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BEST PROXIMITY POINT FOR CERTAIN PROXIMAL CONTRACTION TYPE MAPPINGS

BADR ALQAHTANI, JAVAD HAMZEHNEJADI , ERDAL KARAPINAR, RAHMATOLLAH LASHKARIPOUR.

Abstract. In this paper, we introduce the new notion of generalized proximal α-h-ϕ-contraction mappings and investigate the existence of the best proximity point for such mappings in the complete metric spaces.

1. Introduction

Let A be a nonempty subset of a metric space (X, d) and T be a function that map A into itself. A fixed point of the mapping T is an element $x \in A$ for which $Tx = x$. Fixed point theory plays a crucial role in nonlinear functional analysis and many authors have studied this notion. Fixed point theory has an application in distinct branches of mathematics and also in different sciences, such as engineering, computer science, economics, etc. In 1922, Banach presented convergent and uniqueness of fixed point for contraction mappings(see [8]). Many authors have generalized this result in several directions(see, e.g.,[2]-[11]-[15]-[16]-[26]).
Now, let A and B be two nonempty subsets of a metric space X and $T : A \to B$ be a mapping. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary condition for existence of the fixed point. If $T(A) \cap A = \emptyset$, then the set of fixed points of T is empty. In such a situation, one often attempts to find an element $x^* \in A$ which is in some sense closest to Tx. Best approximation theory and best proximity point analysis have been developed in this direction. A best proximity point of mapping T is a point $x^* \in A$ satisfying the equality $d(x^*, Tx^*) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. We denote by A_0, B_0 and $P_T(A)$ the following sets:

$$A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\},$$
$$P_T(A) := \{x \in A : d(x, Tx) = d(A, B)\}.$$

A best proximity point theorem for cyclic contraction mappings has been detailed in A. Anthony, P. Veeramani [3]. Let $\mathcal{F}$ be a class of those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0.$$

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A mapping $T : A \to B$ is said to be Geraghty-contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in A$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Definition 1. *([1]). Let A and B be two nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to be have the $P-$property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$*

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \implies d(x_1, x_2) = d(y_1, y_2).$$

A best proximity point theorem for Geraghty-contraction mappings is obtained by Caballero, Harjani and Sadarangani in [12]. Also Raj in [27] has discussed best proximity point theorem for weakly contractive non-self mappings. Other works on the existence of a best proximity point for contractions can be seen in ([4]-[5]-[6]-[7]-[9]-[10]-[25]).
Let $\alpha : X \times X \to \mathbb{R}$ be a function. The notion of $\alpha-$proximal admissible mappings is presented in [28], by Jleli and Samet [18], as follows:

Definition 2. *([18]) Let $T : A \to B$ be a map and $\alpha : X \times X \to [0, \infty)$ be a function. The mapping T is said to be $\alpha-$proximal admissible if*

$$\left.\begin{array}{l} \alpha(x, y) \geq 1 \\ d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies \alpha(u, v) \geq 1,$$

for all $x, y, u, v \in A$.

Also, a self map $T : X \to X$ is said to be $\alpha-$admissible, see[28], if

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.$$

Definition 3. *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ be a function, and let $T : X \to X$ be a mapping. The sequence $\{x_n\}$ is said to be $\alpha-$regular if $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, implies that there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.*

An $\alpha-$proximal admissible map T is said to be triangular $\alpha-$proximal admissible [4] if

$$\alpha(x, z) \geq 1 \text{ and } \alpha(z, y) \geq 1 \text{ implies } \alpha(x, y) \geq 1.$$

For more details we refer the reader to some related papers ([13]-[14]-[19]-[20]-[23]-[24]).
In this work, we introduce a new class of generalized contraction mappings to the case of non-self mappings as generalized proximal α-h-ϕ-contraction mappings and study the existence and uniqueness of best proximity points for this type of mappings. Furthermore, we obtain some known and some new results in fixed point theory via the generalized α-h-ϕ-contraction mappings.
Before indicating our main results we shall present the following definition and examples introduced in [22] and needed throughout this paper.

Definition 4. *Let A and B be two nonempty subsets of a metric space (X, d) and $T : A \to B$ be a mapping. We say that T has $RJ-$property if for any sequence $\{x_n\} \subseteq A$,*

$$\left.\begin{array}{l} \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A, B) \\ \lim_{n\to\infty} x_n = x \end{array}\right\} \implies x \in A_0.$$

In order to illustrate $RJ-$property, we present some examples.

Example 1. *Let A and B be two nonempty closed subsets of metric space (X,d) and $T:A\to B$ be a continuous mapping. Let $\lim_{n\to\infty} d(x_{n+1},Tx_n)=d(A,B)$ and $\lim_{n\to\infty} x_n=x$. Since T is continuous, therefore $\lim_{n\to\infty} Tx_n=Tx$. This implies that*

$$d(x,Tx)=\lim_{n\to\infty} d(x_{n+1},Tx_n)=d(A,B).$$

Therefore $x\in A_0$, which implies that T has $RJ-$property.

Let A and B be two nonempty closed subsets of metric space (X,d). B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $\lim_{n\to\infty} d(x,y_n)=d(x,B)$ for some $x\in A$, has a convergent subsequence.

Example 2. *Let A and B be two nonempty closed subsets of metric space (X,d) such that B is approximatively compact with respect to A and $T:A\to B$ be a mapping. Let $\lim_{n\to\infty} x_n=x$ and $\lim_{n\to\infty} d(x_{n+1},Tx_n)=d(A,B)$. For any $n\in\mathbb{N}$, we have*

$$d(A,B)\le d(x,Tx_n)\le d(x,x_{n+1})+d(x_{n+1},Tx_n).$$

Thus $\lim_{n\to\infty} d(x,Tx_n)=d(A,B)$. Also for any $n\in\mathbb{N}$, we have $d(x,B)\le d(x,Tx_n)$. Therefore

$$d(x,B)\le\lim_{n\to\infty} d(x,Tx_n)=d(A,B)\le d(x,B),$$

which implies that $\lim_{n\to\infty} d(x,Tx_n)=d(x,B)$. Since B is approximatively compact with respect to A, there exist subsequence $\{x_{n_k}\}\subseteq\{x_n\}$ and $y\in B$ such that $\lim_{k\to\infty} Tx_{n_k}=y$. Hence

$$d(x,y)=\lim_{k\to\infty} d(x,Tx_{n_k})=d(A,B),$$

which implies that $x\in A_0$. Hence T has $RJ-$property.

Lemma 1.1. *Let $T:A\to B$ be a triangular $\alpha-$proximal admissible mapping. Assume that $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\ge 1$, for all $n\in\mathbb{N}$. Then, we have $\alpha(x_n,x_m)\ge 1$ for all $m,n\in\mathbb{N}$ with $n<m$.*

Let Φ denote the class of all functions $\phi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

(a) ϕ is nondecreasing;
(b) ϕ is continuous;
(c) $\phi(t)=0\Leftrightarrow t=0$.

Also, let Let Ψ denote the class of all functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t)=0\Leftrightarrow t=0$;
(d) ψ is subadditive, that is, $\psi(s+t)\le\psi(s)+\psi(t)$ for all $s,t\ge 1$.

Note that $\Psi\subseteq\Phi$.

Definition 5. *Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a ψ-Geraghty contraction if there exist $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that for all $x, y \in A$,*

$$d(Tx, Ty) \leq \beta(\psi(d(x, y)))\psi(d(x, y)).$$

Definition 6. *Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be weakly contraction type mapping if there exists $\phi \in \Phi$ such that for all $x, y \in A$,*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)).$$

2. Main results

Let $\mathcal{H}(X)$ be the family of functions $h : X \times X \times X \times X \to [0, 1)$ satisfying the following condition:

$$\lim_{n\to\infty} h(x_n, y_n, u_n, v_n) = 1 \implies \lim_{n\to\infty} d(x_n, y_n) = 0,$$

for all sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\} \subseteq X$ that the sequence $\{d(x_n, y_n)\}$ is decreasing and convergent.

Example 3. *Let $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, 1)$, defined by*

(i) $h_1(x, y, u, v) = \dfrac{t}{t + x^2 + y^2 + u^2 + v^2}$, *for some $t \in (0, \infty)$.*

(ii) $h_2(x, y, u, v) = k$, *for some $k \in (0, 1)$.*

Then $h_1, h_2 \in \mathcal{H}(\mathbb{R})$.

Example 4. *Let (X, d) be a metric space and $\beta \in \mathcal{F}$. Define $h : X \times X \to [0, 1)$, by*

$$h(x, y, u, v) = \beta(d(x, y)).$$

If $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}$ be sequences in X such that $\lim_{n\to\infty} h(x_n, y_n, u_n, v_n) = 1$, then

$$\lim_{n\to\infty} \beta(d(x_n, y_n)) = 1.$$

Thus

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$.

We start this section with the following definition.

Definition 7. *Let A and B be two nonempty subsets of a metric space (X, d), and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : A \to B$ is said to be generalized proximal α-h-ϕ-contraction if there exists $h \in \mathcal{H}(\mathcal{X})$ such that for all $x, y, u, v \in A$,*

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies \alpha(x, y)\phi(d(u, v)) \leq h(x, y, u, v)\phi(M_a(x, y, u, v)),$$

where $\phi \in \Phi$ and for any $x, y, u, v \in A$,

$$M_a(x, y, u, v) = \max\left\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\right\}.$$

Now we prove the following theorem that extends, improves and generalizes some earlier results in the literature on fixed point and best proximity point theorems.

Theorem 2.1. *Let A and B be two nonempty closed subsets of complete metric space (X,d), $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a mapping. Let $h \in \mathcal{H}(\mathcal{X})$ and $\phi \in \Phi$ be mappings such that the following conditions are satisfied:*

(i) *T is generalized proximal α-h-ϕ-contraction type mapping and*
$$\lim_{n\to\infty} h(x_n, y_n, u_n, v_n) = 1 \implies \lim_{n\to\infty} d(u_n, v_n) = 0,$$
for all sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\} \subseteq X$;

(ii) *$T(A_0) \subseteq B_0$ and T is triangular $\alpha-$proximal admissible;*

(iii) *T has $RJ-$property;*

(iv) *If $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k;*

(v) *there exist $x_0, x_1 \in A$ such that*
$$d(x_1, Tx_0) = d(A,B) \text{ and } \alpha(x_0, x_1) \geq 1.$$

Then, there exists an element $x^* \in A_0$ such that
$$d(x^*, Tx^*) = d(A,B).$$
Moreover, if $\alpha(x,y) \geq 1$ for all $x, y \in P_T(A)$, then x^* is the unique best proximity point of T.

Proof. Let $x_1, x_0 \in A$ be such that
$$d(x_1, Tx_0) = d(A,B) \text{ and } \alpha(x_0, x_1) \geq 1.$$
Hence $x_1 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Now, we have
$$\begin{cases} \alpha(x_0, x_1) \geq 1, \\ d(x_1, Tx_0) = d(A,B), \\ d(x_2, Tx_1) = d(A,B). \end{cases}$$
Since T is $\alpha-$proximal admissible, then $\alpha(x_1, x_2) \geq 1$. Thus, we have
$$d(x_2, Tx_1) = d(A,B) \text{ and } \alpha(x_1, x_2) \geq 1.$$
Continuing this process, by induction, we can construct a sequence $\{x_n\} \subseteq A_0$ such that
$$d(x_{n+1}, Tx_n) = d(A,B) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.1}$$
Therefore for any $n \in \mathbb{N}$, we have
$$\begin{cases} \alpha(x_{n-1}, x_n) \geq 1, \\ d(x_n, Tx_{n-1}) = d(A,B), \\ d(x_{n+1}, Tx_n) = d(A,B). \end{cases}$$
Since T is a generalized proximal α-h-ϕ-contraction type mapping, we have
$$\begin{aligned} \phi(d(x_n, x_{n+1})) &\leq \alpha(x_{n-1}, x_n)\phi(d(x_n, x_{n+1})) \\ &\leq h(x_{n-1}, x_n, x_n, x_{n+1})\phi(M_a(x_{n-1}, x_n, x_n, x_{n+1})) \\ &< \phi(M_a(x_{n-1}, x_n, x_n, x_{n+1})). \end{aligned} \tag{2.2}$$

Also, we have

$$\begin{aligned} M_a(x_{n-1}, x_n, x_n, x_{n+1}) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\}. \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{2.3}$$

If $M_a(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$, applying (2.2), we deduce that

$$\begin{aligned} \phi(d(x_n, x_{n+1})) &< \phi(M_a(x_{n-1}, x_n, x_n, x_{n+1})) \\ &= \phi(d(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction. Thus, we conclude that

$$M_a(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Now, from (2.2) and (2.4), we get that

$$\phi(d(x_n, x_{n+1})) < \phi(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}.$$

Taking into account that ϕ is nondecreasing, implies that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

We deduce that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and decreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$. This implies that $x_{n_0} = x_{n_0+1}$. Applying (2.1), we deduce that

$$d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B).$$

This is the desired result. Now, let for any $n \in \mathbb{N}$, $d(x_n, x_{n+1}) \neq 0$. In the sequel, we prove that $r = 0$. In the contrary case suppose that $r > 0$. Then from (2.2) and (2.4), we have

$$0 < \frac{\phi(d(x_n, x_{n+1}))}{\phi(d(x_{n-1}, x_n))} \leq h(x_{n-1}, x_n, x_n, x_{n+1}),$$

which implies that $\lim_{n\to\infty} h(x_{n-1}, x_n, x_n, x_{n+1}) = 1$. Therefore

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0.$$

This implies that $r = 0$, which is a contradiction. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Thus, there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$ there exist $m_k \geq n_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \epsilon.$$

Also, choosing m_k as small as possible, it may be assumed that

$$d(x_{n_k}, x_{m_k-1}) < \epsilon.$$

Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \epsilon \leq d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ &\leq \epsilon + d(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon.$$

Note that, for any $k \in \mathbb{N}$,

$$\begin{cases} \alpha(x_{n_k}, x_{m_k}) \geq 1, \\ d(x_{n_k+1}, Tx_{n_k}) = d(A,B), \\ d(x_{m_k+1}, Tx_{m_k}) = d(A,B). \end{cases}$$

Then for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \phi(d(x_{n_k+1}, x_{m_k+1})) &\leq \alpha(x_{n_k}, x_{m_k})\phi(d(x_{n_k+1}, x_{m_k+1})) \qquad (2.5) \\ &\leq h(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})\phi(M_a(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1})). \end{aligned}$$

Also, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} M_a(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) &= \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \\ &\qquad \frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}\Big\} \\ &\leq \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{m_k+1}), \\ &\qquad \frac{d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k+1})}{2} \\ &+ \frac{d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})}{2}\Big\}. \end{aligned}$$

Since $\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = 0$, then

$$\lim_{k\to\infty} M_a(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}). \qquad (2.6)$$

By using the triangular inequality and taking the limit as $n \to \infty$, we derive

$$\begin{aligned} \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) &\leq \lim_{k\to\infty} (d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k})) \\ &= \lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}). \qquad (2.7) \end{aligned}$$

Combining (2.5), (2.6) and (2.7) with the continuity of ϕ, we get

$$\lim_{k\to\infty} \phi(d(x_{n_k}, x_{m_k})) \leq \lim_{k\to\infty} h(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) \lim_{k\to\infty} \phi(d(x_{n_k}, x_{m_k})).$$

Since $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon > 0$, we deduce that

$$\lim_{k\to\infty} h(x_{n_k}, x_{m_k}, x_{n_k+1}, x_{m_k+1}) = 1.$$

Since $h \in \mathcal{H}(X)$, then

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = 0,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subseteq A$ and A is a closed subset of the complete metric space (X,d), there exists $x^* \in A$ such that $\lim_{n\to\infty} x_n = x^*$. $RJ-$property of T, implies that $x^* \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $w \in A_0$ such that $d(w, Tx^*) = d(A,B)$. We shall prove that $w = x^*$. In the contrary case, let $w \neq x^*$. Property (iv) implies that, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. Without loss of generality, we assume that

$$\alpha(x_n, x^*) \geq 1 \text{ for all } n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, we have $d(x_{n+1}, Tx_n) = d(A, B)$ and $d(w, Tx^*) = d(A, B)$. Using the fact that T is a generalized proximal α-h-ϕ-contraction type mapping, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} \phi(d(x_{n+1}, w)) &\leq \alpha(x_n, x^*)\phi(d(x_{n+1}, w)) \\ &\leq h(x_n, x^*, x_{n+1}, w)\phi(M_a(x_n, x^*, x_{n+1}, w)) \\ &< \phi(M_a(x_n, x^*, x_{n+1}, w)). \end{aligned} \tag{2.8}$$

Also, for any $n \in \mathbb{N}$, we have

$$M_a(x_n, x^*, x_{n+1}, w) = \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, w), \frac{d(x_n, w) + d(x^*, x_{n+1})}{2}.\right\}$$

Let there exists subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that

$$M_a(x_{n_k}, x^*, x_{n_k+1}, w) = d(x^*, w), \quad \forall k \in \mathbb{N}.$$

Thus for any $k \in \mathbb{N}$, we have

$$\begin{aligned} \phi(d(x_{n_k+1}, w)) &\leq \alpha(x_{n_k}, x^*)\phi(d(x_{n_k+1}, w)) \\ &\leq h(x_n, x^*, x_{n+1}, w)\phi(d(x^*, w)). \end{aligned}$$

Taking the limit of both sides as $n \to \infty$, implies that $\lim_{n\to\infty} h(x_n, x^*, x_{n+1}, w) = 1$. Therefore $\lim_{n\to\infty} d(x_{n+1}, w) = 0$ and so $x^* = w$, which is a contradiction. Thus there exists $k \in \mathbb{N}$ such that

$$M_a(x_n, x^*, x_{n+1}, w) = \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), \frac{d(x_n, w) + d(x^*, x_{n+1})}{2}\right\}, \quad \forall n > k. \tag{2.9}$$

Together with (2.8), (2.9) and by Taking the limit as $n \to \infty$, we deduce that

$$\phi(d(x^*, w)) < \phi(\frac{d(x^*, w)}{2}).$$

This is a contradiction. Therefore $x^* = w$, which implies that

$$d(x^*, Tx^*) = d(w, Tx^*) = d(A, B).$$

Hence x^* is the best proximity point of T.
For the uniqueness, let $\alpha(x, y) \geq 1$ for all $x, y \in P_T(A)$. Suppose that x_1 and x_2 are two best proximity points of T with $x_1 \neq x_2$. Therefore

$$\begin{cases} d(x_1, Tx_1) = d(A, B), \\ d(x_2, Tx_2) = d(A, B). \end{cases}$$

Also, we have

$$M_a(x_1, x_2, x_1, x_2) = \max\left\{d(x_1, x_2), d(x_1, x_1), d(x_2, x_2), \frac{d(x_1, x_2) + d(x_2, x_1)}{2}\right\} = d(x_1, x_2).$$

Since $\alpha(x_1, x_2) \geq 1$ and T is a generalized proximal α-h-ϕ- contraction type mapping, we get

$$\begin{aligned} \phi(d(x_1, x_2)) &\leq \alpha(x_1, x_2)\phi(d(x_1, x_2) \\ &\leq h(x_1, x_2, x_1, x_2)\phi(d(x_1, x_2)) \\ &< \phi(d(x_1, x_2)), \end{aligned}$$

which is a contradiction. Hence the best proximity point is unique. □

By Example 1 a continuous map has $RJ-$property and if T be continuous, then condition (iv) is not needed.

Theorem 2.2. *Let A and B be two nonempty closed subsets of complete metric space (X,d), $\alpha : X\times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a mapping. Let $h \in \mathcal{H}(\mathcal{X})$ and $\phi \in \Phi$ be mappings such that the following conditions are satisfied:*

(i) *T is generalized proximal α-h-ϕ-contraction type mapping;*
(ii) *The conditions (ii) and (v) of Theorem 2.1 are satisfied;*
(ii) *T is continuous.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A,B).$$

Moreover, if $\alpha(x,y) \geq 1$ for all $x,y \in P_T(A)$, then x^* is the unique best proximity point of T.

Proof. Let $x_0, x_1 \in A$ be such that

$$d(x_1, Tx_0) = d(A,B) \text{ and } \alpha(x_0,x_1) \geq 1.$$

Therefore $x_1 \in A_0$. Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A,B)$. Now, we have

$$\begin{cases} \alpha(x_0,x_1) \geq 1, \\ d(x_1, Tx_0) = d(A,B), \\ d(x_2, Tx_1) = d(A,B). \end{cases}$$

Since T is $\alpha-$proximal admissible, then $\alpha(x_1,x_2) \geq 1$. Thus, we have

$$d(x_2, Tx_1) = d(A,B) \text{ and } \alpha(x_1,x_2) \geq 1.$$

Continuing this process, by induction, we can construct a sequence $\{x_n\} \subseteq A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A,B) \text{ and } \alpha(x_n,x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}.$$

Following the lines in the proof of Theorem 2.1, there exists a sequence $\{x_n\}$ such that $d(x_{n+1}, Tx_n) = d(A,B)$ for all $n \in \mathbb{N}$, and the sequence $\{x_n\}$ converges to some $x^* \in A$. Since T is continuous, then

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A,B).$$

Therefore x^* is the best proximity point of T. If $\alpha(x,y) \geq 1$ for all $x,y \in P_T(A)$, following the lines in the proof of Theorem 2.1, we get that the best proximity point is unique. □

In Theorem 2.1, if we take $\phi(t) = t$ for all $t \geq 0$, and $\alpha(x,y) = 1$ for all $x,y \in X$ then we have the following corollary.

Corollary 2.3. *Let A and B be two nonempty closed subsets of complete metric space (X,d), and $\alpha : X\times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a mapping. Let $h \in \mathcal{H}(\mathcal{X})$ be a mapping such that the following conditions are satisfied:*

(i) *T is a generalized proximal h-contraction type mapping, that is*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies d(u,v) \leq h(x,y,u,v)M_a(x,y,u,v),$$

where $M_a(x,y,u,v) = \max\left\{d(x,y), d(x,u), d(y,v), \dfrac{d(x,v)+d(y,u)}{2}\right\}$, *for all $x,y,u,v \in A$;*

(ii) *The mapping T is continuous or for all sequences $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\} \subseteq X$,*

$$\lim_{n\to\infty} h(x_n, y_n, u_n, v_n) = 1 \implies \lim_{n\to\infty} d(u_n, v_n) = 0;$$

(iii) *$T(A_0) \subseteq B_0$ and T has RJ-property;*

(iv) *there exist $x_0, x_1 \in A$ such that*

$$d(x_1, Tx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1.$$

Then, there exists a unique element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B).$$

To illustrate our results given in Theorem 2.1, we present the following example.

Example 5. *Consider $X = \mathbb{R}^2$ with the usual metric. Let A and B be the subsets of X defined by*

$$A = \{0\} \times [0, 5] \text{ and } B = \{1\} \times [0, 5].$$

Obviously, $d(A, B) = 1$ and A, B are nonempty closed subsets of X. Moreover, it is easily seen that $A_0 = A$ and $B_0 = B$. Let $T : A \to B$ be the mapping defined as

$$T(0, x) = (1, \frac{1}{3}\ln(1 + x)), \quad \text{for all } (0, x) \in A.$$

Also, define $\alpha : X \times X \to \mathbb{R}$ as follows

$$\alpha(x, y) = \begin{cases} 1 & x, y \in A \\ 0 & \text{otherwise.} \end{cases}$$

In the sequel, we check that T is a generalized proximal α-h-ϕ- contraction type mapping. Define $h : X \times X \times X \times X \to [0, 1)$ and $\phi : [0, \infty) \to [0, \infty)$ by

$$h(x, y, u, v) = \begin{cases} \dfrac{1}{1 + \frac{1}{2}|x - y|} & x \neq y \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } \phi(t) = \frac{1}{2}t, \quad \text{for all } t \geq 0.$$

Then $h \in \mathcal{H}$ and $\phi \in \Phi$. If $x, y \in A$, then $t = d(x, y) \in [0, 5]$. Also, it is easy to show that

$$\frac{1}{6}(\ln(1 + t)) \leq \frac{t}{2 + t} \text{ for all } t \in [0, 5]. \tag{2.10}$$

Let $x = (0, x_0), y = (0, y_0), u = (0, u_0), v = (0, v_0) \in A$ satisfy the following conditions

$$\begin{cases} d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B). \end{cases}$$

Then $u_0 = \frac{1}{3}\ln(x_0+1)$ *and* $v_0 = \frac{1}{3}\ln(y_0+1)$. *Since* $f(t) = \dfrac{t}{2+t}$ *on* $[0,\infty)$ *is a nondecreasing function, thus from (2.10), we have*

$$\begin{aligned}\alpha(x,y)\phi(d(u,v)) &= \frac{1}{2}(\frac{1}{3}|\ln(\frac{1+x_0}{1+y_0})| \\ &\leq \frac{1}{2}(\frac{1}{3}\ln(1+|x_0-y_0|) \\ &\leq \frac{|x_0-y_0|}{2+|x_0-y_0|} \\ &= (\frac{1}{1+\frac{1}{2}d(x,y)})\frac{1}{2}d(x,y) \\ &\leq h(x,y,u,v)(\frac{1}{2}M_a(x,y,u,v)) \\ &= h(x,y,u,v)\phi(M_a(x,y,u,v)).\end{aligned}$$

Hence T is a generalized proximal α-h-ϕ-contraction type mapping. Obviously, other conditions of Theorem 2.1 are satisfied. Therefore T has a unique best proximity point. Note that $x^* = (0,0)$ is the best proximity point of T.

3. Particular cases

As applications of our results we consider some special cases and prove some several well-known best proximity point theorems.

Corollary 3.1. *Let A and B be two nonempty closed subsets of complete metric space (X,d), $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a mapping. Suppose that the following conditions are satisfied:*

(i) *There exist $\phi \in \Phi$ and $\beta \in \mathcal{F}$ such that for all $x,y,u,v \in A$,*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B)\end{array}\right\} \implies \alpha(x,y)\phi(d(u,v)) \leq \beta(\phi(M_a(x,y,u,v)))\phi(M_a(x,y,u,v));$$

(ii) *The conditions $(ii),(iii),(iv)$ and (v) of Theorem 2.1 are satisfied or the conditions $(ii),(v)$ of Theorem 2.1 are satisfied and T is continuous.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $\alpha(x,y) \geq 1$ for all $x,y \in P_T(A)$, then x^* is the unique best proximity point of T.

Proof. Define $h : X \times X \times X \times X \to [0,\infty)$ by

$$h(x,y,u,v) = \beta(\phi(M_a(x,y,u,v)), \quad \text{for all } x,y,u,v \in X.$$

suppose that $\{x_n\},\{y_n\},\{u_n\},\{v_n\} \subseteq X$ are such that $\lim_{n\to\infty} h(x_n,y_n,u_n,v_n) = 1$. Then

$$\lim_{n\to\infty}\phi(M_a(x_n,y_n,u_n,v_n) = 0.$$

Since ϕ is continuous and $\phi^{-1}\{0\} = 0$, then $\lim_{n\to\infty} M_a(x_n,y_n,u_n,v_n) = 0$. This implies that

$$\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(u_n,v_n) = 0. \tag{3.1}$$

Hence $h \in \mathcal{H}(X)$, and so by (3.1), we have

$$\alpha(x,y)\phi(d(u,v)) \leq h(x,y,u,v)\phi(M_a(x,y,u,v)), \text{ for all } x,y,u,v \in A.$$

Therefore T is generalized proximal α-h-ϕ-contraction type mapping and all hypotheses of Theorems 2.1 and 2.2 are satisfied. Thus T has a best proximity point $x^* \in X$. □

Corollary 3.2. *(*[22]*) Let A and B be two nonempty closed subsets of complete metric space (X,d), $\alpha : X \times X \to \mathbb{R}$ be a function, and let $T : A \to B$ be a mapping. Suppose that the following conditions are satisfied:*

(i) *There exist $\phi \in \Phi$ and $\beta \in \mathcal{F}$ such that for all $x,y,u,v \in A$,*

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \implies \alpha(x,y)\phi(d(u,v)) \leq \beta(\phi(M(x,y,u,v)))\phi(M(x,y,u,v)),$$

where

$$M(x,y,u,v) = \max\{d(x,y), d(x,u), d(y,v)\};$$

(ii) *The conditions $(ii),(iii),(iv)$ and (v) of Theorem 2.1 are satisfied or the conditions $(ii),(v)$ of Theorem 2.1 are satisfied and T is continuous.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*,Tx^*) = d(A,B).$$

Moreover, if $\alpha(x,y) \geq 1$ for all $x,y \in P_T(A)$, then x^* is the unique best proximity point of T.

Proof. Define $h : X \times X \times X \times X \to [0,\infty)$ by

$$h(x,y,u,v) = \beta(\phi(M(x,y,u,v)), \quad \text{for all } x,y,u,v \in X.$$

Similarly the proof of Corollary 3.1, we can show that all hypotheses of Theorem 2.1 and Theorem 2.2 are satisfied, which implies that T has a best proximity point $x^* \in X$. □

Corollary 3.3. *(*[27]*) Let (A,B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A,B) has the P-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*) = dist(A,B)$.*

Proof. Let $\phi(t) = t$, for all $t \in [0,\infty)$, and $\alpha(x,y) = 1$, for all $x,y \in X$. Define $h : X \times X \times X \times X \to [0,1)$, by

$$h(x,y,u,v) = \begin{cases} \dfrac{d(x,y) - \psi(d(x,y))}{d(x,y)} & if\ x \neq y, \\ 0 & if\ x = y. \end{cases} \tag{3.2}$$

Let $\{x_n\},\{y_n\},\{u_n\},\{v_n\} \subseteq X$ be such that sequence $\{d(x_n,y_n)\}$ is decreasing and $\lim_{n\to\infty} d(x_n,y_n) = r$. Suppose that $\lim_{n\to\infty} h(x_n,y_n,u_n,v_n) = 1$. We show that $\lim_{n\to\infty} d(x_n,y_n) = 0$. In the contrary case, let $\lim_{n\to\infty} d(x_n,y_n) = r > 0$. Since ψ is continuous, thus

$$\lim_{n\to\infty} h(x_n,y_n,u_n,v_n) = \lim_{n\to\infty} \frac{d(x_n,y_n) - \psi(d(x_n,y_n))}{d(x_n,y_n)} = \frac{r - \psi(r)}{r} = 1,$$

which implies that $\psi(r) = 0$, and so $r = 0$. This is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$. Now, let for some $x, y, u, v \in A$,

$$d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B).$$

The pair (A, B) has the P-property and T is a weakly contractive mapping which implies that

$$\begin{aligned} d(u, v) &= d(Tx, Ty) \\ &\leq d(x, y) - \psi(d(x, y)) \\ &= \frac{d(x, y) - \psi(d(x, y))}{d(x, y)} d(x, y) \\ &= h(x, y, u, v) d(x, y) \\ &\leq h(x, y, u, v) M_a(x, y, u, v). \end{aligned}$$

Therefore T is a continuous generalized proximal h-contraction type mapping. Since $A_0 \neq \emptyset$ and $T(A_0) \subseteq B_0$, therefore there exist $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$. Therefore all hypotheses of Corollary 2.3 are satisfied and so the mapping T has a unique best proximity point. □

Corollary 3.4. *([21]) Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a ψ-Geraghty contraction type mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

Proof. Let $\alpha(x, y) = 1$, for all $x, y \in X$. Define $h : X \times X \times X \times X \to [0, 1)$, by

$$h(x, y, u, v) = \begin{cases} \beta(\psi(d(x, y))) & if\ x \neq y, \\ 0 & if\ x = y. \end{cases} \tag{3.3}$$

Let $\{x_n\}, \{y_n\}, \{u_n\}, \{v_n\} \subseteq X$ be such that sequence $\{d(x_n, y_n)\}$ is decreasing and $\lim_{n\to\infty} d(x_n, y_n) = r$. Suppose that $\lim_{n\to\infty} h(x_n, y_n, u_n, v_n) = 1$. Then $\lim_{n\to\infty} \beta(\psi(d(x_n, y_n))) = 1$. This implies that $\lim_{n\to\infty} \psi(d(x_n, y_n)) = 0$. Therefore $\lim_{n\to\infty}(d(x_n, y_n)) = 0$. which implies that $\psi(r) = 0$, and so $r = 0$. This is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

This implies that $h \in \mathcal{H}(X)$. Now, let for some $x, y, u, v \in A$,

$$d(u, Tx) = d(A, B) \text{ and } d(v, Ty) = d(A, B).$$

Since the pair (A, B) has the P-property and T is a weakly contractive mapping , therefore

$$\begin{aligned} \psi(d(u, v)) &= \psi(d(Tx, Ty)) \\ &\leq \beta(\psi(d(x, y)))\psi(d(x, y)) \\ &= h(x, y, u, v)\psi(d(x, y)) \\ &\leq h(x, y, u, v)\psi(M_a(x, y, u, v)). \end{aligned}$$

Therefore T is a continuous generalized proximal α-ϕ-h-contraction type mapping. We can show that all hypotheses of Theorem 2.1 are satisfied. Therefore T has a unique best proximity point. □

Similarly we can prove next theorem

Corollary 3.5. *([12]) Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty contraction type mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = dist(A, B)$.*

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Badr Alqahtani,
Department of Mathematics, King Saud University, Riyadh, Saudi Arabia.
E-mail address: balqahtani1@ksu.edu.sa

Javad Hamzehnejadi,
Department of Mathematics University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: javad.math@pgs.usb.ac.ir

Erdal Karapinar,
Atilim University, Department of Mathematics,06836, İncek, Ankara, Turkey
&
China Medical University, Taiwan
E-mail address: erdalkarapinar@yahoo.com

Rahmatollah Lashkaripour,
Department of Mathematics University of Sistan and Baluchestan, Zahedan, Iran
E-mail address: lashkari@hamoon.usb.ac.ir