A NEW THREE-STEP ITERATION METHOD FOR 
\(\alpha\)-NONEXPANSIVE MAPPINGS AND VARIATIONAL 
INEQUALITIES

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Abstract. In this paper, we find a common element for the set of fixed points 
of an \(\alpha\)-nonexpansive mapping and the set of solutions for variational inequal-
ities by some three-steps iteration scheme. Moreover, the strong convergence 
to common element of two sets under some constraints are established.

1. Introduction

Let \(K\) be a nonempty closed and convex set in a real Hilbert space, whose inner 
product and norm are denoted by \(\langle ., . \rangle\) and \(|.|\), respectively. Let \(T : K \rightarrow K\) be 
a nonlinear operator and \(S\) be a nonexpansive operator. Let \(P_K\) be the projection 
from \(H\) onto the convex set \(K\). A mapping \(T : K \rightarrow H\) is called monotone, if 
\(\langle Tu - Tv, u - v \rangle \geq 0\).

\(T\) is an \(\alpha\)-inverse strongly monotone, if there exists a positive real number \(\alpha\) such 
that 
\(\langle Tu - Tv, u - v \rangle \geq \alpha |Tu - Tv|^2,\)

for all \(u, v \in K\). A self mapping \(S\) on \(K\) is called nonexpansive if 
\(|Su - Sv| \leq |u - v|\) for all \(u, v \in K\).

We take into account the following problem:

Find \(u \in K\) such that 
\(\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K,\)  \hspace{1cm} (1.1)

which is introduced and studied by Stampacchia [9] in 1964. Iterative methods 
has been extensively studied by many authors (see [1, 2, 6] and the references therein). 
The following definitions and results from [7] are needed to go on:

Lemma 1.1. For each \(z \in H\), there exists \(u \in K\) such that 
\(\langle u - z, v - u \rangle \geq 0,\)

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for all \( v \in K \), if and only if, \( u = P_K[z] \) in which \( P_K \) is the projection from \( H \) onto the closed convex set \( K \).

Using Lemma 1.1, one can show that the variational inequality (1.1) is equivalent to a fixed-point problem.

**Lemma 1.2.** The function \( u \in K \) is a solution of the variational inequality (1.1), if and only if, \( u \in K \) satisfies the relation

\[
u = P_K[u - \rho Tu],
\]

where \( \rho > 0 \) is a constant.

It is clear from Lemma 1.2 that the variational inequalities and the fixed point problems are equivalent. This alternative equivalent formulation plays a significant role in the studies of variational inequalities and related optimization problems.

Let \( S \) be a nonexpansive mapping. We denote the set of all fixed points of \( S \) by \( F(S) \) and the set of all solutions of the variational inequalities (1.1) by \( VI(K,T) \).

We can characterize the problem. If \( x^* \in F(S) \cap VI(K,T) \), then by Lemma 1.2 we have

\[
x^* = Sx^* = P_K[x^* - \rho Tx^*] = SP_K[x^* - \rho Tx^*],
\]

where \( \rho > 0 \) is a constant.

This fixed point formulation is used to suggest the following iterative methods for finding a common element for two different sets of fixed points of nonexpansive mappings and solutions of the variational inequalities.

**Algorithm 1.3.** For given \( x_0 \in K \), consider the sequence \( x_n \) by the following iterative scheme:

\[
z_n = (1 - c_n)x_n + c_n SP_K[x_n - \rho T x_n], \tag{1.2}
\]

\[
y_n = (1 - b_n)x_n + b_n SP_K[z_n - \rho T z_n], \tag{1.3}
\]

\[
x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - \rho T y_n], \tag{1.4}
\]

where \( a_n,b_n,c_n \in [0,1] \), for all \( n \geq 0 \) and \( S \) is a nonexpansive operator. Algorithm 1.3 is a three-steps predictor-corrector method. For \( S = I \) (the identity operator) Algorithm 1.3 has been investigated by Noor [4].

Note that, for \( c_n \equiv 0 \), Algorithm 1.3 reduces as follows:

**Algorithm 1.4.** For an arbitrarily chosen initial point \( x_0 \in K \), consider the sequence \( x_n \) by the following iterative scheme:

\[
y_n = (1 - b_n)x_n + b_n SP_K[x_n - \rho T x_n],
\]

\[
x_{n+1} = (1 - a_n)x_n + a_n SP_K[y_n - \rho T y_n],
\]

where \( a_n,b_n \in [0,1] \), for all \( n \geq 0 \) and \( S \) is the nonexpansive operator. Algorithm 1.4 is called two-steps iterative method, which has been considered and studied by Huang and Noor [3]. For \( b_n \equiv 1, a_n \equiv 1 \), Algorithm 1.4 reduces as follows:

**Algorithm 1.5.** For an arbitrarily chosen initial point \( x_0 \in K \), consider the sequence \( x_n \) by the following iterative scheme:

\[
y_n = SP_K[x_n - \rho T x_n],
\]

\[
x_{n+1} = SP_K[y_n - \rho T y_n].
\]
Remark. It is worth mentioning that our Algorithm 1.3 is a two-steps method, which may be regarded as a predictor-corrector method. Moreover, Algorithm 1.4 covers the case in Algorithm 1.5 and Algorithm 1.5 can be written as

\[ x_{n+1} = P_K[P_K[x_n - \rho T x_n] - \rho TP_K[x_n - \rho T x_n]], \]

which is called extragradient Algorithm. We refer to [4], for more details about the convergence of Algorithm 1.3.

For \( b_n = 0, c_n = 0 \), Algorithm 1.3 is simplified in the following iterative method:

**Algorithm 1.6.** For given \( x_0 \in K \), consider the sequence \( x_n \) by the following iterative scheme:

\[ x_{n+1} = (1 - a_n) x_n + a_n SP_K[x_n - \rho T x_n], \tag{1.5} \]

which is known as one-step iteration method. We refer to Huang and Noor [4] for the convergence of Algorithm 1.6.

In particular, three-step method is quite general and includes several new and recent known algorithms for solving variational inequalities and nonexpansive mappings.

**Theorem 1.7.** [8 Theorem 3.1] Let \( K \) be a closed convex subset of a real Hilbert space \( H \). Let \( T : K \to H \) be an \( \alpha \)-inverse-monotone and let \( S \) be a nonexpansive mapping from \( K \) into itself such that \( F(S) \cap VI(K, T) \neq \emptyset \). Let \( u \in C \) be fixed and let \( x_0 \in C \) be given and let the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are generated by the following:

\[
\begin{align*}
  y_n &= \sigma_n x_n + (1 - \sigma_n)SP_K[x_n - \rho_n T x_n] \\
  z_n &= \mu_n x_n + (1 - \mu_n)SP_K[y_n - \rho_n T y_n] \\
  x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SP_K[z_n - \rho_n T z_n], \quad n \geq 0,
\end{align*}
\]

where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\} \) and \( \{\sigma_n\} \) are five sequences in \([0, 1]\) and \( \{\rho_n\} \) is a sequence in \([0, 2\alpha]\). As long as, \( \rho_n \in [a, b] \) in which \( 0 < a < b < 2\alpha \) and

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \),
(ii) \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=0}^{\infty} \alpha_n = \infty \)
(iii) \( 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \) and \( \lim_{n \to \infty} \mu_n = 1 \)
(iv) \( \lim_{n \to \infty}(\rho_{n+1} - \rho_n) = 0 \),

then \( \{x_n\} \) is defined by (1.6) which converges strongly to \( x^* \in P_{F(S) \cap VI(K, T)} u \).

2. Preliminaries

Gobel and Pineda [3], studied \( \alpha \)-nonexpansive mapping, which is a generalization of the nonexpansive one. Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), where \( \alpha_i \geq 0, i = 1, 2, \ldots, n \) and \( \sum_{i=1}^{n} \alpha_i = 1 \) and let \( K \) be a nonempty closed and convex subset of a Banach space \( X \). A mapping \( T : K \to K \) is said to be \( \alpha \)-nonexpansive if

\[
\sum_{i=1}^{n} \alpha_i \|T^i x - T^i y\| \leq \|x - y\|, \tag{2.1}
\]

for all \( x, y \in K \).
For some technical reasons we always assume that $\alpha_1 > 0$. In this case the mapping $T$ satisfies the Lipschitz condition
\[ \|Tx - Ty\| \leq \frac{1}{\alpha_1} \|x - y\|, \]
for all $x, y \in K$.

Taking into account (2.1), one can investigate that considering $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ conducive to nonexpansivity of
\[ T_\alpha x = \sum_{i=1}^{n} \alpha_i T^i x, \]
for all $x \in K$. However, nonexpansivity of $T_\alpha$ is much weaker than (2.1), it does not yield the continuity of $T$.

**Lemma 2.1.** Suppose that $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfies in the following inequality
\[ \delta_{k+1} \leq (1 - \lambda_k)\delta_k + \sigma_k, \quad k \geq 0, \]
where $\lambda_k \in [0,1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sigma_k = o(\lambda_k)$. Then $\lim_{k \to \infty} \delta_k = 0$.

**Theorem 2.2.** Let $K$ be a closed convex subset of Banach space $X$ and let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, for all $n \in \mathbb{N}$, be such that $\alpha_i \geq 0, i = 1, 2, \cdots, n, \alpha_1 > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Let $T$ be an $\alpha$-nonexpansive mapping from $K$ into itself. If $\alpha_1 > \frac{1}{\sqrt{2}}$, then $F(T) = F(T_\alpha)$.

### 3. Main results

In this section, we introduced a new definition which is generalization of the $\alpha$-inverse-monotone mapping.

**Definition 3.1.** We say that a mapping $T : K \to H$ satisfies in $\theta$-property, if there exist $\rho > 0$ and a function $0 < \theta(\rho) < \infty$ such that
\[ \|(x - y) - \rho(Tx - Ty)\| \leq \theta(\rho) \|x - y\|, \]
for all $x, y \in K$.

**Example 3.2.** Let $H = K := l^2$ and $Tx = \frac{x}{\rho + 1}$, for some $\rho > 0$. Then we put $\theta(\rho) = \frac{1}{\rho + 1} < 1$.

**Example 3.3.** Let $H = K := l^2$, $Tx := x$ and $0 < \rho < 1$. Then $\theta(\rho) = 1 - \rho < 1$.

**Example 3.4.** Let $H = [0, \infty)$ and $K = [0, 1]$. Define $T : K \to H$ as follows:
\[ T(x) = \frac{x^2}{1 + x} \quad \forall x \in [0, 1]. \]

Put $\theta(\rho) = 1 + 3\rho$. Clearly,
\[
\|(x - y) - \rho(Tx - Ty)\| = \|(x - y) - \rho\left(\frac{x^2}{1 + x} - \frac{y^2}{1 + y}\right)\|
\leq \|x - y\| + \rho \left|\frac{x^2}{1 + x} - \frac{y^2}{1 + y}\right|
\leq \|x - y\| + \rho \|x^2(1 + y) - y^2(1 + x)\|
\leq (1 + 3\rho) \|x - y\|
\]
Therefore, $T$ satisfies in $\theta$-property.

**Example 3.5.** Let $H = [0, \infty)$ and $K = [0, 1]$. The mappings $T(x) = x$, $T(x) = 1-x$ and $T(x) = \beta x$ satisfy in $\theta$-property by $\theta(\rho) = 1-\rho$, for $0 < \rho < 1$, $\theta(\rho) = 1+\rho$ and $\theta(\rho) = 1$, respectively.

**Theorem 3.6.** Let $K$ be a closed convex subset of a real Hilbert space $H$. Let $T : K \to H$ satisfies in $\theta$-property and $S$ be an $\alpha$-nonexpansive mapping from $K$ into itself such that $\alpha_1 > \frac{1}{1+\sqrt{2}}$ and $F(S) \cap VI(K,T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by Algorithm 1.3 for any initial point $x_0 \in K$ where $0 < \theta(\rho) < 1, a_n, b_n, c_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence $x_n$, which is derived from Algorithm 1.3, converges strongly to $x^* \in F(S) \cap VI(K,T)$.

**Proof.** Pick $x^* \in F(S) \cap VI(K,T)$. Since $F(S) = F(S_\alpha)$, by Theorem 2.2, we have $x^* \in K \cap F(S_\alpha) \cap VI(K,T)$. Therefore,

$$x^* = (1 - c_n)x^* + c_n S_\alpha P_K[x^* - \rho T x^*]$$ (3.1)

$$= (1 - b_n)x^* + b_n S_\alpha P_K[x^* - \rho T x^*]$$ (3.2)

$$= (1 - a_n)x^* + a_n S_\alpha P_K[x^* - \rho T x^*]$$ (3.3)

where $a_n, b_n, c_n \in [0, 1]$. We shall show that $x_{n+1} \to x^*$ as $n \to \infty$. From (3.4), (3.3) and the nonexpansive property of the projection $P_K$ and the nonexpansive mapping $S_\alpha$, we have

$$\|x_{n+1} - x^*\| = \|x_n - x^* - a_n S_\alpha P_K[y_n - \rho Ty_n] - (1 - a_n)x_n - a_n S_\alpha P_K[x^* - \rho T x^*]|$$

$$\leq (1 - a_n)\|x_n + a_n S_\alpha P_K[y_n - \rho Ty_n] - (1 - a_n)x_n - a_n S_\alpha P_K[x^* - \rho T x^*]|$$

$$\leq (1 - a_n)\|x_n - x^*\| + a_n\|S_\alpha P_K[y_n - \rho Ty_n] - S_\alpha P_K[x^* - \rho T x^*]|$$

From $\theta$-property on $T$,

$$\|y_n - x^* - \rho(Ty_n - Tx^*)\| \leq \theta(\rho)\|y_n - x^*\|,$$ (3.5)

where $\theta(\rho) < 1$.

Combining (3.4) and (3.5), we have

$$\|x_{n+1} - x^*\| \leq (1 - a_n)\|x_n - x^*\| + a_n\|x^* - \rho T x^*|$$ (3.6)

From (1.2), (3.3) and $\alpha$-nonexpansivity of the operators $S_\alpha$ and $P_K$, we have

$$\|y_n - x^*\| = \|(1 - b_n)x_n + b_n S_\alpha P_K[z_n - \rho T z_n] - (1 - b_n)x_n + b_n S_\alpha P_K[x^* - \rho T x^*]|$$

$$\leq (1 - b_n)\|x_n - x^*\| + b_n\|S_\alpha P_K[z_n - \rho T z_n] - S_\alpha P_K[x^* - \rho T x^*]|$$

$$\leq (1 - b_n)\|x_n - x^*\| + b_n\|z_n - \rho T z_n - x^* - \rho T x^*|.$$ (3.7)

Now from $\theta$-property on $T$, it yields that

$$\|z_n - x^* - (1 - \theta(\rho))(Tz_n - Tx^*)\| \leq \theta(\rho)\|z_n - x^*\|.$$ (3.8)

Analogously, (1.2) and (3.1) conclude that

$$\|z_n - x^*\| \leq (1 - c_n)\|x_n - x^*\| + c_n \theta(\rho)\|x_n - x^*\| = (1 - c_n(1 - \theta(\rho)))\|x_n - x^*\| \leq \|x_n - x^*\|.$$ (3.9)

Then from (3.7) and (3.8), we have

$$\|y_n - x^*\| \leq (1 - b_n)\|x_n - x^*\| + b_n \theta(\rho)\|z_n - x^*\|$$

$$\leq (1 - b_n)\|x_n - x^*\| + b_n \theta(\rho)\|x_n - x^*\|$$

$$\leq \|x_n - x^*\|.$$ (3.10)
From (3.6), (3.9) and (3.10), one can obtain
\[ ||x_{n+1} - x^*|| \leq (1 - a_n) ||x_n - x^*|| + a_n \theta(\rho) ||y_n - x^*|| \]
\[ \leq (1 - a_n) ||x_n - x^*|| + a_n \theta(\rho) ||x_n - x^*|| = [1 - a_n(1 - \theta(\rho))] ||x_n - x^*||, \]
and hence by Lemma 2.1 \( \lim_{n \to \infty} ||x_n - x^*|| = 0. \)

**Example 3.7.** Let \( H := l^2, K := B_{l^2}, \) where \( B_{l^2} \) denote the closed unit ball of \( l^2. \)

Define the mapping \( T : H \to K \) by:
\[ T(x) = (x - \frac{1}{2})(\frac{1}{1 + \rho}), \]
for all \( x \in B_{l^2}, S : K \to K \) by \( S(x) = 1 - x \) for all \( x \in K \) and the projection operator \( P_K : H \to K \) define as follows:
\[ P_K(x) = \frac{x}{\max\{1, ||x||_2\}}, \]
for all \( x \in K. \) Clearly \( S \) is an \( (\frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}) \)-nonexpansive mapping and \( \frac{1}{2} \in F(S) \cap \text{VI}(K,T). \) If \( a_n = \frac{1}{n^2}, b_n = \frac{1}{n^2}, c_n = \frac{1}{n^2} \) and \( x_0 = 0, \) then the sequence \( x_n, \) which is derived from Algorithm 1, converges strongly to \( \frac{1}{2} \in F(S) \cap \text{VI}(K,T). \)

Solving the above example in Mathematica software as follows:

```plaintext
Clear ["\*""]
n  \[ x_0 = 0;  \\
\rho = 1;  \\
n_{n-1} = 1/(n + 1)
\]
b\[ n_{-1} = 1/2^n;  \\
c\[ n_{-1} = 1/3^n;  \\
S\[ n_{-1} = 1/Sqrt[2] (1 - x) + (1 - 1/Sqrt[2]) x  \\
(1 - x)/Sqrt[2] + (1 - 1/Sqrt[2]) x  \\
T\[ n_{-1} = 1/(\rho + 1)(x - 1/2);  \\
Subscript[P, k] [n_{-1}] = x;  \\
Z\[ n_{-1} := (1 - c[n])X[n] + c[n]S[Subscript[P, k] [X[n] - \rho T[X[n]]]] // N  \\
Y\[ n_{-1} := (1 - b[n])X[n] + b[n]S[Subscript[P, k] [Z[n] - \rho T[Z[n]]]] // N  \\
X[(n + 1)_{-1}] := (1 - a[n])X[n] + a[n]S[Subscript[P, k] [Y[n] - \rho T[Y[n]]]] // Simplify // N  \\
For [i = 0, i <= 8, i + 1 + , , X[i + 1] = (1 - a[i])X[i] + a[i]S[Subscript[P, k] [Y[i] - \rho T[Y[i]]]]]
Table[X[i + n], {i, 0, 10}]
{0, 0.504442, 0.499961, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5}

If \( c_n \equiv 0 \) then the following result is a special case of Theorem 3.6.

**Corollary 3.8.** Let \( K \) be a closed convex subset of a real Hilbert space \( H. \) Let \( T : K \to H \) satisfies in \( \theta \) property and \( S \) be an \( \alpha \)-nonexpansive mapping of \( K \) into \( K \) such that \( \alpha_1 > \frac{1}{n - 1/\sqrt{2}} \) and \( F(S) \cap \text{VI}(K,T) \neq \emptyset. \) Let \( \{x_n\} \) be a sequence
defined by Algorithm 1.4 for any initial point \( x_0 \in K \), with condition \( 0 < \theta(\rho) < 1 \), \( a_n, b_n \in [0,1] \) and \( \sum_{n=0}^{\infty} a_n = \infty \), then \( \{x_n\} \) obtained from Algorithm 1.4 converges strongly to \( x^* \in F(S) \cap VI(K, T) \).

Now, we prove the strong convergence theorem of Algorithm 1.6.

**Theorem 3.9.** Let \( K \) be a closed convex subset of a real Hilbert space \( H \). Let \( T : K \to H \) satisfies in \( \theta \)-property and \( S \) be an \( \alpha \)-nonexpansive mapping of \( K \) into \( K \) such that \( \alpha \neq 0 \). Let \( \{x_n\} \) be the approximate solution obtained from Algorithm 1.6 for any initial point \( x_0 \in K \), with condition \( 0 < \theta(\rho) < 1 \). Then the sequence \( \{x_n\} \) converges strongly to \( x^* \in F(S) \cap VI(K, T) \).

**Proof.** Consider
\[
\|x_n - x^* - \rho[Tx_n - Tx^*]\| \leq \theta(\rho)\|x_n - x^*\|. 
\]
From (1.5), (3.9), (3.11) and the \( \alpha \)-nonexpansive property of the operators \( S_\alpha \) and \( P_K \), we have
\[
\|x_{n+1} - x^*\| = \|(1 - a_n)x_n + a_nS_\alpha P_K[x_n - \rho Tx_n] - (1 - a_n)x^* - a_nS_\alpha P_K[x^* - \rho Tx^*]\|
\leq (1 - a_n)\|x_n - x^*\| + a_n\|S_\alpha P_K[x_n - \rho Tx_n] - S_\alpha P_K[x^* - \rho Tx^*]\|
\leq (1 - a_n)\|x_n - x^*\| + a_n\|x_n - x^* - \rho(Tx_n - Tx^*)\|
\leq (1 - a_n)\|x_n - x^*\| + a_n\theta(\rho)\|x_n - x^*\|
= (1 - a_n(1 - \theta(\rho)))\|x_n - x^*\|.
\]
Therefore by Lemma 2.1, \( \lim_{n \to \infty} \|x_n - x^*\| = 0 \). \qed

4. **Another New Algorithm**

The following equality is established in a Hilbert space
\[
\|(x - y) - \rho(Tx - Ty)\|^2 = \|x - y\|^2 + \rho^2\|Tx - Ty\|^2 - 2\rho(x - y, Tx - Ty).
\]
If the mapping \( T : K \to H \) be an \( \alpha \)-inverse strongly monotone, then
\[
\|(x - y)^2 + \rho^2\|Tx - Ty\|^2 - ((x - y) - \rho(Tx - Ty))^2 \geq 2\rho\alpha\|Tx - Ty\|^2.
\]
Hence
\[
\rho^2 - 2\rho\alpha\|Tx - Ty\|^2 + \|x - y\|^2 \geq \|(x - y) - \rho(Tx - Ty)\|^2.
\]
That is, if \( \alpha < \frac{\rho}{2} \), then \( T \) is not satisfies in \( \theta \)-property. The following theorem shows that, if \( T \) satisfies in \( \theta \)-property, then \( T \) is an \( \alpha \)-inverse strongly monotone mapping. Therefore it is a conclusion of Theorem 1.7.

**Theorem 4.1.** Let \( K \) be a closed convex subset of a real Hilbert space \( H \). Let \( T : K \to H \) satisfies in \( \theta \)-property with condition \( 0 < \theta(\rho) < 1 \) and \( S \) be an \( \alpha \)-nonexpansive mapping of \( K \) into \( K \) such that \( \alpha_1 > \frac{1}{\rho - \sqrt{\rho^2}} \) and \( F(S) \cap VI(K, T) \neq \emptyset \). For fixed \( u \in K \) and given \( x_0 \in K \) arbitrary, let the sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are generated by
\[
\begin{align*}
\{y_n\} = \{ & \sigma_n x_n + (1 - \sigma_n)S_\alpha P_K[x_n - \rho_n Tx_n] \\
\{z_n\} = \{ & \mu_n x_n + (1 - \mu_n)S_\alpha P_K[y_n - \rho_n Ty_n] \\
x_{n+1} = \{ & \alpha_n u + \beta_n x_n + \gamma_n S_\alpha P_K[z_n - \rho_n Tz_n], n \geq 0,
\end{align*}
\]
where \( \{\sigma_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\} \) and \( \{\sigma_n\} \) are five sequences in \([0,1]\) and \( \theta(\rho_n) \) is a sequence in \((0,1)\). If for these sequences \( \rho_n \in (0,2) \) and
Proof. For fixed $T$:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$ and $\lim_{n \to \infty} \mu_n = 1$
(iv) $\lim_{n \to \infty} (\rho_{n+1} - \rho_n) = 0$.

Then $\{x_n\}$ defined by (4.1) converges strongly to $x^* \in P_{F(S) \cap VI(K, T)} u$.

Corollary 4.2. Let $K$ be a closed convex subset of a real Hilbert space $H$. Let $T : K \to H$ satisfies in $\theta$-property with condition $0 < \theta(\rho) < 1$ and $VI(K, T) \neq \emptyset$. For fixed $u \in K$ and given $x_0 \in K$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by

\[
\begin{align*}
\begin{cases}
  y_n = \sigma_n x_n + (1 - \sigma_n) P_K [x_n - \rho_n T x_n] \\
  z_n = \mu_n x_n + (1 - \mu_n) P_K [y_n - \rho_n T y_n] \\
  x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_K [z_n - \rho_n T z_n], n \geq 0,
\end{cases}
\end{align*}
\]

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0, 1]$ and $\theta(\rho_n)$ is a sequence in $(0, 1)$. If for these sequences $\rho_n \in (0, 2)$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$ and $\lim_{n \to \infty} \mu_n = 1$
(iv) $\lim_{n \to \infty} (\rho_{n+1} - \rho_n) = 0$.

Then $\{x_n\}$ defined by (4.3) converges strongly to $x^* \in P_{VI(K, T)} u$.

Theorem 4.3. Let $K$ be a closed convex subset of a real Hilbert space $H$. Let $T : K \to H$ satisfies in $\theta$-property with condition $0 < \theta(\rho) < 1$ and $S$ be an $\alpha$-nonexpansive mapping of $K$ into $K$ such that $\alpha_1 > \frac{1}{\pi - 1}$ and $F(S) \cap T^{-1} \neq \emptyset$. For fixed $u \in K$ and given $x_0 \in K$ arbitrary, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by

\[
\begin{align*}
\begin{cases}
  y_n = \sigma_n x_n + (1 - \sigma_n) S_{\alpha_n} P_K [x_n - \rho_n T x_n] \\
  z_n = \mu_n x_n + (1 - \mu_n) S_{\alpha_n} P_K [y_n - \rho_n T y_n] \\
  x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S_{\alpha_n} P_K [z_n - \rho_n T z_n], n \geq 0,
\end{cases}
\end{align*}
\]

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\sigma_n\}$ are five sequences in $[0, 1]$ and $\theta(\rho_n)$ is a sequence in $(0, 1)$. If for these sequences $\rho_n \in (0, 2)$ and

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$ and $\lim_{n \to \infty} \mu_n = 1$
(iv) \( \lim_{n \to \infty} (\rho_{n+1} - \rho_n) = 0 \).

Then \( \{x_n\} \) defined by (4.1) converges strongly to \( x^* \in P_{F(S) \cap T^{-1}u} \).

Proof. Since \( T^{-1}0 = VI(K,T) \), putting \( P_K = I \), by Theorem [4.1] we can obtain the conclusion. This completes the proof. \( \Box \)

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