

STRONG CONVERGENCE FOR SYMMETRIC RANDOM VARIABLES

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ABSTRACT. In this study we investigate the general strong law of large numbers for symmetric sequence of independent identically distributed random variables. Necessary and sufficient conditions for the SLLN are obtained.

1. INTRODUCTION AND PRELIMINARIES

Let $\{u_i, i \geq 1\}$ and $\{w_i, i \geq 1\}$ are two increasing sequences of positive reals such that

$$\lim_{i \rightarrow \infty} u_i = \infty \quad \text{and} \quad \left\{ \frac{w_i}{u_i}, i \geq 1 \right\} \text{ is nondecreasing sequence.} \quad (1.1)$$

Set $T(1) = \{n, w_n \leq 2\}$ and $T(t) = \{n, 2^{t-1} < w_n \leq 2^t\}, t \geq 2$. Which $T(t), t \geq 1$ are mutually exclusive sets. Then (1.1) implies that $0 < w_i \nearrow \infty$ and $\bigcup_{t=1}^{\infty} T(t) = \{1, 2, 3, \dots, i, \dots\}$. Hence, there exists positive itegers $l_i, t_i, i \geq 1$ such that $l_1 < l_2 < \dots < l_i < \dots, t_1 < t_2 < \dots < t_i < \dots, \bigcup_{i=1}^{\infty} T(t_i) = \{1, 2, 3, \dots, i, \dots\}$ and $T(t_1) = \{1, \dots, l_1\}, T(t_i) = \{l_{i-1} + 1, \dots, l_i\}, i \geq 2$.

In this paper, we examine the relationship between general strong law of large numbers [1, 2, 3, 4, 5] and SLLN to a sequence of symmetric i.i.d random variables.

Lemma 1.1. *There exists two functions $f(\cdot)$ and $g(\cdot)$ which are continuous and increasing on $[0, \infty)$ such that*

$$\lim_{m \rightarrow \infty} f(m) = \infty \quad \text{and} \quad \frac{g(\cdot)}{f(\cdot)} \text{ is a nondecreasing function.} \quad (1.2)$$

and

$$f(0) = g(0) = 0, \quad f(i) = u_i, \quad g(i) = w_i, \quad i \geq 1. \quad (1.3)$$

Proof. Assume $u_0 = w_0 = 0$. Let

$$f(m) = u_{i-1} + (u_i - u_{i-1})(m - i + 1), \quad i - 1 \leq m < i, \quad i \geq 1$$

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and

$$g(m) = w_{i-1} + (w_i - w_{i-1})(m - i + 1), \quad i - 1 \leq m < i, \quad i \geq 1.$$

Obviously, $f(\cdot)$ and $g(\cdot)$ are increasing and continuous functions defined on $[0, \infty)$ satisfying (1.3).

By (1.1) for $i - 1 < m < i$ and $i \geq 1$, we have

$$\begin{aligned} \left(\frac{g(m)}{f(m)} \right)' &= \frac{g'(m)f(m) - g(m)f'(m)}{f^2(m)} \\ &= \frac{u_{i-1}w_i - w_{i-1}u_i}{f^2(m)} = \frac{u_{i-1}u_i \left(\frac{w_i}{u_i} - \frac{w_{i-1}}{u_{i-1}} \right)}{f^2(m)} \geq 0, \end{aligned}$$

where $\frac{w_0}{u_0} \simeq \frac{w_1}{u_1}$. Then (1.2) holds. \square

Lemma 1.2. *Let $\{\xi_n\}$ is a sequence of symmetric random variables. For every l , set $S_l = \sum_{n=1}^l \xi_n$. Then for each integer M and any $m > 0$,*

$$P \left(\max_{l \leq M} |S_l| > m \right) \leq 2P(|S_M| > m).$$

Proof. Assume $\rho = \inf\{l \leq M, |S_l| > m\}$. We have $P(|S_N| > m) = \sum_{l=1}^M P(|S_M| > m, \rho = l)$.

Since for any l , $\{\rho = l\}$ depends on ξ_1, \dots, ξ_l , we get $P(|S_N| > m) = \sum_{l=1}^M P(|S_l - D_l| > m, \rho = l)$ where $D_l = S_M - S_l$, $l \leq M$. By the triangle inequality

$$\begin{aligned} 2|S_l| &= |S_l + D_l + S_l - D_l| < |S_l + D_l| + |S_l - D_l| \\ &= |S_M| + |S_l - D_l| \end{aligned}$$

Hence

$$2P(|S_M| > m) \geq \sum_{l=1}^M P(\rho = l) = P \left(\max_{l \leq M} |S_l| > m \right)$$

\square

Lemma 1.3. *Assume $\{\xi, i \geq 1\}$ is a sequence of symmetric and independent random variables. Let $l_0 = 0$. Then*

(i) *If*

$$\lim_{i \rightarrow \infty} \frac{\sum_{n=1}^i \xi_n}{u_i} = 0 \quad \text{a.s.} \quad (1.4)$$

Hence for each sequence $\{\beta_i\}$ of integers that

$$0 = \beta_0 < \beta_1 < \dots < \beta_i < \dots,$$

$$\sum_{i=1}^{\infty} P \left(\left| \sum_{n=\beta_{i-1}+1}^{\beta_i} \xi_n \right| > \varepsilon u_{\beta_i} \right) < \infty, \quad \forall \varepsilon > 0. \quad (1.5)$$

(ii)

$$\lim_{i \rightarrow \infty} \frac{\sum_{n=1}^i \xi_n}{w_i} = 0 \quad \text{a.s.} \quad (1.6)$$

if and only if

$$\sum_{i=1}^{\infty} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \xi_n \right| > \varepsilon w_i \right) < \infty, \quad \forall \varepsilon > 0. \quad (1.7)$$

Proof. First we prove (i). Obviously by (1.4) we can get

$$\frac{\left| \sum_{n=\beta_{i-1}+1}^{\beta_i} \xi_n \right|}{u_{\beta_i}} \leq \frac{\left| \sum_{n=1}^{\beta_i} \xi_n \right|}{u_{\beta_i}} + \left(\frac{u_{\beta_i-1}}{u_{\beta_i}} \right) \frac{\sum_{n=1}^{\beta_i-1} \xi_n}{u_{\beta_i-1}} \rightarrow 0 \quad \text{a.s., } i \rightarrow \infty \quad (1.8)$$

By using the Borel-Cantelli Lemma, (1.8) implies (1.5). Now we prove part (ii). By (i) it suffices to prove that (1.7) implies (1.6). By Lemma 1.2 for every $i \geq 1$, we have

$$P \left(\max_{l_{i-1} < l \leq l_i} \left| \sum_{n=l_{i-1}+1}^l \xi_i \right| > \varepsilon w_l \right) \leq 2P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \xi_n \right| > \varepsilon w_l \right) \rightarrow 0 \quad \forall \varepsilon > 0.$$

Then by (1.7) we have

$$\sum_{i=1}^{\infty} P \left(\max_{l_{i-1} < l \leq l_i} \left| \sum_{n=l_{i-1}+1}^l \xi_i \right| > \varepsilon w_l \right) < \infty, \quad \forall \varepsilon > 0.$$

Hence

$$L = \frac{\max_{l_{i-1} < l \leq l_i} \left| \sum_{n=l_{i-1}+1}^l \xi_n \right|}{w_l} \rightarrow 0 \quad \text{a.s.} \quad (1.9)$$

using (1.9) we get

$$\begin{aligned} \max_{l_{i-1} < l \leq l_i} \frac{\left| \sum_{n=1}^l \xi_n \right|}{w_l} &\leq 2 \max_{l_{i-1} < l \leq l_i} \frac{\left| \sum_{n=1}^l \xi_n \right|}{w_{l_i}} \\ &\leq 2 \left(\sum_{\beta=1}^{i-1} \frac{\left| \sum_{n=l_{\beta-1}+1}^{l_{\beta}} \xi_n \right|}{w_{l_i}} + L_i \right) \\ &\leq 2 \sum_{\beta=1}^i \left(\frac{w_{l_{\beta}}}{w_{l_i}} \right) L_{\beta} \\ &\leq 2 \sum_{\beta=1}^i \left(\frac{2^{t_{\beta}}}{2^{t_i}} \right) L_{\beta} \xrightarrow{\text{a.s.}} 0, \quad i \rightarrow \infty. \end{aligned}$$

Then (1.6) holds. \square

Lemma 1.4. Assume $\{\xi_i, i \geq 1\}$ is a sequence of random variables, such that $|\xi_i| < w, i \geq 1$. Let $f(\cdot)$ and $g(\cdot)$ are increasing and continuous functions on $[0, \infty)$ such that $f(0) = g(0) = 0$ and satisfying (1.2). Set $\frac{f(0)}{g(0)} = \lim_{m \rightarrow 0^+} \frac{f(m)}{g(m)}$. For $i \geq 1$, we define $u_i = f(i)$ and $w_i = g(i)$. Assume $\{Y, Y_i, i \geq 1\}$ is a Bernoulli sequence such that is independent of $\{\xi_i, i \geq 1\}$. Then for any $i \geq 1$ and each

$m \geq 0$,

$$P\left(\left|\sum_{n=1}^i Y_n \xi_n\right| > mw_i\right) \leq 2P\left(\left|\sum_{n=1}^i Y_n f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}\right| > mu_i\right) \quad (1.10)$$

We have $f(g^{-1}(|0|)) \frac{0}{|0|} = 0$ since $\lim_{\xi \rightarrow 0} f(g^{-1}(|\xi|)) \frac{\xi}{|\xi|} = 0$.

Proof. Obviously, for any $i \geq 1$ and each $m \geq 0$,

$$\begin{aligned} & P\left(\left|\sum_{n=1}^i Y_n \xi_n\right| > mw_i\right) \\ &= P\left(\left|\sum_{n=1}^i \left(\frac{u_i}{w_i} \cdot \frac{g(g^{-1}(|\xi|))}{f(g^{-1}(|\xi|))}\right) Y_n \frac{f(g^{-1}(|\xi_n|))}{g(g^{-1}(|\xi_n|))} \xi_n\right| > mu_i\right) \\ &= P\left(\left|\sum_{n=1}^i \left(\frac{f(i)}{g(i)} \cdot \frac{g(g^{-1}(|\xi_n|))}{f(g^{-1}(|\xi_n|))}\right) Y_n f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}\right| > mu_i\right) \end{aligned}$$

Since $f(\cdot)$ and $g(\cdot)$ are increasing and continuous functions on $[0, \infty)$ such that $f(0) = g(0) = 0$ and (1.1) holds. Then $f^{-1}(\cdot)$ is a increasing and continuous function on $[0, \infty)$ and we have $g^{-1}(0) = 0$, $\lim_{m \rightarrow \infty} g^{-1}(m) = \infty$, and $0 \leq \frac{g(g^{-1}(m))}{f(g^{-1}(m))} \leq \frac{g(i)}{f(i)} = \frac{w_i}{u_i}$ if $0 \leq m \leq g(i) = w_i$ we have $|\xi| \leq w_i = g(i)$, $i \geq 1$.

Then it implies that, for any $i \geq 1$,

$$0 \leq \frac{g(g^{-1}(|\xi|))}{f(g^{-1}(|\xi_n|))} \leq \frac{g(i)}{f(i)}, \quad n = 1, 2, \dots, i$$

and for each $i \geq 1$,

$$0 \leq \frac{f(i)}{g(i)} \cdot \frac{g(g^{-1}(|\xi_n|))}{f(g^{-1}(|\xi_n|))} \leq 1, \quad n = 1, 2, \dots, i.$$

By using Lemma 1.1 we conclude for any $i \geq 1$ and each $m \geq 0$,

$$\begin{aligned} & P\left(\left|\sum_{n=1}^i \left(\frac{f(i)}{g(i)} \cdot \frac{g(g^{-1}(|\xi_n|))}{f(g^{-1}(|\xi_n|))}\right) Y_n f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}\right| > mu_i\right) \\ & \leq 2P\left(\left|\sum_{n=1}^i Y_n f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}\right| > mu_i\right). \end{aligned}$$

□

Lemma 1.5. *Let $\{\xi_i, i \geq 1\}$ is a sequence of independent and symmetric random variables. Suppose $f(\cdot)$ and $g(\cdot)$ are increasing and continuous functions on $[0, \infty)$ such that $f(0) = g(0) = 0$ and (1.2) holds, then for each $i \geq 1$ and every $m > 0$,*

$$\begin{aligned} P\left(\left|\sum_{n=1}^i \xi_n\right| > mw_i\right) & \leq 4P\left(\left|\sum_{n=1}^i f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}\right| > mu_i\right) \\ & \quad + \sum_{n=1}^i P(|\xi_n| > w_i). \end{aligned} \quad (1.11)$$

Proof. For each $i \geq 1$, let

$$\begin{aligned}\xi_{i,n} &= \xi_i I(|\xi_n| \leq w_i), & M_n &= f(g^{-1}(|\xi_n|)) \frac{\xi_n}{|\xi_n|}, \\ M_{i,n} &= f(g^{-1}(|\xi_{i,n}|)) \frac{\xi_{i,n}}{|\xi_{i,n}|}, & n &= 1, \dots, i.\end{aligned}$$

Hence, for each $m \geq 0$,

$$P\left(\left|\sum_{n=1}^i \xi_n\right| > mw_i\right) \leq P\left(\left|\sum_{n=1}^i \xi_{i,n}\right| > mw_i\right) + \sum_{n=1}^i P(|\xi_n| > w_i). \quad (1.12)$$

Obviously, we have

$$\begin{aligned}\{|\xi_n| \leq w_i\} &= \{g^{-1}(|\xi_n|) \leq i\} = \{f(g^{-1}(|\xi_n| \leq u_i))\} \\ &= \{|M_n| \leq u_i\} \quad n = 1, \dots, i.\end{aligned}$$

Then, we have

$$M_{i,n} = M_n I\{|M_n| \leq u_i\} \quad n = 1, \dots, i.$$

Since $\{\xi_n, n \geq 1\}$ is a sequence of symmetric and independent random variables $\{\xi_{i,n}, n = 1, \dots, i\}$, $\{M_n, n = 1, \dots, i\}$, and $\{M_{i,n}, n = 1, \dots, i\}$ are symmetric and independent random variables. Assume $\{Y_i, i \geq 1\}$ is a Bernoulli sequence such that is independent of $\{\xi_i, i \geq 1\}$. Hence $\{Y_n \xi_{i,n}, n = 1, \dots, i\}$ and $\{\xi_{i,n}, n = 1, \dots, i\}$ have the same distribution and $\{Y_n M_{i,n}, n = 1, \dots, i\}$ and $\{M_{i,n}, n = 1, \dots, i\}$ have the same distribution. Since $|\xi_{i,n}| \leq w_i, n = 1, \dots, i$, by using (1.10), we obtain that, for each $m \geq 0$,

$$\begin{aligned}P\left(\left|\sum_{n=1}^i \xi_{i,n}\right| > mw_i\right) &= E\left\{P\left(\left|\sum_{n=1}^i Y_n \xi_{i,n}\right| > mw_i \mid \xi_1, \dots, \xi_i\right)\right\} \\ &\leq 2E\left\{P\left(\left|\sum_{n=1}^i Y_n M_{i,n}\right| > mu_i \mid \xi_1, \dots, \xi_i\right)\right\} \quad (1.13) \\ &= 2P\left(\left|\sum_{n=1}^i Y_n M_{i,n}\right| > mu_i\right) \\ &= 2P\left(\left|\sum_{n=1}^i M_{i,n}\right| > mu_i\right)\end{aligned}$$

Since $\{M_n, n = 1, \dots, i\}$ for every $i \geq 1$, is a sequence of symmetric and independent random variables, then

$$\{M_n I\{|M_n| \leq u_i\} - M_n I\{|M_n| > u_i\}, \quad n = 1, \dots, i\}$$

and $\{M_n, n = 1, \dots, i\}$ have the same distributions.

Indeed,

$$\sum_{n=1}^i M_{i,n} = \frac{\sum_{n=1}^i M_n + \sum_{n=1}^i (M_n I\{|M_n| \leq u_i\} + M_n I\{|M_n| > u_i\})}{2}, \quad i \geq 1.$$

Hence, for each $i \geq 1$ and every $m \geq 0$,

$$\begin{aligned} P\left(\left|\sum_{n=1}^i M_{i,n}\right| > mu_i\right) &\leq P\left(\left|\sum_{n=1}^i M_n\right| > mu_i\right) \\ &+ P\left(\left|\sum_{n=1}^i M_n I\{|M_n| < u_i\} - M_n I\{|M_n| > u_i\}\right| > mu_i\right) \\ &= 2P\left(\left|\sum_{n=1}^i M_n\right| > mu_i\right) \end{aligned} \quad (1.14)$$

using (1.12), (1.13) and (1.14) we see that (1.11) holds. \square

Lemma 1.6 ([6]). *Suppose $\{w_i, i \geq 1\}$ is a sequence of positive reals such that $\lim_{i \rightarrow \infty} w_i = \infty$. Assume $\{\xi, \xi_i, i \geq 1\}$ is a sequence of i.i.d random variables. Suppose $\{\xi'_i, i \geq 1\}$ is an independent copy of $\{\xi_i, i \geq 1\}$. Set*

$$S_i = \sum_{n=1}^i \xi_n, \quad S'_i = \sum_{n=1}^i \xi'_n, \quad i \geq 1.$$

Then

$$\lim_{i \rightarrow \infty} \frac{S_i - iE(\xi I\{|\xi| \leq w_i\})}{w_i} = 0 \quad \text{a.s.}$$

if and only if

$$\lim_{i \rightarrow \infty} \frac{S_i - S'_i}{w_i} = 0 \quad \text{a.s.}$$

2. MAIN RESULTS

The following theorem is our main result.

Theorem 2.1. *Assume $\{u_i, i \geq 1\}$ and $\{w_i, i \geq 1\}$ are two increasing sequence of positive reals satisfying (1.1). Let $\{\xi, \xi_i, i \geq 1\}$ is a symmetric sequence of i.i.d random variables, such that*

$$\lim_{i \rightarrow \infty} \frac{S_i}{u_i} = 0 \quad \text{a.s.} \quad \text{if} \quad \sum_{i=1}^{\infty} P(|\xi| > u_i) < \infty \quad (2.1)$$

Let

$$\Delta_i = iE(\xi I\{|\xi| < w_i\}), \quad i \geq 1,$$

then, for each sequence $\{\xi, \xi_i, i \geq 1\}$ of i.i.d random variables

$$\lim_{i \rightarrow \infty} \frac{S_i - \Delta_i}{w_i} = 0 \quad \text{a.s.} \quad \text{or} \quad \limsup_{i \rightarrow \infty} \frac{S_i - \Delta_i}{w_i} = \infty \quad \text{a.s.} \quad (2.2)$$

Corresponding to

$$\sum_{i=1}^{\infty} P(|\xi| > w_i) < \infty \quad \text{or} \quad = \infty \quad (2.3)$$

Proof. It suffices to show that, the following are equivalent:

$$\lim_{i \rightarrow \infty} \frac{S_i - \Delta_i}{w_i} = 0 \quad \text{a.s.}, \quad (2.4)$$

$$\limsup_{i \rightarrow \infty} \frac{S_i - \Delta_i}{w_i} < \infty \quad \text{a.s.}, \quad (2.5)$$

$$\sum_{i=1}^{\infty} P(|\xi| > w) < \infty. \quad (2.6)$$

We first show that (2.4) \Leftrightarrow (2.6).

Suppose $\{\xi, \xi_i, i \geq 1\}$ is a sequence of i.i.d random variables such that (2.4) holds. By (2.4) and Lemma 1.6, we have

$$\lim_{i \rightarrow \infty} \frac{\sum_{n=1}^i (\xi_n - \xi'_n)}{w_i} = 0 \quad \text{a.s.}$$

if follows that

$$\frac{\xi_i - \xi'_i}{w_i} \rightarrow 0 \quad \text{a.s.} \quad (2.7)$$

Hence, by the Borel-Cantelli Lemma, (2.7) implies that

$$\sum_{i=1}^{\infty} P(|\xi - \xi'| > \varepsilon w_i) = \sum_{i=1}^{\infty} P(|\xi_i - \xi'_i| > \varepsilon w_i) < \infty, \quad \forall \varepsilon > 0, \quad (2.8)$$

we have

$$\left\{ |\xi'| < \frac{w_i}{2}, |\xi| > w_i \right\} \subseteq \left\{ |\xi - \xi'| > \frac{w_i}{2} \right\}$$

and $\lim_{i \rightarrow \infty} P(|\xi'| \leq \frac{w_i}{2}) = 1$.

Then for all large i , we can see that

$$P(|\xi| > w_i) \leq 2P(|\xi - \xi'| > \frac{w_i}{2})$$

Thus (2.8), implies (2.6). This implies that (2.4) \Rightarrow (2.6).

We now show that (2.6) \Rightarrow (2.4).

Let $\{\xi, \xi_i, i \geq 1\}$ is a sequence of i.i.d random variables such that (2.6) holds. Since $\{u_i, i \geq 1\}$ and $\{w_i, i \geq 1\}$ are increasing sequences of positive reals such that (1.1) holds, by 1.1 Lemma, we can find two increasing and continuous functions $f(\cdot)$ and $g(\cdot)$ defined on $[0, \infty)$ such that (1.2) and (1.3) holds. Set

$$\xi^s = \frac{\xi - \xi'}{2}, \quad \xi_i^s = \frac{\xi_i - \xi'_i}{2}, \quad i \geq 1$$

and

$$\eta = f(g^{-1}(|\xi^s|)) \frac{\xi^s}{|\xi^s|}, \quad \eta_i = f(g^{-1}(|\xi_i^s|)) \frac{\xi_i^s}{|\xi_i^s|}, \quad i \geq 1.$$

Hence $\{\xi^s, \xi_i^s, i \geq 1\}$ is a sequence of symmetric i.i.d random variables such that

$$\sum_{i=1}^{\infty} P(|\xi^s| > w_i) \leq 2 \sum_{i=1}^{\infty} P(|\xi| > w_i) < \infty \quad (2.9)$$

and $\{\eta, \eta_i, i \geq 1\}$ is a sequence of symmetric i.i.d random variables such that

$$\sum_{i=1}^{\infty} P(|\eta| > u_i) = \sum_{i=1}^{\infty} P(|\xi^s| > w_i) < \infty \quad (2.10)$$

Then, by Lemma 1.3 (i), (2.10) and (1.10), we have

$$\sum_{i=1}^{\infty} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \eta_n \right| > \varepsilon u_i \right) < \infty, \quad \varepsilon > 0 \quad (2.11)$$

By Lemma 1.5, for every $i \geq 1$, we get,

$$\begin{aligned} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \xi_n^s \right| > \varepsilon w_i \right) &\leq 4P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \eta_n \right| > \varepsilon u_i \right) + \sum_{n=l_{i-1}+1}^{l_i} P(|\xi_n^s| > w_i) \\ &\leq 4P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \eta_n \right| > \varepsilon u_i \right) + \sum_{n=l_{i-1}+1}^{l_i} P(|\xi_n^s| > w_n) \quad \forall \varepsilon > 0. \end{aligned}$$

By (2.10) and (2.11) we have

$$\begin{aligned} &\sum_{i=1}^{\infty} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \xi_i^s \right| > \varepsilon w_i \right) \\ &\leq 4 \sum_{i=1}^{\infty} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \eta_n \right| > \varepsilon u_i \right) + \sum_{i=1}^{\infty} \sum_{n=l_{i-1}+1}^{l_i} P(|\xi_n^s| > w_n) \quad (2.12) \\ &= 4 \sum_{i=1}^{\infty} P \left(\left| \sum_{n=l_{i-1}+1}^{l_i} \eta_n \right| > \varepsilon u_i \right) + \sum_{i=1}^{\infty} P(|\xi_n^s| > w_i) < \infty, \quad \forall \varepsilon > 0. \end{aligned}$$

By Lemma 1.3 (ii), (2.12) implies that

$$\frac{S_i - S'_i}{2w_i} = \frac{\sum_{n=1}^i \xi_n^s}{w_i} \rightarrow 0 \quad \text{a.s.}$$

Then

$$\frac{S_i - S'_i}{w_i} \rightarrow 0 \quad \text{a.s.}$$

By Lemma 1.6, (2.4) hold. Thus (2.6) \Rightarrow (2.4).

We prove that (2.4) \Leftrightarrow (2.5).

It is clearly seen that (2.4) \Rightarrow (2.5).

We now show that (2.5) \Rightarrow (2.4).

By (2.5) we have

$$\limsup_{i \rightarrow \infty} \frac{\left| \sum_{n=1}^i (\xi_n - \xi'_n) \right|}{w_i} < \infty \quad \text{a.s.}$$

Then

$$\limsup_{i \rightarrow \infty} \frac{|\xi_i - \xi'_i|}{w_i} < \infty \quad \text{a.s.} \quad (2.13)$$

By the Borel-Cantelli Lemma, for some constant $0 < \alpha < \infty$, (2.13) implies that $\sum_{i=1}^{\infty} P(|\xi_i - \xi'_i| > \alpha w_i) < \infty$; then $\sum_{i=1}^{\infty} P\left(\left|\frac{\xi_i - \xi'_i}{\alpha}\right| > w_i\right) < \infty$. That is, (2.6)

holds when ξ replaced by $\frac{\xi - \xi'}{\alpha}$. Since (2.4) and (2.6) are two equivalent statements, then

$$\left(\frac{1}{\alpha}\right) \frac{S_i - S'_i}{w_i} = \frac{\sum_{n=1}^i \frac{\xi_n - \xi'_n}{\alpha}}{w_i} \rightarrow 0 \quad \text{a.s.}$$

Hence

$$\frac{S_i - S'_i}{w_i} \rightarrow 0 \quad \text{a.s.}$$

Then, by Lemma 1.6, (2.4) holds.

The proof is complete. \square

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