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GLOBAL OPTIMAL SOLUTIONS FOR HYBRID GERAGHTY-SUZUKI PROXIMAL CONTRACTIONS

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Abstract. Best proximity point theorem furnishes sufficient conditions for the existence and computation of an approximate solution x that is optimal in the sense that the error $d(x, Tx)$ assumes the global minimum value $d(A, B)$. In this paper, in the setting of semi-preordered metric spaces, we introduce a new notion of hybrid weak Geraghty and Suzuki Type proximal contractions and establish certain best proximity point results for such contractions. Further, we deduce new fixed point results in semi-preordered metric spaces and discuss some illustrative examples to highlight the realized improvements. Presented theorems extend and improve certain well known results from the literature.

1. Introduction

Banach contraction principle states that a self-mapping f on a complete metric space (X, d) such that $d(fx, fy) \leq c\, d(x, y)$ for all $x, y \in X$, where $c \in [0, 1)$, has a unique fixed point. Since then, the Banach contraction principle has been generalized in several directions, see [14, 18, 19, 23, 29] and references cited therein.

On the other hand, Kirk [23] explored several significant generalizations of the Banach contraction principle to the case of non-self mappings. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is called a k-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in A$. Notice that k-contraction coincides with Banach contraction mapping if one take $A = B$.
Moreover, a non-self contraction mapping may not have a fixed point. In this case, it is quite natural to find an element x such that $d(x, Tx)$ is minimum, which implies that x and Tx are in close proximity to each other. Precisely, in light of the fact that $d(x, Tx)$ is at least $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$, we are interested in establishing the existence of an element x for which $d(x, Tx) = d(A, B)$. Such an element is designated as a best proximity point of the non-self-mapping T. Obviously, a best proximity point reduces to a fixed point if the considered mapping is a self-mapping.

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This research subject has attracted attention of many authors, as confirmed referring to [3, 4, 5, 7, 9, 12, 13, 15, 16, 17, 20, 24, 25, 26, 27, 33, 35]. It should be noted that best proximity point theorems furnish an approximate solution to the equation $Tx = x$, when T has no fixed point.

Here, we collect some notions and notations which will be used throughout the rest of this work. We denote by A_0 and B_0 the following sets:

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned} \tag{1.1}$$

In 2003, Kirk et al. [24] presented sufficient conditions for determining when the sets A_0 and B_0 are nonempty.

Let $\mathcal{F}$ denote the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to +\infty. \tag{1.2}$$

Definition 1.1 ([12]). *Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \mathcal{F}$ such that*

$$d(Tx, Ty) \leq \beta((d(x, y))\, d(x, y), \text{ for all } x, y \in A. \tag{1.3}$$

In [26], Raj defined the following definition.

Definition 1.2. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$\left.\begin{aligned} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{aligned}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2). \tag{1.4}$$

Recently, Zhang et al. [35] introduced the following notion which is weaker than P-property.

Definition 1.3. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,*

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2). \tag{1.5}$$

Existence of best proximity and fixed points in partially ordered metric spaces has been considered recently by many authors (see, [1, 2, 17] and references therein). In this paper, in the setting of semi-preordered metric spaces, we introduce a new notion of ordered hybrid weak Geraghty-Suzuki contraction mapping and establish some best proximity point theorems for such mappings. Further, we deduce new fixed point results in semi-preordered metric spaces and discuss some illustrative examples to highlight the realized improvements. Presented theorems extend and improve certain well known results from the literature.

2. Best Proximity Point Result

Let Θ denote the set of all functions $\theta : {R^+}^4 \to R^+$ satisfying:
(Θ_1) θ is continuous and increasing in all its variables;
(Θ_2) $\theta(t_1, t_2, t_3, t_4) = 0$ iff $t_1.t_2.t_3.t_4 = 0$.
Also for more details on Θ see [22].

We extend the above notion as follow which is suitable for best proximity point contractions.

Let (X,d) be a metric space and A and B are two subsets of X and $\Theta^\star$ denote the set of all functions $\theta^\star : [d(A,B),\infty)^4 \to R^+$ satisfying:
$(\Theta_1^\star)$ $\theta^\star$ is continuous and increasing in all its variables;
$(\Theta_2^\star)$ $\theta^\star(t_1,t_2,t_3,t_4)=0$ iff $t_1\vee t_2\vee t_3\vee t_4=d(A,B)$.

Let X be a nonempty set. A preorder $\preceq$ on X is a binary relation which is reflexive and transitive [3]. Let $(X,\preceq)$ be a preordered set and let $T:X\to X$ be a mapping. We say that T is non-decreasing if for each $x,y\in X$, $x\preceq y\Rightarrow Tx\preceq Ty$.

A semi-perorder $\preceq$ on X is a binary relation which is transitive. Let $(X,\preceq)$ be a semi-preordered set and $T:X\to X$ be a mapping. We say that T is non-decreasing if for each $x,y\in X$, $x\preceq y\Rightarrow Tx\preceq Ty$.

Definition 2.1. *Let $(X,d,\preceq)$ be a semi-preordered metric space. A non-self mapping $T:A\to B$ is said to be proximally non-decreasing if and only if*

$$\begin{cases} d(x_1,Ty_1)=d(A,B),\\ d(x_2,Ty_2)=d(A,B), \qquad \Longrightarrow x_1\preceq x_2,\\ y_1\preceq y_2,\end{cases}$$

where $x_1,x_2,y_1,y_2\in A$.

Definition 2.2. *Let A,B be two nonempty subsets of a semi-preordered metric space $(X,d,\preceq)$ where $A_0\neq\emptyset$. A mapping $T:A\to B$ is said to be a hybrid weak Geraghty-contraction if there exists $\beta\in\mathcal{F}$ and $\theta^\star\in\Theta^\star$ such that*

$$d(Tx,Ty)\le\beta\big(M(x,y)\big)M(x,y)+\gamma\big(N(x,y,\theta^\star)\big)N(x,y,\theta^\star)\ \text{ for all }\ \mathrm{x,y}\in \mathrm{A}\textit{with}\ \mathrm{x}\preceq \mathrm{y} \tag{2.1}$$

where $\gamma:[0,\infty)\to[0,\infty)$ is a bounded function,

$$M(x,y)=\max\Big\{d(x,y),\qquad \frac{d(x,Tx)+d(y,Ty)}{2}-d(A,B),\qquad \frac{d(x,Ty)+d(y,Tx)}{2}-d(A,B)\Big\}$$

and

$$N(x,y,\theta)=\theta^\star\Big(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\Big).$$

Note that a semi-preordered metric space $(X,d,\preceq)$ is called totally semi-preordered if for all $x,y\in X$ we have $x\preceq y$ or $y\preceq x$.

Theorem 2.3. *Let A and B be nonempty subsets of a semi-preordered metric space $(X,d,\preceq)$ such that A_0 is nonempty and complete. Assume that T is hybrid weak Geraghty-contraction mapping satisfying the following assertions:*

(i) *$T(A_0)\subseteq B_0$ and the pair (A,B) satisfies the weak P-property;*
(ii) *T is a proximally non-decreasing;*
(iii) *there exist elements x_0 and x_1 in A_0 such that,*

$$d(x_1,Tx_0)=d(A,B)\ \textit{and}\ x_0\preceq x_1;$$

(iv) *if $\{x_n\}$ is an increasing in A_0 such that $x_n\to x\in A_0$ as $n\to\infty$, then $x_n\preceq x$ for all $n\in\mathbb{N}$.*

Then there exists an x^* in A such that $d(x^*, Tx^*) = d(A, B)$. Moreover, best proximity point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

Proof. By (iii) there exist elements x_0 and x_1 in A_0 such that,

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1$$

On the other hand $T(A_0) \subseteq B_0$ so, there exists $x_2 \in A_0$ such that,

$$d(x_2, Tx_1) = d(A, B).$$

As T is proximally non-decreasing, so we have $x_1 \preceq x_2$. That is,

$$d(x_2, Tx_1) = d(A, B),\ \ x_1 \preceq x_2$$

Again, as $T(A_0) \subseteq B_0$, so there exists $x_3 \in A_0$ such that,

$$d(x_3, Tx_2) = d(A, B).$$

Thus we have,

$$d(x_2, Tx_1) = d(A, B),\ \ d(x_3, Tx_2) = d(A, B),\ \ x_1 \preceq x_2.$$

Similarly we have, $x_2 \preceq x_3$. Hence,

$$d(x_3, Tx_2) = d(A, B),\ \ x_2 \preceq x_3.$$

Finally, we deduce,

$$d(x_{n+1}, Tx_n) = d(A, B),\ \ x_n \preceq x_{n+1} \text{ for all } n \in \mathbb{N} \cup 0. \tag{2.2}$$

Since (A, B) has the weak P-property, we derive that

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \ \text{ for any } \ n \in \mathbb{N}. \tag{2.3}$$

Applying, (2.2) and (2.3) we obtain,

$$\begin{aligned}M(x_{n-1}, \quad x_n) &= \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)}{2} - d(A, B),\\ &\frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} - d(A, B)\Big\}\\ &\le \max\Big\{d(x_{n-1}, x_n),\\ &\frac{d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) + d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n)}{2} - d(A, B),\\ &\frac{d(x_{n-1}, x_{n+1}) + d(x_{n+1}, Tx_n) + d(x_n, Tx_{n-1})}{2} - d(A, B)\Big\}\\ &= \max\Big\{d(x_{n-1}, x_n),\\ &\frac{d(x_{n-1}, x_n) + d(A, B) + d(x_n, x_{n+1}) + d(A, B)}{2} - d(A, B),\\ &\frac{d(x_{n-1}, x_{n+1}) + d(A, B) + d(A, B)}{2} - d(A, B)\Big\}\\ &= \max\Big\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\Big\}\\ &\le \max\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\}\\ &\le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.\end{aligned}$$

Thus,

$$M(x_{n-1}, x_n) \le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{2.4}$$

Also,

$$\begin{aligned}N(x_{n-1}, x_n, \theta) \quad &= \theta^\star\Big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n),\\ &\qquad d(x_n, Tx_{n-1})\Big)\\ &= \theta^\star\Big(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\\ &, d(x_{n-1}, Tx_n), d(A, B)\Big) = 0.\end{aligned} \tag{2.5}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}), \tag{2.6}$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$. Therefore, we conclude that

$$d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0}). \tag{2.7}$$

That is, x_0 is best proximity point of T. For this we suppose $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$. In view of the fact that T is a hybrid weak Geraghty-contraction, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \\ &\leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) + \gamma(N(x_{n-1}, x_n, \theta^\star))N(x_{n-1}, x_n, \theta^\star) \\ &< M(x_{n-1}, x_n) + \gamma(N(x_{n-1}, x_n, \theta^\star))N(x_{n-1}, x_n, \theta^\star). \end{aligned} \tag{2.8}$$

From (2.4), (2.5) and (2.8) we deduce,

$$d(x_n, x_{n+1}) < \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Now if, $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ then,

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is a contradiction. Hence,

$$d(x_{n-1}, x_n) \leq M(x_{n-1}, x_n) \leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n),$$

and so

$$M(x_{n-1}, x_n) = d(x_{n-1}, x_n) \tag{2.9}$$

for all $n \in \mathbb{N}$. Now, by (2.8) we get,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n) \end{aligned} \tag{2.10}$$

for all $n \in \mathbb{N}$. Consequently, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence which is bounded below and so $\lim_{n\to\infty} d(x_n, x_{n+1}) := L$ exists. Let $L > 0$. Then, from (2.10), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1$$

for each $n \geq 1$, which implies

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1})) = 1.$$

On the other hand, since $\beta \in \mathcal{F}$, we conclude

$$L = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

Since, $d(x_n, Tx_{n-1}) = d(A, B)$ holds for all $n \in \mathbb{N}$ and (A, B) satisfies the weak P-property, so for all $m,\ n \in \mathbb{N}$ with $m < n$ we obtain, $d(x_m, x_n) \leq d(Tx_{m-1}, Tx_{n-1})$. Note that

$$\begin{aligned}
M(x_m, \quad x_n) &= \max\Big\{d(x_m,x_n), \frac{d(x_m,Tx_m)+d(x_n,Tx_n)}{2} - d(A,B), \\
&\frac{d(x_m,Tx_n)+d(x_n,Tx_m)}{2} - d(A,B)\Big\} \\
&\leq \max\Big\{d(x_m,x_n), \\
&\frac{d(x_m,x_{m+1})+d(x_{m+1},Tx_m)+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)}{2} - d(A,B), \\
&\frac{d(x_m,x_{n+1})+d(x_{n+1},Tx_n)+d(x_n,x_{m+1})+d(x_{m+1},Tx_m)}{2} - d(A,B)\Big\} \\
&= \max\Big\{d(x_m,x_n), \frac{d(x_m,x_{m+1})+d(x_n,x_{n+1})}{2}, d(x_m,x_{n+1})\Big\} \\
&\leq \max\Big\{d(x_m,x_n), \frac{d(x_m,x_{m+1})+d(x_n,x_{n+1})}{2}, d(x_m,x_n)+d(x_n,x_{n+1})\Big\}.
\end{aligned}$$

As $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have,

$$\lim_{m,n\to\infty} d(x_m,x_n) \leq \lim_{m,n\to\infty} M(x_m,x_n) \leq \lim_{m,n\to\infty} d(x_m,x_n),$$

that is,

$$\lim_{m,n\to\infty} M(x_m,x_n) = \lim_{m,n\to\infty} d(x_m,x_n). \tag{2.12}$$

Also,

$$\begin{aligned}
0 \leq N(x_m,x_n,\theta^\star) &= \theta^\star\Big(d(x_m,Tx_m), d(x_n,Tx_n), d(x_m,Tx_n), d(x_n,Tx_m)\Big) \\
&\leq \theta^\star\Big(d(x_m,x_{m+1})+d(x_{m+1},Tx_m), \\
&\quad d(x_n,Tx_n), d(x_m,Tx_n), d(x_n,Tx_m)\Big) \\
&= \theta^\star\Big(d(x_m,x_{m+1})+d(A,B), d(x_n,Tx_n), d(x_m,Tx_n), d(x_n,Tx_m)\Big)
\end{aligned} \tag{2.13}$$

and so, by $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$, we deduce,

$$\begin{aligned}
0 \leq \lim_{m,n\to\infty} N(x_m,x_n,\theta^\star) &\leq \lim_{m,n\to\infty} \theta^\star\Big(d(x_m,x_{m+1})+d(A,B), d(x_n,Tx_n), d(x_m,Tx_n), \\
&d(x_n,Tx_m)\Big) \\
&\leq \lim_{m,n\to\infty} \theta^\star\Big(d(A,B), d(x_n,Tx_n), d(x_m,Tx_n), d(x_n,Tx_m)\Big) = 0.
\end{aligned}$$

That is,

$$\lim_{m,n\to\infty} N(x_m,x_n,\theta^\star) = 0. \tag{2.14}$$

Now on the contrary, assume that

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n,x_m) > 0. \tag{2.15}$$

By using the triangle inequality, we can write

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{m+1}) + d(x_{m+1},x_m). \tag{2.16}$$

From (2.29) and (2.16) we have,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\le d(x_n, x_{n+1}) + \beta(M(x_n, x_m))M(x_n, x_m) \\ &+\gamma(N(x_n, x_m, \theta^\star))N(x_n, x_m, \theta^\star) + d(x_{m+1}, x_m). \end{aligned} \tag{2.17}$$

Now, (2.12), (2.14), (2.17) and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, imply

$$\begin{aligned} \lim_{m,n\to\infty} d(x_n, x_m) &\le \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} M(x_m, x_n) \\ &+ \lim_{m,n\to\infty} \gamma(N(x_n, x_m, \theta^\star)) \lim_{m,n\to\infty} N(x_m, x_n, \theta^\star) \\ &= \lim_{m,n\to\infty} \beta(M(x_n, x_m)) \lim_{m,n\to\infty} d(x_m, x_n). \end{aligned}$$

By (2.15) we get

$$1 \le \lim_{m,n\to\infty} \beta(M(x_n, x_m)).$$

Therefore, $\lim_{m,n\to\infty} \beta(M(x_n, x_m)) = 1$, and so $\lim_{m,n\to\infty} M(x_n, x_m) = 0$. This implies, $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since $(x_n) \subset A_0$ and (A_0, d) is a complete metric space, we can find $x^* \in A_0$ such that $x_n \to x^*$ as $n \to \infty$. From (iv) we know that, $x_n \preceq x$ for all $n \in \mathbb{N}$. We shall show that $d(x^*, Tx^*) = d(A, B)$. Suppose, to contrary that, $d(x^*, Tx^*) \ne d(A, B)$. From (2.29) with $x = x_n$ and $y = x^*$ we get,

$$d(Tx_n, Tx^*) \le \beta\big(M(x_n, x^*)\big)M(x_n, x^*) + \gamma\big(N(x_n, x^*, \theta)\big)N(x_n, x^*, \theta). \tag{2.18}$$

On the other hand,

$$\begin{aligned} M(x_n, x^*) &= \max\left\{d(x_n, x^*), \frac{d(x_n, Tx_n) + d(x^*, Tx^*)}{2} - d(A, B),\right. \\ &\left.\frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2} - d(A, B)\right\} \\ &\le \max\left\{d(x_n, x^*), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) + d(x^*, Tx^*)}{2} - d(A, B),\right. \\ &\left.\frac{d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n)}{2} - d(A, B)\right\} \\ &= \max\left\{d(x_n, x^*), \frac{d(x_n, x_{n+1}) + d(A, B) + d(x^*, Tx^*)}{2} - d(A, B),\right. \\ &\left.\frac{d(x_n, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+1}) + d(A, B)}{2} - d(A, B)\right\} \end{aligned}$$

and so,

$$\lim_{k\to\infty} M(x_n, x^*) \le \frac{d(x^*, Tx^*) - d(A, B)}{2}. \tag{2.19}$$

Also we have,

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, Tx_n) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + d(A, B) + d(Tx_n, Tx^*). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality we have,

$$d(x^*, Tx^*) - d(A, B) \leq \lim_{n\to\infty} d(Tx_n, Tx^*). \tag{2.20}$$

Further we get,

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A, B).$$

Taking limit as $n \to \infty$ in the above inequality we get

$$\lim_{n\to\infty} d(x_n, Tx_n) \leq d(A, B),$$

and so $\lim_{n\to\infty} d(x_n, Tx_n) = d(A, B)$. Now we have

$$\lim_{n\to\infty} N(x_n, x^*, \theta^*) =$$

$$= \theta^*\Bigg(\lim_{n\to\infty} d(x_n, Tx_n), d(x^*, Tx^*), \lim_{n\to\infty} d(x_n, Tx^*),$$

$$\lim_{n\to\infty} d(x^*, Tx_n)\Bigg)$$

$$= \theta^*\Bigg(d(A, B), d(x^*, Tx^*), \lim_{n\to\infty} d(x_n, Tx^*), \lim_{n\to\infty} d(x^*, Tx_n)\Bigg) = 0$$

that is,

$$\lim_{n\to\infty} N(x_n, x^*, \theta) = 0. \tag{2.21}$$

From (2.18), (2.19), (2.20) and (2.21) we deduce that

$$\begin{aligned} d(x^*, Tx^*) - d(A, B) &\leq \lim_{n\to\infty} d(Tx_n, Tx^*) \\ &\leq \lim_{n\to\infty} \beta(M(x_n, x^*)) \lim_{n\to\infty} M(x_n, x^*) \\ &+ \lim_{n\to\infty} \gamma(N(x_n, x^*, \theta^*)) \lim_{n\to\infty} N(x_n, x^*, \theta^*) \\ &= \lim_{n\to\infty} \beta(M(x_n, x^*))(\frac{d(x^*, Tx^*) - d(A, B)}{2}) \\ &< d(x^*, Tx^*) - d(A, B), \end{aligned} \tag{2.22}$$

which is a contradiction. Therefore, $d(x^*, Tx^*) = d(A, B)$, and x^* is a best proximity point of T. We now show the uniqueness of the best proximity point of T. Suppose that x^* and y^* are two distinct best proximity points of T. This implies

$$d(x^*, Tx^*) = d(A, B) = d(y^*, Ty^*). \tag{2.23}$$

Using the weak P-property, we have

$$d(x^*, y^*) \leq d(Tx^*, Ty^*), \tag{2.24}$$

which implies

$$
\begin{aligned}
M(x^*,y^*) = \max\Big\{&d(x^*,y^*), \frac{d(x^*,Tx^*)+d(y^*,Ty^*)}{2} - d(A,B),\\
&\frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2} - d(A,B)\Big\}\\
&= \max\Big\{d(x^*,y^*), 0, \frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2} - d(A,B)\Big\}\\
&\leq \max\Big\{d(x^*,y^*), 0,\\
&\frac{d(x^*,Tx^*)+d(Tx^*,Ty^*)+d(y^*,Ty^*)+d(Ty^*,Tx^*)}{2} - d(A,B)\Big\}\\
&\leq \max\Big\{d(x^*,y^*), 0,\\
&\frac{d(A,B)+d(x^*,y^*)+d(A,B)+d(y^*,x^*)}{2} - d(A,B)\Big\} = d(x^*,y^*).
\end{aligned}
$$

Also,

$$
\begin{aligned}
N(x^*,y^*,\theta^*) &= \theta\Big(d(x^*,Tx^*), d(y^*,Ty^*), d(x^*,Ty^*), d(y^*,Tx^*)\Big)\\
&= \theta^*\Big(d(A,B), d(A,B), d(x^*,Ty^*), d(y^*,Tx^*)\Big) = 0
\end{aligned}
$$

As T is hybrid weak Geraghty-contraction, we obtain

$$
\begin{aligned}
d(x^*,y^*) &\leq d(Tx^*,Ty^*)\\
&\leq \beta(M(x^*,y^*))M(x^*,y^*)\\
&+ \gamma(N(x^*,y^*,\theta^*))N(x^*,y^*,\theta^*)\\
&= \beta(d(x^*,y^*))d(x^*,y^*)\\
&\leq \beta(d(x^*,y^*))d(x^*,y^*) < d(x^*,y^*)
\end{aligned}
$$

which is a contradiction. This completes the proof of the theorem. □

If in Theorem 2.3 we take, $\beta(t) = r$ where $r \in [0,1)$ and $\gamma(t) = L$ where $L \geq 0$, then we obtain following best proximity point result.

Corollary 2.4. *Let (A,B) be a pair of nonempty subsets of a semi-preordered metric space $(X,d,\preceq)$ such that A_0 is nonempty and complete. Let $T : A \to B$ be non-self mapping satisfying $T(A_0) \subseteq B_0$ and*

$$d(Tx,Ty) \leq rM(x,y) + LN(x,y,\theta^*)$$

for all $x,y \in A$ with $x \preceq y$, where $r \in [0,1)$, $L \geq 0$, $\theta^* \in \Theta^*$. Suppose that the pair (A,B) has the weak P-property and following assertions hold:

(i) *T is a proximally non-decreasing;*
(ii) *there exist elements x_0 and x_1 in A_0 such that,*

$$d(x_1,Tx_0) = d(A,B) \text{ and } x_0 \preceq x_1;$$

(iii) *if $\{x_n\}$ is an increasing in A_0 such that $x_n \to x \in A_0$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists an x^* in A such that $d(x^*, Tx^*) = d(A, B)$. Moreover, best proximity point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

If in Corollary 2.4 we take, $\theta(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\} - d(A, B)$, we obtain following best proximity result.

Corollary 2.5. *Let (A, B) be a pair of nonempty subsets of a semi-preordered metric space $(X, d, \preceq)$ such that A_0 is nonempty and complete. Let $T : A \to B$ be non-self mapping satisfying $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq rM(x, y) + LN(x, y)$$

for all $x, y \in A$ with $x \preceq y$, where $r \in [0, 1)$, $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \quad \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \quad \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\right\}$$

and

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} - d(A, B).$$

Suppose that the pair (A, B) has the weak P-property and following assertions hold:

(i) *T is a proximally non-decreasing;*
(ii) *there exist elements x_0 and x_1 in A_0 such that,*

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1;$$

(iii) *if $\{x_n\}$ is an increasing in A_0 such that $x_n \to x \in A_0$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists an x^* in A such that $d(x^*, Tx^*) = d(A, B)$. Moreover, best proximity point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

Now we give an example to illustrate our results.

Example 2.6. *Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$ be a metric on X and $x \preceq y$ if and only if $(x, y) \in [0, 4] \times [0, 4]$. Clearly $\preceq$ is a semi-perorder. Indeed, if $x \preceq y$ and $y \preceq z$, then $(x, y), (y, z) \in [0, 4] \times [0, 4]$ and so, $x, y, z \in [0, 4]$. That is, $x \preceq z$. Also suppose $A = [-4, 4]$ and $B = (-\infty, -5] \cup [5, \infty)$ are two subsets of $\mathbb{R}$. Define $T : A \to B$ by*

$$Tx = \begin{cases} -5, & if x \in [-4, -3) \\ 30, & if x \in [-3, -2) \\ x^2 + 10 & if x \in [-2, -1) \\ x^4 + 14 & if x \in [-1, 0) \\ 5, & if x \in [0, 4] \end{cases}$$

, $\beta : (0, \infty) \to [0, 1)$ by $\beta(t) = \frac{t}{1+t}$, $\theta^* : [1, \infty)^4 \to R^+$ by $\theta^*(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\} - 1$ and $\gamma : [0, \infty) \to [0, \infty)$ by $\gamma(t) = 4$. Clearly, $d(A, B) = 1$, $A_0 = \{-4, 4\}$, $B_0 = \{-5, 5\}$ and $T(A_0) \subseteq B_0$. Further, the pair (A, B) satisfies the weak P-property but it does not satisfy P-property. Since, $d(4, -4) \neq d(5, -5)$ but $d(4, 5) = d(-4, -5) = d(A, B) = 1$. Now, if $x, y \in A$ with $x \preceq y$. Then, $x, y \in [0, 4]$. Hence,

$$d(Tx, Ty) \leq \beta\big(M(x, y)\big)M(x, y) + \gamma\big(N(x, y, \theta^*)\big)N(x, y, \theta^*)$$

holds true. That is, T is a hybrid weak Geraghty-contraction. Now, if,

$$\begin{cases} d(x_1, Ty_1) = d(A,B) = 1 \\ d(x_2, Ty_2) = d(A,B) = 1 \\ y_1 \preceq y_2, \end{cases}$$

then,

$$\begin{cases} d(x_1, Ty_1) = d(A,B) = 1 \\ d(x_2, Ty_2) = d(A,B) = 1 \\ y_1, y_2 \in [0,4], \end{cases}$$

and so,

$$\begin{cases} d(x_1, 5) = d(A,B) = 1 \\ d(x_2, 5) = d(A,B) = 1 \\ y_1, y_2 \in [0,4], \end{cases}$$

that is, $x_1 = x_2 = 4 \in [0,4]$. i.e., $x_1 \preceq x_2$. Hence, all conditions of Theorem 2.3 hold and T has a best proximity point. Here, $x = -4$ and $x = 4$ are two best proximity points of T, as the defined semi-preorder is not total.

Definition 2.7. *Let A, B be two nonempty subsets of a metric space (X, d) where $A_0 \neq \emptyset$. A mapping $T : A \to B$ is said to be a hybrid weak Geraghty-contraction if there exists $\beta \in \mathcal{F}$ and $\theta^* \in \Theta^*$ such that*

$$d(Tx, Ty) \leq \beta\big(M(x,y)\big)M(x,y) + \gamma\big(N(x,y,\theta^*)\big)N(x,y,\theta) \text{ for all } \mathrm{x, y \in A}, \tag{2.25}$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a bounded function,

$$M(x,y) = \max\bigg\{d(x,y), \qquad \frac{d(x,Tx) + d(y,Ty)}{2} - d(A,B), \\ \frac{d(x,Ty) + d(y,Tx)}{2} - d(A,B)\bigg\}$$

and

$$N(x,y,\theta^*) = \theta\bigg(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigg).$$

If in Theorem 2.3 we take $\preceq = X \times X$, then we deduce following Theorem.

Theorem 2.8. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty and complete. Let $T : A \to B$ be hybrid weak Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the weak P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

If in Theorem 2.8 we take, $\beta(t) = r$ where $r \in [0, 1)$ and $\gamma(t) = L$ where $L \geq 0$, then we obtain following best proximity point result.

Corollary 2.9. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d) such that A_0 is nonempty and (A_0, d) is a complete metric space. Let $T : A \to B$ be non-self mapping satisfying $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq rM(x,y) + LN(x,y,\theta^*)$$

for all $x, y \in A_0$, where $r \in [0, 1)$, $L \geq 0$, $\theta^* \in \Theta^*$. Suppose that the pair (A, B) has the weak P-property. Then there exists a unique point x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

If in Corollary 2.9 we take, $\theta(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}-d(A,B)$, we have following Corollary.

Corollary 2.10. *Let (A,B) be a pair of nonempty subsets of a metric space (X,d) such that A_0 is nonempty and (A_0,d) is a complete metric space. Let $T:A\to B$ be non-self mapping satisfying $T(A_0)\subseteq B_0$ and*

$$d(Tx,Ty)\leq rM(x,y)+LN(x,y)$$

for all $x,y\in A_0$, where $r\in[0,1)$, $L\geq 0$,

$$M(x,y)=\max\left\{d(x,y),\quad \frac{d(x,Tx)+d(y,Ty)}{2}-d(A,B),\right.$$
$$\left.\frac{d(x,Ty)+d(y,Tx)}{2}-d(A,B)\right\}$$

and

$$N(x,y)=\min\left\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right\}-d(A,B)$$

for all $x,y\in A$. Suppose that the pair (A,B) has the weak P-property. Then there exists a unique point x^* in A such that $d(x^*,Tx^*)=d(A,B)$.

Corollary 2.11. *(Zhang et al. [35]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) such that A_0 is nonempty. Let $T:A\to B$ be non-self mapping satisfying $T(A_0)\subseteq B_0$ and*

$$d(Tx,Ty)\leq rd(x,y)$$

for all $x,y\in A$, where $r\in[0,1)$. Suppose that the pair (A,B) has the weak P-property and A is complete. Then there exists a unique point x^* in A such that $d(x^*,Tx^*)=d(A,B)$.

Corollary 2.12. *(Theorem 9 of Suzuki [32]) Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d) such that A_0 is nonempty and (A_0,d) is a complete metric space. Let $T:A\to B$ be non-self mapping satisfying $T(A_0)\subseteq B_0$ and*

$$d(Tx,Ty)\leq r\left[\frac{d(x,Tx)+d(y,Ty)}{2}-d(A,B)\right] \tag{2.26}$$

for all $x,y\in A$ where $r\in[0,1)$. Suppose that the pair (A,B) has the weak P-property. Then there exists a unique point x^* in A such that $d(x^*,Tx^*)=d(A,B)$.

Corollary 2.13. *Let (A,B) be a pair of nonempty subsets of a complete metric space (X,d) such that A_0 is nonempty and (A_0,d) is a complete metric space. Let $T:A\to B$ be non-self mapping satisfying $T(A_0)\subseteq B_0$ and*

$$d(Tx,Ty)\leq r\left[\frac{d(x,Ty)+d(y,Tx)}{2}-d(A,B)\right] \tag{2.27}$$

for all $x,y\in A_0$ where $r\in[0,1)$. Suppose that the pair (A,B) has the weak P-property. Then there exists a unique point x^* in A such that $d(x^*,Tx^*)=d(A,B)$.

Also, if in Theorem 2.8 we take, $\beta(t)=\frac{1}{K+t}$, $\theta(t_1,t_2,t_3,t_4)=\min\{t_1,t_2,t_3,t_4\}$ and $\gamma(t)=\frac{1}{K+t}$ where $K>1$, then we deduce following result.

Corollary 2.14. *Let (A, B) be a pair of nonempty subsets of a complete metric space (X, d) such that A_0 is nonempty and (A_0, d) is a complete metric space. Let $T : A \to B$ be non-self mapping satisfying $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq \frac{M(x, y)}{K + M(x, y)} + \frac{N(x, y, \theta)}{K + N(x, y, \theta)} \tag{2.28}$$

for all $x, y \in A_0$ where $K > 1$, $\theta \in \Theta$,

$$M(x, y) = \max\left\{d(x, y), \quad \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \quad \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\right\}$$

and

$$N(x, y, \theta) = \theta\Big(d(x, Tx) - d(A, B), d(y, Ty) - d(A, B), d(x, Ty) - d(A, B), d(y, Tx) - d(A, B)\Big)$$

for all $x, y \in A$. Suppose that the pair (A, B) has the weak P-property. Then there exists a unique point x^* in A such that $d(x^*, Tx^*) = d(A, B)$.

Now we provide some Suzuki type best proximity results in the setting of semi-preordered metric spaces.

Definition 2.15. *Let A, B be two nonempty subsets of a semi-preordered metric space $(X, d, \preceq)$ where $A_0 \neq \emptyset$. A mapping $T : A \to B$ is said to be a ordered weak Geraghty-Suzuki contraction if there exists $\beta \in \mathcal{F}$ such that*

$$\frac{1}{2} d^*(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \beta(M(x, y))[M(x, y) - d(A, B)], \tag{2.29}$$

for all $x, y \in A$ with $x \preceq y$, where $d^*(x, y) = d(x, y) - d(A, B)$ and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}.$$

Theorem 2.16. *Let A and B two be nonempty subsets of a semi-preordered metric space $(X, d, \preceq)$ such that A_0 is nonempty and complete. Assume that T is ordered weak Geraghty-Suzuki contraction satisfying the following assertions:*

(i) *$T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the weak P-property;*
(ii) *T is a proximally non-decreasing;*
(iii) *there exist elements x_0 and x_1 in A_0 such that,*

$$d(x_1, Tx_0) = d(A, B) \text{ and } x_0 \preceq x_1;$$

(iv) *if $\{x_n\}$ is an increasing in A_0 such that $x_n \to x \in A_0$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists an x^* in A such that $d(x^*, Tx^*) = d(A, B)$. Moreover, best proximity point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

Proof. The argument goes along similar lines as the proof of Theorem 2.3 and main theorem of [31]. □

Definition 2.17. *Let (A, B) be a pair of nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a generalized Geraghty-Suzuki contraction if there exists $\beta \in \mathcal{F}$ such that*

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \beta(M(x, y))[M(x, y) - d(A, B)],$$

for all $x, y \in A$, where $d^*(x, y) = d(x, y) - d(A, B)$ and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}.$$

If in Theorem 2.16 we take $\preceq = X \times X$, then we deduce following Corollary.

Corollary 2.18. *(Main Theorem of [31]) Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a generalized Geraghty-Suzuki contraction such that $T(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

3. Fixed point results

As an application of our results established above, we deduce new fixed point theorems. By taking $A = B = X$ in Theorem 2.3, we obtain following fixed point results.

Theorem 3.1. *Let $(X, d, \preceq)$ be a complete semi-preordered metric space and $T : X \to X$ be a nondecreasing hybrid weak Geraghty-contraction mapping satisfying the following assertions:*

(i) *there exists x_0 in X such that, $x_0 \preceq Tx_0$;*
(ii) *if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists an x^* in X such that $x^* = Tx^*$. Moreover, fixed point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

Corollary 3.2. *Let $(X, d, \preceq)$ be a complete semi-preordered metric space and $T : X \to X$ be a nondecreasing mapping satisfying the following assertions:*

(i) *there exists x_0 in X such that, $x_0 \preceq Tx_0$;*
(ii) *if $\{x_n\}$ is an increasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*
(iii)
$$d(Tx, Ty) \leq rM(x, y) + LN(x, y)$$
for all $x, y \in X$ with $x \preceq y$,

where $r \in [0, 1)$, $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \quad \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \quad \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\right\}$$

and

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} - d(A, B).$$

Then there exists a fixed point of T in X. Moreover, fixed point of T is unique whenever $(X, d, \preceq)$ is a totally semi-preordered metric space.

Theorem 3.3. *Let* (X,d) *be a complete metric space. Assume that* $T : X \to X$ *is a self-mapping satisfying,*

$$d(Tx,Ty) \leq \beta(M(x,y))M(x,y) + \gamma(N(x,y,\theta))N(x,y,\theta) \tag{3.1}$$

for all $x, y \in X$*, where* $\gamma : [0,\infty) \to [0,\infty)$ *is a bounded function,* $\beta \in \mathcal{F}$*,* $\theta \in \Theta$*,*

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

and

$$N(x,y,\theta) = \theta\bigg(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigg).$$

for all $x, y \in X$*. Then* T *has a unique fixed point.*

Corollary 3.4. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be self-mapping such that*

$$d(Tx,Ty) \leq \frac{M(x,y)}{K+M(x,y)} + \frac{N(x,y)}{K+N(x,y)} \tag{3.2}$$

for all $x, y \in X$ *where* $K > 1$*,*

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \\ N(x,y) &= \min\left\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}. \end{aligned}$$

Then T *has a unique fixed point.*

Corollary 3.5. *(Theorem 2* [10]*, Theorem 2.4* [11]*) Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a self-mapping such that*

$$d(Tx,Ty) \leq rd(x,y) + L \cdot N(x,y)$$

for all $x, y \in X$*, where* $r \in [0,1)$*,* $L \geq 0$*, and*

$$N(x,y) = \min\left\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}.$$

Then T *has a unique fixed point.*

Finally, from our results, as a corollary, we obtain the next well known Zamfirescu theorem ([28],[34]).

Corollary 3.6. *(Zamfirescu* [34]*) Let* (X,d) *be a complete metric space and* $T : X \mapsto X$ *a map for which there exist real numbers* a, b *and* c *satisfying* $0 \leq a < 1, 0 \leq b, c < 1/2$ *such that for each pair* $x, y \in X$*, at least one of the following is true:*

(1) $d(Tx,Ty) \leq ad(x,y)$;

(2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$;

(3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T *has a unique fixed point* p *and the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by* $x_{n+1} = Tx_n, n = 0, 1, 2, ...$ *converges to* p*, for any* $x_0 \in X$*.*

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References

[1] Abkar A., Gabeleh M., *Best proximity points for cyclic mappings in ordered metric spaces*. J. Optim. Theory. Appl. **150(1)**, 188–193 (2011). doi:10.1007/s10957-011-9818-2

[2] Ravi P. Agarwal, N. Hussain and M.A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstract and Applied Analysis, vol. **2012**, Article ID 245872, 15 pp.

[3] A. Amini-Harandi, M. Fakhar, H. R. Hajisharifi, N. Hussain, *Some new results on fixed and best proximity points in preordered metric spaces*, Fixed Point Theory and Applications, **2013**, 2013:263.

[4] A. Amini-Harandi, N. Hussain and F Akbar, *Best proximity point results for generalized contractions in metric spaces*, Fixed Point Theory and Applications, 2013, 2013:**164**.

[5] J. Anuradha, P. Veeramani, *Proximal pointwise contraction*. Topol. Appl. 156, 2942-2948 (2009)

[6] M.A.Al-Thagafi, N.Shahzad, *Convergence and existence results for best proximity points*. Nonlinear Anal. 70,3665-3671 (2009)

[7] H. Aydi, E. Karapinar, I. M. Erhan, P. Salimi, *Best proximity points of generalized almost ψ-Geraghty contractive non-self-mappings*, Fixed Point Theory and Appl. 2014, 2014:**32**

[8] S.Sadiq Basha, *Best proximity point theorems generalizing the contraction principle*, Nonlinear Analysis, **74** (2011), 5844–5850.

[9] S.S.Basha, P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres*. J. Approx. Theory **103**, 119-129 (2000)

[10] V. Berinde, *Approximating fixed points of weak contractions using the Picard iterations*, Nonlinear Analysis Forum, **9** (2004), 43-53.

[11] V. Berinde, *Some remarks on fixed points theorem for Ćirić-type almost contractions*, Carpathian J. Math. **25**(2009) 157-162.

[12] J Caballero, J Harjani and K Sadarangani, *A best proximity point theorem for Geraghty-contractions*, Fixed Point Theory and Applications 2012, 2012: **231**

[13] A.A.Eldred, P. Veeramani, *Existence and convergence of best proximity points*. J. Math. Anal. Appl. **323**, 1001-1006 (2006)

[14] M. Geraghty, *On contractive mappings*. Proc. Am. Math. Soc. **40**, 604-608 (1973)

[15] N. Hussain, A. Latif, P. Salimi, *Best proximity point results for modified Suzuki $\alpha-\psi$-proximal contractions*, Fixed Point Theory and Appl., 2014, 2014:**10**

[16] N. Hussain, A. Latif, P. Salimi, *Best Proximity Point Results in G-Metric Spaces*, Abstract and Applied Analysis, Article ID 837943, 8 pages.

[17] N. Hussain, M. A. Kutbi, P.Salimi, *Best proximity point results for modified α-ψ-proximal rational contractions*, Abstract and Applied Analysis, Vol., **2013**, Article ID 927457.

[18] N. Hussain, M. A. Kutbi and P.Salimi, *Fixed point theory in α-complete metric spaces with applications*, Abstract and Applied Analysis, Vol. **2014**, Article ID 280817, 11 pp.

[19] N. Hussain, M. A. Kutbi, S. Khaleghizadeh and P. Salimi, *Discussions on recent results for α-ψ-contractive mappings*, Abstract and Applied Analysis, Vol. **2014**, Article ID 456482, 13 pp.

[20] E. Karapınar, *Best proximity points Of cyclic mappings*, Appl. Math. Lett., **25** (2012), No: 11, 1761-1766.
342–353.

[21] E. Karapınar, *Best proximity points of Kannan type cylic weak ϕ-contractions in ordered metric spaces*, Analele Stiintifice ale Universitatii Ovidius Constanta, **20**(2012), no:3, 51-64.

[22] P. Kumam, A. Roldán, *On existence and uniqueness of g-best proximity points under $(\varphi,\theta,\alpha,g)$-contractivity conditions and consequences*, Abstract and Applied Analysis, Volume **2014**, Article ID 234027, 14 pp.

[23] W.A. Kirk, *Contraction mappings and extensions, in: W.A. Kirk, B. Sims (Eds.)*, Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, Dordrecht, **2001**, pp. 1-34.

[24] W. A. Kirk, S. Reich, P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal.Optim. **24** (2003) 851-862.

[25] V. S. Raj, *A best proximity theorem for weakly contractive non-self mappings*. Nonlinear Anal. **74**, 4804-4808 (2011)

[26] V. S. Raj, *Banach's contraction principle for non-self mappings*,(preprint).

[27] V. S. Raj, P. Veeramani, *Best proximity pair theorems for relatively nonexpansive mappings*. Appl. Gen. Topol. **10**, 21-28 (2009)
[28] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.
[29] P.Salimi, E.Karapinar, *Suzuki-Edelstein type contractions via auxiliary functions*, Mathematical Problems in Engineering, **2013**, Article ID 648528.
[30] P.Salimi, A.Latif, N.Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory and Applications 2013, 2013:**151**.
[31] P. Salimi, P. Vetro, *A Best Proximity Point Theorem for Generalized Geraghty–Suzuki Contractions*, Bulletin of the Malaysian Mathematical Sciences Society, 2015, 1-12
[32] T. Suzuki, *The existence of best proximity points with the weak P-property*, Fixed Point Theory and Applications 2013, 2013:**259**.
[33] C. Vetro, *Best proximity points: convergence and existence theorems for p-cyclic mappings*. Nonlinear Anal. **73**, 2283-2291 (2010).
[34] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. (Basel), **23** (1972), 292-298
[35] J.Zhang, Y.Su, Q. Cheng, *A note on 'A best proximity point theorem for Geraghty-contractions'*, Fixed Point Theory and Applications, 2013, 2013: **99**.

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