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ON THE GENERAL METHOD OF SUMMABILITY

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Abstract. This paper deals with the generalization of an absolute summability theorem of infinite series. A theorem concerned with $\varphi-\mid A,\beta;\delta\mid_k$ summability is proved. Also, some known results are deduced.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants K and M such that $Kc_n \leq b_n \leq Mc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums $s_n=\sum_{v=1}^{n} a_v$. The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n, \beta; \delta \mid_k$, $k\geq 1$, $\delta \geq 0$ and β is a real number, if (see [6])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)} \mid z_n - z_{n-1}\mid^k < \infty, \tag{1.1}$$

where (p_n) is a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to \infty \quad as \quad n\to\infty, \tag{1.2}$$

and

$$z_n=\frac{1}{P_n}\sum_{v=1}^{n} p_v s_v. \tag{1.3}$$

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is summable $\varphi - \mid A,\beta;\delta\mid_k$, $k\geq 1$, $\delta\geq 0$ and β is a real number, if

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k+k-1)} \mid \bar{\Delta} A_n(s)\mid^k < \infty, \tag{1.4}$$

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where

$$A(s) \sim \sum a_{nv} s_v, \quad A_n(s) = \sum_{v=1}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.5}$$

and

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.6}$$

For $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 1$, $\varphi - \mid A, \beta; \delta \mid_k$ summability reduces to $\mid A, p_n; \delta \mid_k$ summability (see [10]). For $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$ and $\delta = 0$, $\varphi - \mid A, \beta; \delta \mid_k$ summability reduces to $\mid A, p_n \mid_k$ summability (see [18]). Also, by taking $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, we get $|\bar{N}, p_n|_k$ summability (see [2]). Furthermore, by taking $\varphi_n = n$, $\beta = 1$, $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we obtain $|C, 1|_k$ summability (see [5]).

In [3], the following theorem concerned with $\left|\bar{N}, p_n\right|_k$ summability factors of an infinite series has already been proved.

Theorem 1.1. *Let (X_n) be a positive non-decreasing sequence and let there be sequences (γ_n) and (λ_n) such that*

$$|\Delta \lambda_n| \leq \gamma_n, \tag{1.7}$$

$$\gamma_n \to 0 \quad as \quad n \to \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} n \left|\Delta \gamma_n\right| X_n < \infty, \tag{1.9}$$

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty \tag{1.10}$$

are satisfied. If (p_n) is a sequence of positive numbers such that

$$P_n = O(n p_n) \quad as \quad n \to \infty, \tag{1.11}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \left|s_n\right|^k = O\left(X_m\right) \quad as \quad m \to \infty, \tag{1.12}$$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k, k \geq 1$, where $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

2. Main result

The purpose of this paper is to obtain a general theorem concerned with $\varphi - \mid A, \beta; \delta \mid_k$ summability by using almost increasing sequences. Before we give the generalization of Theorem 1.1, let us mention some notations. Let $A = (a_{nv})$ be a normal matrix, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad and \quad \bar{a}_{nv} = 0 \quad for \quad v > n \tag{2.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{2.2}$$

It is well-known that

$$A_n(s) = \sum_{v=1}^{n} a_{nv} s_v = \sum_{v=1}^{n} \bar{a}_{nv} a_v \tag{2.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=1}^{n} \hat{a}_{nv} a_v. \tag{2.4}$$

Recently, almost increasing sequences have been used to obtain some different absolute summability theorems (see [4], [7]-[8], [12]-[14], [16]-[17], [19]). Now, we prove the following theorem.

Theorem 2.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \ n = 0, 1, ..., \tag{2.5}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{2.6}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{2.7}$$

$$|\hat{a}_{n,v+1}| = O(v|\Delta_v(\hat{a}_{nv})|). \tag{2.8}$$

Let (X_n) be an almost increasing sequence and the sequences (γ_n) and (λ_n) satisfy the conditions (1.7)-(1.10). If the condition $\varphi_n p_n = O(P_n)$ and the conditions

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} |s_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{2.9}$$

$$\sum_{n=v+1}^{\infty} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| = O\left(\varphi_v^{\beta(\delta k+k-1)-k}\right) \tag{2.10}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\varphi - | A, \beta; \delta |_k$, $k \geq 1$, $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$, where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$.

Lemma 2.2. *([7]) Under the conditions on (X_n), (γ_n) and (λ_n) as taken in the statement of Theorem 2.1, the following conditions hold:*

$$n\gamma_n X_n = O(1) \quad as \quad n \to \infty, \tag{2.11}$$

$$\sum_{n=1}^{\infty} \gamma_n X_n < \infty. \tag{2.12}$$

3. Proof of Theorem 2.1

Let (M_n) denotes A-transform of the series $\sum a_n \lambda_n$, namely,

$$M_n = A_n(s\lambda) = \sum_{v=1}^{n} a_{nv}(s\lambda)_v = \sum_{v=1}^{n} \bar{a}_{nv} a_v \lambda_v,$$

by (2.3), where $(s\lambda)_n = \sum_{v=1}^{n} a_v \lambda_v$.

Now, using (2.4) and Abel's transformation, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\hat{a}_{nv}\lambda_v\right) \sum_{k=1}^{v} a_k + \hat{a}_{nn}\lambda_n \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \left(\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1}\right) s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \left(\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v\right) s_v + a_{nn}\lambda_n s_n \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v + a_{nn}\lambda_n s_n \\
&= M_{n,1} + M_{n,2} + M_{n,3}.
\end{aligned}$$

To prove that $\sum a_n \lambda_n$ is summable $\varphi - \mid A, \beta; \delta \mid_k$, by using (1.4), it is enough to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} \mid M_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3.$$

First, from Hölder's inequality, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\mid M_{n,1}\mid^k &\leq \sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|\,|s_v|\right)^k\\ &\leq \sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|s_v|^k\right)\\ &\quad\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}.\end{aligned}$$

Now, since

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv}-\hat{a}_{n,v+1}\\ &= \bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v}\end{aligned}\tag{3.1}$$

by (2.1) and (2.2), then we have

$$\begin{aligned}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| &= \sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\\ &= \sum_{v=0}^{n-1}a_{n-1,v}-a_{n-1,0}-\sum_{v=0}^{n}a_{nv}+a_{n0}+a_{nn}\\ &= \bar{a}_{n-1,0}-a_{n-1,0}-\bar{a}_{n0}+a_{n0}+a_{nn}\\ &\leq a_{nn}\end{aligned}\tag{3.2}$$

by using (2.1), (2.5) and (2.6). Hence, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\mid M_{n,1}\mid^k &\leq \sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|s_v|^k\right)\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|^k\,|s_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{\beta(\delta k+k-1)-k}|\lambda_v|^{k-1}|\lambda_v||s_v|^k.\end{aligned}$$

Since (X_n) is an almost increasing sequence and $|\lambda_n|X_n = O(1)$ by (1.10), then $|\lambda_n| = O\left(\frac{1}{X_n}\right) = O(1)$ and $|\lambda_n|^{k-1} = O(1)$. Hence, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,1} \mid^k &= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} \, |\lambda_v| \, |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k} |s_r|^k \\
&\quad + O(1)|\lambda_m| \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} |s_n|^k \\
&= O(1) \sum_{v=1}^{m-1} \gamma_v X_v + O(1)|\lambda_m| X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by using (1.7), (1.10), (2.9) and (2.12).

Again, using Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,2} \mid^k &\leq \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta\lambda_v| |s_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} v \, |\Delta_v(\hat{a}_{nv})| \, |\Delta\lambda_v| |s_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} v \, |\Delta_v(\hat{a}_{nv})| |\Delta\lambda_v| \, |s_v|^k \right) \\
&\quad \times \left(\sum_{v=1}^{n-1} v |\Delta_v(\hat{a}_{nv})| \gamma_v \right)^{k-1} .
\end{aligned}$$

Here, since (X_n) is an almost increasing sequence and $v\gamma_v X_v = O(1)$ by Lemma 2.2, then $v\gamma_v = O\left(\frac{1}{X_v}\right) = O(1)$. Also, by using (3.2), we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} v \left| \Delta_v(\hat{a}_{nv}) \right| \gamma_v |s_v|^k \right) \\
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} \left(\sum_{v=1}^{n-1} v \left| \Delta_v(\hat{a}_{nv}) \right| \gamma_v |s_v|^k \right) \\
&= O(1) \sum_{v=1}^{m} v\gamma_v |s_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)-k+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} v\gamma_v |s_v|^k \varphi_v^{\beta(\delta k+k-1)-k} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\gamma_v) \sum_{r=1}^{v} \varphi_r^{\beta(\delta k+k-1)-k} |s_r|^k \\
&\quad + O(1) m\gamma_m \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)-k} |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta\gamma_v| X_v + O(1) \sum_{v=1}^{m-1} \gamma_v X_v \\
&+ O(1) m\gamma_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by means of (1.7), (1.9), (2.7), (2.8), (2.9), (2.10), (2.11) and (2.12).

Finally, by using Abel's formula and same operations as in $M_{n,1}$, we get

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \mid M_{n,3} \mid^k &= \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} a_{nn}^k |\lambda_n|^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n} \right)^k |\lambda_n|^{k-1} |\lambda_n| |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)-k} |\lambda_n| \, |s_n|^k \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by using (1.7), (1.10), (2.7), (2.9) and (2.12).
This completes the proof of Theorem 2.1.

4. Conclusions

By taking (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $\beta = 1$, we get a theorem dealing with $|A, p_n, \delta|_k$ summability (see [15]). If we take, (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$ and $\delta = 0$, then we get a theorem dealing with $|A, p_n|_k$ summability (see [11]). Also, if we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$, $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.1. In this case, the condition (2.9) reduces to the condition (1.12). Also, when we take (X_n) as a positive non-decreasing sequence, $\varphi_n = n$, $\beta = 1$, $\delta = 0$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we have a result for $|C, 1|_k$ summability (see [9]).

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