy)].$$

All the required conditions of Theorem 3.3 are satisfied. f and g have a coincidence point in X.

Corollary 3.6. *Let $(X, d, \preceq)$ be an $\overline{O}$-complete ordered metric space. Consider f, $g : X \to X$. Assume there exist an ultra-altering distance function ϕ and an altering distance function ψ such that*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)) \; \forall x, y \in X \text{ with } gx \prec\succ gy. \tag{3.34}$$

Suppose the following conditions hold:
(a) $f(X) \subseteq g(X)$;
(b) f is g-increasing;
(c) $\exists\, x_0 \in X$ *s.t* $g(x_0) \preceq f(x_0)$,
(d) either
(d_1) *f and g are $\overline{O}$-compatible;*
(d_2) *g is $\overline{O}$-continuous;*
(d_3) *either f is $\overline{O}$-continuous or$(X, d, \preceq)$ has g-ICC property,*
or alternatively,
(d')
(d'_1) *there exists an $\overline{O}$-closed subspace Y of X s.t $f(X) \subseteq Y \subset g(X)$;*
(d'_2) *either f is $(g,\overline{O})$-continuous or f and g are continuous or $(Y, d, \preceq)$ has ICC-property,*
Then, f and g have a coincidence point.

Proof. The result corresponding to part (d) follows easily by setting $Y = X$ in Theorem 3.3, while the same result in the presence of (d') follows using proposition 2.11.

□

Definition 3.7. [2]. *Let $(X, \preceq)$ be an ordered set and f, g be two self-mappings on X. Then $(X, \preceq)$ is said to be (f, g)-directed if for each pair $x, y \in X$, there exists $z \in X$ such that $f(x) \prec\succ g(z)$ and $f(y) \prec\succ g(z)$. In the case $g = I$ (identity mapping), $(X, \preceq)$ is called f-directed.*

Definition 3.8. [2]. *Let $(X, \preceq)$ be an ordered set, $E \subseteq X$ and $a, b \in E$. A subset $\{e_1, e_2, ..., e_k\}$ of E is called a $\prec\succ$-chain between a and b in E, if*
(i) $k \geq 2$;
(ii) $\mathrm{e_1 = a}$ and $\mathrm{e_k = b}$;
(iii) $e_1 \prec\succ e_2 \prec\succ ... \prec\succ e_k$.
We denote by $C(a, b, \prec\succ, E)$ the class of all $\prec\succ$-chains between a and b in E. In particular, for $E = X$ we write $C(x, y, \prec\succ)$ instead of $C(x, y, \prec\succ, X)$.

Notation 2. [2]. *For a pair f, g of self-mappings on a non-empty set X, we denote by $C(f, g) = \{x \in X : fx = gx\}$ the set of all coincidence points of f and g. Also, let $\overline{C}(f, g) = \{\overline{x} \in X : \exists\, x \in X \text{ s.t } fx = gx = \overline{x}\}$ be the set of all points of coincidence of f and g.*

Theorem 3.9. *In addition to assumptions of Theorem 3.3, suppose the following condition holds:*
(f) $C(x, y, \prec\succ, gX)$ is non-empty for each $x, y \in X$. Then f and g have a unique point of coincidence.

Proof. In Theorem 3.3, we have proved the existence of a coincidence point of f and g in X, *i.e.*, there exists $x \in X$ such that $fx = gx = k$ (say) $\in X$. Thus, $k \in \overline{C}(f,g)$ and so $\overline{C}(f,g) \neq \emptyset$. If $\overline{C}(f,g)$ is a singleton, the proof is completed. Otherwise, let $\overline{x}, \overline{y} \in \overline{C}(f,g)$. So there exist $x, y \in X$ such that

$$f(x) = g(x) = \overline{x} \text{ and } f(y) = g(y) = \overline{y}. \tag{3.35}$$

We have to show that $\overline{x} = \overline{y}$. As $f(x), f(y) \in f(X) \subseteq g(X)$, then by (f), there exists $\prec\succ -chain\{gt_1, gt_2, ..., gt_k\}$ between $f(x)$ and $f(y) \in g(X)$, where $t_1, t_2, ..., t_k \in X$. By (3.35), without loss of generality, we can choose

$$t_1 = x \textit{ and } t_k = y,$$

$$g(t_i) \prec\succ g(t_{i+1}) \textit{ for each } i \ (1 \le i \le k-1). \tag{3.36}$$

Now, we define constant sequences,

$$t_n^1 = t_1 = x \textit{ and } t_n^k = t_k = y. \tag{3.37}$$

Then using (3.35), we have $g(t_{n+1}^1) = f(t_n^1)$ and $g(t_{n+1}^k) = f(t_n^k)$ for all $n \in \mathbb{N}_0$. Put $t_0^2 = t_2, t_0^3 = t_3, ..., t_0^{k-1} = t_{k-1}$. Since $f(X) \subseteq g(X)$, we can define sequences $\{t_n^2\}, \{t_n^3\}, \cdots, \{t_n^{k-1}\}$ in X such that $g(t_{n+1}^2) = f(t_n^2)$, $g(t_{n+1}^3) = f(t_n^3)$,...,$g(t_{n+1}^{k-1}) = f(t_n^{k-1})$ for all $n \in \mathbb{N}_0$. So, we have

$$g(t_{n+1}^i) = f(t_n^i) \ \forall \ n \in \mathbb{N}_0 \textit{ and for each } i \ (1 \le i \le k-1). \tag{3.38}$$

Now, we claim that

$$g(t_n^i) \prec\succ g(t_n^{i+1}) \ \forall \ n \in \mathbb{N}_0 \textit{ and for each } i \ (1 \le i \le k-1). \tag{3.39}$$

We will prove it by induction. From (3.36), we can say that (3.39) holds for $n = 0$. We assume that (3.39) holds for $n = r > 0$, *i.e.*,

$$g(t_r^i) \prec\succ g(t_r^{i+1}) \textit{ for each } i \ (1 \le i \le k-1).$$

Since f is g-increasing, we obtain

$$f(t_r^i) \prec\succ f(t_r^{i+1}) \textit{ for each } i \ (1 \le i \le k-1).$$

Using (3.38),

$$g(t_{r+1}^i) \prec\succ g(t_{r+1}^{i+1}) \textit{ for each } i \ (1 \le i \le k-1).$$

This shows that (3.39) also holds for $n = r + 1$. By principle of mathematical induction, (3.39) holds for all $n \in \mathbb{N}_0$. Now, for each $n \in \mathbb{N}_0$ and $1 \le i \le k-1$, define

$$R_n^i = d(gt_n^i, gt_n^{i+1}).$$

We claim that

$$\lim_{n\to\infty} R_n^i = 0 \textit{ for each } i \ (1 \le i \le k-1). \tag{3.40}$$

By fixing i, two cases arise. Firstly, suppose that $R_{n_0}^i = d(gt_{n_0}^i, gt_{n_0}^{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by Lemma 2.19, we have $d(ft_{n_0}^i, ft_{n_0}^{i+1}) = 0$. Consequently, by using (3.38), we get

$$\begin{aligned} R_{n_0+1}^i &= d(gt_{n_0+1}^i, gt_{n_0+1}^{i+1}) \\ &= d(ft_{n_0}^i, ft_{n_0}^{i+1}) \\ &= 0. \end{aligned}$$

By induction, $R_n^i = 0$ for all $n \geq n_0$, so $\lim_{n\to\infty} R_n^i = 0$. Secondly, suppose $R_n > 0$ for all $n \in \mathbb{N}_0$. Using (3.38), (3.39) and (3.1) of Theorem 3.3, we have

$$\begin{aligned}\psi(R_n^i) &= \psi(d(gt_n^i, gt_n^{i+1}))\\ &= \psi(d(ft_{n-1}^i, ft_{n-1}^{i+1}))\\ &\leq \psi(d(gt_{n-1}^i, gt_{n-1}^{i+1})) - \phi((d(gt_{n-1}^i, gt_{n-1}^{i+1})))\\ &= \psi(R_{n-1}^i) - \phi(R_{n-1}^i),\end{aligned}$$

that is,

$$\psi(R_n^i) \leq \psi(R_{n-1}^i) - \phi(R_{n-1}^i). \tag{3.41}$$

Using properties of ϕ, we have for all $n \geq 1$,

$$\psi(R_n^i) \leq \psi(R_{n-1}^i).$$

Since ψ is monotonic increasing, we get

$$R_n^i \leq R_{n-1}^i \ \forall\ n \textit{ and each } i\ (1 \leq i \leq k-1).$$

Therefore, $\{R_n^i\}$ is a monotonic decreasing sequence of non-negative real numbers, thus there exists $r \geq 0$ such that

$$R_n^i \to r \textit{ as } n \to \infty. \tag{3.42}$$

Taking $\lim_{n\to\infty}$ in (3.41) and using (3.42) and continuities of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which it is a contradiction, unless $r = 0$. Thus,

$$R_n^i \to 0 \text{ as } n \to \infty\ \forall\ i\ (1 \leq i \leq k-1).$$

In both cases, (3.40) is proved. Now, by triangular inequality, we have

$$d(\overline{x}, \overline{y}) \leq d(\overline{x}, gt_n^2) + d(gt_n^2, gt_n^3) + ... + d(gt_n^{k-1}, \overline{y}). \tag{3.43}$$

Using (3.35) and (3.37), we have

$$\overline{x} = g(x) = g(t_n^1) \textit{ and } \overline{y} = g(t_n^k). \tag{3.44}$$

Using (3.44) in (3.43), by (3.40), we get

$$\begin{aligned}d(\overline{x}, \overline{y}) &= d(gt_n^1, gt_n^2) + d(gt_n^2, gt_n^3) + ... + d(gt_n^{k-1}, gt_n^k)\\ &= R_n^1 + R_n^2 + ... + R_n^{k-1}\\ &\to 0 \textit{ as } n \to \infty.\end{aligned}$$

Thus $d(\overline{x}, \overline{y}) = 0$, which implies that $\overline{x} = \overline{y}$. □

Example 3.10. *Let $X = [0, \infty)$ be endowed with the usual metric and the natural ordering $\leq$. Take $Y = [0, 1]$. Clearly, Y is an $\overline{O}$-complete subspace of ordered metric space $(X, d, \preceq)$. Define the self mappings $f, g : X \to X$ by $f(x) = \frac{x}{x+1}$ and $g(x) = 2x$, So that $f(X) = [0, 1)$ and $g(X) = [0, \infty)$. It is easy to verify the conditions (a), (b) and (c) of Theorem 3.3. Also we can see that $Y \subseteq g(X)$ and f*

and g are continuous. It remains to prove (3.1). Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(x) = \frac{x}{2}$ and $\phi(x) = \frac{x}{5}$. Now, for $x, y \in X$ such that $gx \prec\succ gy$, we have

$$\begin{aligned} \psi[d(gx,gy)] - \phi[d(gx,gy)] &= \psi[d(2x,2y)] - \phi[d(2x,2y)] \\ &= \psi(2\mid x-y\mid) - \phi(2\mid x-y\mid) \\ &= \mid x-y\mid - \frac{2}{5}\mid x-y\mid \\ &= \frac{3}{5}\mid x-y\mid \\ &\geq \frac{1}{2}\mid x-y\mid. \end{aligned} \tag{3.45}$$

Again, we have

$$\begin{aligned} \psi[d(fx,fy)] &= \psi[d(\frac{x}{x+1},\frac{y}{y+1})] \\ &= \psi(\frac{\mid x-y\mid}{\mid x+1\mid\mid y+1\mid}) \\ &= \frac{\mid x-y\mid}{2\mid x+1\mid\mid y+1\mid} \\ &\leq \frac{1}{2}\mid x-y\mid. \end{aligned} \tag{3.46}$$

We deduce that

$$\psi[d(fx,fy)] \leq \psi[d(gx,gy)] - \phi[d(gx,gy)].$$

Hence all the required conditions of Theorem 3.3 are satisfied. Thus, f and g have a coincidence point in X. Further, it is clear that for each $x, y \in X$, $C(x, y, \prec\succ, gX)$ is non-empty, so by Theorem 3.9, f and g have a unique point of coincidence.

Theorem 3.11. *Theorem 3.9 remains also true if we replace condition (f) of Theorem 3.9 by one of the following conditions:*
(f_1): $(fX, \preceq)$ is totally ordered;
(f_2): $(X, \preceq)$ is (f, g)-directed.

Proof. Suppose that (f_1) holds. For $x, y \in X$ such that $f(x) \prec\succ f(y)$, we have $\{fx, fy\}$ is a $\prec\succ$-chain between fx *and* fy *in* $g(X)$. Thus $C(fx, fy, \prec\succ, g(X))$ is non-empty for each $x, y \in X$, *i.e.*, (f) holds. Hence Theorem 3.9 is applicable. Next, assume (f_2) holds, then for all $x, y \in X$, there exists $z \in X$ such that

$$f(x) \prec\succ g(z) \prec\succ f(y).$$

This implies that $\{fx, gz, fy\}$ is a $\prec\succ$-chain between fx and fy in $g(X)$. It follows that $C(fx, fy, \prec\succ, g(X))$ is non-empty for each $x, y \in X$, *i.e.*, (f) holds and so Theorem 3.9 is applicable. □

On setting $g = I$ in earlier results, we deduce the following corresponding fixed point theorems.

Theorem 3.12. *Let $(X, d, \preceq)$ be an ordered metric space, Y be an $\overline{O}$-complete subspace of X and f be a self-mapping on X such that $f(X) \subseteq Y$. Assume there exist an ultra-altering distance function ϕ and an altering distance function ψ such that*

$$\psi(d(fx,fy)) \leq \psi(d(x,y)) - \phi(d(x,y)) \ \forall\ x,y \in X \text{ with } x \prec\succ y. \tag{3.47}$$

Also suppose the following conditions hold:
(a) f is increasing;
(b) there exists $x_0 \in X$ such that $x_0 \preceq f(x_0)$;
(c) either f is $\overline{O}$-continuous, or $(Y, d, \preceq)$ has g-ICC property.
Then f has a fixed point.

Theorem 3.13. *Theorem 3.12 remains true if certain involved terms, namely $\overline{O}$-complete, $\overline{O}$-continuous and ICC-property are replaced by respective $\underline{O}$-complete, $\underline{O}$-continuous and DCC property provided assumption (b) is replaced by the following condition:*
(b') there exists $x_0 \in X$ such that $x_0 \succeq f(x_0)$.

Theorem 3.14. *Theorem 3.12 remains true if certain involved terms, namely $\overline{O}$-complete, $\overline{O}$-continuous and ICC-property are replaced by respective O-complete, O-continuous and MCC property provided assumption (b) is replaced by the following condition:*

(b') there exists $x_0 \in X$ such that $x_0 \prec\succ f(x_0)$.

4. Conclusion

In our results, another metrical notions (such as completeness, continuity, g-continuity and compatibility) are compatible with a partial ordering. In the similar manner, for a possible problem, the reader mainly sharpened the existing results under different kinds of contractions.

Competing interests

The authors declare that they have no competing interests.

Authors Contributions

All the authors contributed equally. All the authors read and approved the final paper.

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