ON THE INCLUSIONS OF SOME LORENTZ MIXED NORMED SPACES AND WIENER-DITKIN SETS

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Abstract. In this paper, we consider sufficient and necessary condition for the inclusion between Lorentz spaces with mixed norms for two different product measures on \((X \times Y, A \times B)\). Later, we discuss the Wiener-Ditkin sets of Lorentz spaces with mixed norms and Lorentz spaces.

1. Introduction

Let \((X, A, \mu)\) be a measure space such that \(\mu\) is a nonnegative measure. For a measurable function \(f\) on \(X\), the distribution function is

\[
\lambda_f(y) = \mu\left(\{x \in X \mid |f(x)| > y\}\right), \quad (y > 0).
\]

Its rearrangement is defined by

\[
f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\}, \quad (t > 0),
\]

and its average function is given by

\[
f^{**}(x) = \frac{1}{x} \int_0^x f^*(t)dt, \quad (x > 0).
\]

Note that \(\lambda_f\), \(f^*\) and \(f^{**}\) are nonincreasing and right continuous functions on \((0, \infty)\). If \(\lambda_f(y)\) is continuous and strictly decreasing, then \(f^*\) is the inverse function of \(\lambda_f\). The most important property of \(f^*\) is that it has the same distribution function as \(f\). It follows that

\[
\left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} = \left(\int_0^\infty [f^*(t)]^p dt\right)^{\frac{1}{p}}. \tag{1.1}
\]

The Lorentz space denoted by \(L(p, q)(X, \mu)\) (shortly \(L(p, q)\)) is defined to be vector space of all (equivalence classes) of measurable functions \(f\) such that \(\| f \|_{pq} < \infty\), where

\[
\| f \|_{pq} = \begin{cases}
\left(\frac{q}{p} \int_0^\infty t^{\frac{q}{p}-1} [f^*(t)]^q dt\right)^{\frac{1}{q}}, & 0 < p < q < \infty \\
\sup_{t>0} t^{\frac{q}{p}} f^*(t), & 0 < p \leq q = \infty.
\end{cases}
\]

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For any one of the cases $p \leq 2 \ A. \ SANDIKÇI$

Lorentz space by measurable function on the product space $(L^p_{\nu})$. In this theory are due to Miamee (see [9]) for Lebesgue spaces with two different norms $P,Q$ (see [3, 8, 11]). Let $(X, A, \mu)$ and $(Y, B, \nu)$ be $\sigma$-finite measure spaces, $f$ be a complex valued measurable function on the product space $(X \times Y, A \times B, \mu \times \nu)$, $1 < P = (p_1, p_2) < \infty$ and $1 \leq Q = (q_1, q_2) \leq \infty$, that is $p_i$ and $q_i$, $i = 1, 2$, are between 1 and $\infty$. The Lorentz mixed norm space $L(P, Q) = L(P, Q)(X \times Y)$ is defined by

$$L(P, Q) = L(p_2, q_2)[L(p_1, q_1)] = \left\{ f : \| f \|_P = \| f \|_{L(p_2, q_2)}[L(p_1, q_1)] = \| f \|_{p_1, q_1} \right\}_{p_2, q_2} < \infty \right\}.$$

So, $L(P, Q)$ occurs by taking an $L(p_1, q_1)$-norm with respect to first variable and an $L(p_2, q_2)$-norm with respect to second variable. The $L(P, Q)$ space is a Banach space under the norm $\| . \|_{PQ}$ (see [2, 3, 10]).

In section 2, we prove that the inclusion $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$ will hold under which condition for two different product measures. The first results in this theory are due to Miamee (see [9]) for Lebesgue spaces with two different measures. In [3], Gürkanlı extended the results of Mamee to the Lorentz spaces. Our theorems generalize previous results from [9] and [6]. In the third section, we investigate the Wiener-Ditkin sets of Lorentz spaces with mixed norms and Lorentz spaces.

In this paper, we denote the Lebesgue space with mixed norm by $L^{p,q}$; the Lorentz space by $L(p, q)$; the Lorentz mixed norm space $L(P, Q)$ or sometimes $L(p_2, q_2)[L(p_1, q_1)]$, where $P = (p_1, p_2)$ and $Q = (q_1, q_2)$.

2. The inclusions of some Lorentz mixed normed spaces

Throughout the paper, the letters $P$ and $Q$ will denote 2-tuples $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, where $p_i$ and $q_i$, $i = 1, 2$, are between 1 and $\infty$. When $i = 1$, we shall write $P = p$ and $Q = q$. Moreover, $P \leq Q$ will mean $p_i \leq q_i$ for $i = 1, 2$. Further, in this section, the symbol $\mu \ll \nu$ is used to indicate the absolute continuity of $\mu$ with respect to $\nu$.

**Lemma 2.1.** Let $0 < P_1, P_2 < \infty$ and $0 < Q_1, Q_2 \leq \infty$, where $P_i = (p_i^1, p_i^2)$, $Q_i = (q_i^1, q_i^2)$, $i = 1, 2$. Then the inclusion $L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2)$ holds in the sense of equivalence classes if and only if $\mu_1 \times \nu_1$ and $\mu_2 \times \nu_2$ are
absolute continuous with respect to each other and \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \) in the sense of individual functions.

**Proof.** The 'if' part is easy. To prove the 'only if' part, let us take any \( E \in \mathcal{A} \times \mathcal{B} \) with \( (\mu_1 \times \nu_1)(E) = 0 \). Then we have \( \mu_1(E_y) = 0 \) for \( \nu_1 \)-a.e. \( y \in Y \) and \( \nu_1(E_x) = 0 \) for \( \mu_1 \)-a.e. \( x \in X \), where \( E_y \) and \( E_x \) are the \( x \)-section and the \( y \)-section, respectively, of \( E \). Hence \( \chi_{E_y} = 0 \) \( \nu_1 \)-a.e. and the rearrangement of \( \chi_{E_y} \) is

\[
\chi_{E_y}^*(s) = \begin{cases} 1, & 0 < s < \mu_1(E_y) \\ 0, & s \geq \mu_1(E_y) \end{cases}
\]

If \( 0 < P_1, Q_1 < \infty \), then we have

\[
\| \chi_E \|_{p_1, Q_1}^* = \left\| \chi_{E_y} \right\|_{p_1, q_1}^* = \left\| \left( \frac{q_1^1}{p_1^1} \int_0^\infty \frac{s^{q_1^1-1}}{s^{p_1^1}} |\chi_{E_y}(s)|^{q_1^1} ds \right)^{\frac{1}{q_1^1}} \right\|_{p_1^2, q_1^2}^* = \left( \frac{q_1^1}{p_1^1} \int_0^\infty \frac{\mu_1(E_y)}{s^{p_1^1}} ds \right)^{\frac{1}{q_1^1}} = 0
\]

since \( \mu_1(E_y) = 0 \) a.e. with respect to \( \nu_1 \). Similarly, if \( 0 < P_1 < \infty \) and \( Q_1 = (\infty, \infty) \), we obtain

\[
\| \chi_E \|_{P_1, Q_1}^* = \left\| \chi_{E_y} \right\|_{P_1, \infty}^* = \left\| \left( \mu_1(E_y) \right)^{\frac{1}{p_1^1}} \right\|_{P_1^2, \infty}^* = 0.
\]

So, \( \chi_E \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \) for \( 0 < P_1 < \infty \) and \( 0 < Q_1 \leq \infty \). Then \( \chi_E \) is in the equivalence class 0 of \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \), and so, by the hypothesis, this equivalence class 0 is in the space \( L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \). Hence \( \chi_E \) is in the equivalence class 0 of \( L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \). Thus we write

\[
\| \chi_E \|_{P_2, Q_2}^* = \left\| \left( \mu_2(E_y) \right)^{\frac{1}{p_2^1}} \right\|_{P_2^2, q_2^2}^* = 0.
\]

It follows from here \( \mu_2(E_y) = 0 \) a.e. with respect to \( \nu_2 \). That means \( (\mu_2 \times \nu_2)(E) = \int \mu_2(E_y) d\nu_2 = 0 \). Thus the absolute continuity of \( \mu_2 \times \nu_2 \) follows. Similarly, \( \mu_1 \times \nu_1 \) is absolute continuous with respect to \( \mu_2 \times \nu_2 \). \( \square \)

We shall need the following lemma for the next theorem. In this lemma, we will state the existence of a.e. convergent subsequence of a convergent sequence \( (f_n) \) in \( L(P, Q)(X \times Y, \mu \times \nu) \). This mode of convergence is true for Banach function spaces \([2]\). Since \( L(P, Q)(X \times Y, \mu \times \nu) \) is a Banach function space \([2]\), it is also true for \( L(P, Q)(X \times Y, \mu \times \nu) \). But let us provide the details anyway for the sake of completeness.

**Lemma 2.2.** Let \( 1 < P = (p_1, p_2) < \infty, 1 \leq Q = (q_1, q_2) \leq \infty \). If \( (f_n) \) is a sequence in \( L(P, Q)(X \times Y, \mu \times \nu) \) and \( \| f_n - f \|_{PQ} \to 0 \), where \( f \in L(P, Q)(X \times Y, \mu \times \nu) \), then \( (f_n) \) contains a subsequence that converges almost everywhere to the limit function.
Proof. Suppose \( \| f_n - f \|_{PQ} \to 0 \) as \( n \to \infty \). Then \( \| f_n - f \|_{P_iQ_i} \to 0 \) as \( n \to \infty \).
By using the inequality \( \| f \|_{P_iQ_i}^* \leq \| f \|_{P_iQ_i} \), we get \( \| f_n - f \|_{P_iQ_i}^* \to 0 \) as \( n \to \infty \). Thus we write
\[
\| f_n - f \|_{P_iQ_i}^* = \sup_{s > 0} \lambda \left( (f_n - f)^*_y(s) = \sup_{\lambda > 0} \lambda \left( \mu((f_n - f)_y)(\lambda) \right)^{\frac{1}{\lambda}} \right.
\]
and so \( \mu \left\{ x \in X : \left| (f_n - f)_y \right| > \lambda \right\} \to 0 \) as \( n \) tends to infinity. This implies \( (f_n)_y \) converges to \( f_y \) in measure with respect to \( \mu \). Hence there is a subsequence \( (f_{n_k})_y \subset (f_n)_y \) such that \( (f_{n_k})_y \) converges \( \mu \)-a.e. to \( f_y \). Finally, by holding \( x \) fixed and repeating the same procedure with respect to second variable \( y \), we obtain a subsequence \( (f_{n_k}) \) of \( (f_n) \) converges almost everywhere to the limit function \( f \).

Theorem 2.3. Let \( 1 < P_1, P_2 < \infty \) and \( 1 \leq Q_1, Q_2 \leq \infty \), where \( P_i = (p_i^1, p_i^2) \), \( Q_i = (q_i^1, q_i^2) \), \( i = 1, 2 \). Then \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subset L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \)
if and only if \( \mu_1 \times \nu_1 \) is absolute continuous with respect to \( \mu_2 \times \nu_2 \) and there exists a constant \( C > 0 \) such that
\[
\| f \|_{P_2Q_2, \mu_2 \times \nu_2} \leq C \| f \|_{P_1Q_1, \mu_1 \times \nu_1} \tag{2.2}
\]
for all \( f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \).

Proof. Suppose that \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \subset L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \). Then \( \mu_1 \times \nu_1 \) and \( \mu_2 \times \nu_2 \) are absolute continuous with respect to each other by Lemma 2.1.
In order to show (2.2), define the linear operator \( I(f) = f \) from \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \) into \( L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \). Then the graph of \( I \) is
\[
G(I) = \{ (f, I(f)) : f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \}
\]
and it is a vector subspace of \( L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \times L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \)
since \( I \) is linear. Now we will show that \( G(I) \) is close. Assume that \( (f_n, I(f_n)) \subset G(I) \)
converges to \( (f, g) \), where \( f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \), \( g \in L(P_2, Q_2)(X \times Y, \mu_2 \times \nu_2) \).
Then, for any \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that
\[
\| (f_n, I(f_n)) - (f, g) \| = \| f_n - f \|_{P_1Q_1, \mu_1 \times \nu_1} + \| I(f_n) - g \|_{P_2Q_2, \mu_2 \times \nu_2} < \epsilon
\]
for all \( n \geq n_0 \). Thus we get \( \| f_n - f \|_{P_1Q_1, \mu_1 \times \nu_1} \to 0 \) and \( \| I(f_n) - g \|_{P_2Q_2, \mu_2 \times \nu_2} \to 0 \). Then by Lemma 2.2, there exists a subsequence \( (f_{n_k}) \) of \( (f_n) \) such that
\[
f_{n_k} \to f, \quad \mu_1 \times \nu_1 - a.e. \quad \text{and} \quad I(f_{n_k}) = f_{n_k} \to g, \quad \mu_2 \times \nu_2 - a.e.
\]
Since \( \mu_2 \times \nu_2 \ll \mu_1 \times \nu_1 \), we have
\[
f_{n_k} \to f, \quad \mu_2 \times \nu_2 - a.e. \quad \text{and} \quad f_{n_k} \to g, \quad \mu_2 \times \nu_2 - a.e.
\]
Thus \( f = g, \mu_2 \times \nu_2 - a.e. \) and so \( (f, g) \in G(I) \). That means \( G(I) \) is close. Hence the linear operator \( I \) is bounded by closed graph theorem. Then there exists \( C > 0 \) such that
\[
\| f \|_{P_2Q_2, \mu_2 \times \nu_2} \leq C \| f \|_{P_1Q_1, \mu_1 \times \nu_1}
\]
for all \( f \in L(P_1, Q_1)(X \times Y, \mu_1 \times \nu_1) \).

The if-part of this theorem is obvious.
Proposition 2.4. Let $0 < P = (p_1, p_2) < \infty$, $0 < Q = (q_1, q_2) \leq \infty$ and $f \in L(P, Q) (X \times Y, \mu_1 \times \nu_1)$ be a real valued measurable function. If there exist constants $m, n > 0$ such that $\mu_2 (A) \leq m \mu_1 (A)$ for all $A \in \mathcal{A}$ and $\nu_2 (B) \leq n \nu_1 (B)$ for all $B \in \mathcal{B}$, then we have the inequality

$$
\| f \|^{*}_{pQ, \mu_2 \times \nu_2} \leq m^{\frac{1}{p_1}} n^{\frac{1}{p_2}} \| f \|^{*}_{pQ, \mu_1 \times \nu_1}.
$$

Moreover, there is the inclusion $L(P, Q) (X \times Y, \mu_1 \times \nu_1) \subseteq L(P, Q) (X \times Y, \mu_2 \times \nu_2)$.

Proof. Let $f \in L(P, Q) (X \times Y, \mu_1 \times \nu_1)$. Then $f_y \in L(p_1, q_1) (X, \mu_1)$ and $\| f_y \|_{p_1, q_1, \mu_1} \in L(p_2, q_2) (Y, \nu_1)$. Since $f_y \in L(p_1, q_1) (X, \mu_1)$ is a measurable real valued function, then

$$
E_{\sigma} = \{ x \in X : f_y (x) > \sigma \} \in \mathcal{A}
$$

for all real number $\sigma$. Also let $M$ and $N$ such that

$$
M = \{ \sigma > 0 : \lambda_{f_y}^{\mu_2} (\sigma) \leq t \}
$$

and

$$
N = \{ \sigma > 0 : \lambda_{f_y}^{\mu_1} (\sigma) \leq t \}
$$

for all $t > 0$, where $\lambda_{f_y}^{\mu_2} (\sigma) = \mu_2 (E_{\sigma})$ and $\lambda_{f_y}^{\mu_1} (\sigma) = (m \mu_1) (E_{\sigma})$. Since $\mu_2 (A) \leq m \mu_1 (A)$, we get $\lambda_{f_y}^{\mu_2} (\sigma) \leq \lambda_{f_y}^{\mu_1} (\sigma)$ and so $N \subseteq M$. Thus we can write

$$
f_{y, \mu_2}^{*, \ast} (t) = \inf_{\sigma} M \leq \inf_{\sigma} N = f_{y, \mu_1}^{*, \ast} (t). \tag{2.3}
$$

Moreover, since

$$
\{ \sigma > 0 : \lambda_{f_y}^{m \mu_1} (\sigma) \leq t \} = \{ \sigma > 0 : (m \mu_1) (E_{\sigma}) \leq t \}
$$

$$
= \left\{ \sigma > 0 : (\mu_1) (E_{\sigma}) \leq \frac{t}{m} \right\}, \tag{2.4}
$$

we have by (2.3) and (2.4)

$$
f_{y, \mu_2}^{*, \ast} (t) \leq f_{y, \mu_1}^{*, \ast} (t) = f_{y, \mu_1}^{*, \ast} \left( \frac{t}{m} \right).
$$

Thus we get

$$
\| f_y \|^{*}_{p_1, q_1, \mu_1} = \left( \frac{q_1}{p_1} \int_0^\infty t^{\frac{q_1}{p_1} - 1} [f_{y, \mu_2}^{*, \ast} (t)]^{q_1} \, dt \right)^{\frac{1}{q_1}}
$$

$$
\leq \left( \frac{q_1}{p_1} \int_0^\infty \left( \frac{t}{m} \right)^{\frac{q_1}{p_1} - 1} [f_{y, \mu_1}^{*, \ast} (\frac{t}{m})]^{q_1} \, dt \right)^{\frac{1}{q_1}}
$$

$$
= \left( \frac{q_1}{p_1} \int_0^\infty (m u)^{\frac{q_1}{p_1} - 1} \left( f_{y, \mu_1}^{*, \ast} (u) \right)^{q_1} m \, du \right)^{\frac{1}{q_1}}
$$

$$
= m^{\frac{1}{p_1}} \| f_y \|^{*}_{p_1, q_1, \mu_1}. \tag{2.5}
$$

Also since $\| f_y \|_{p_1, q_1, \mu_1} \in L(p_2, q_2) (Y, \nu_1)$ and $\nu_2 (B) \leq n \nu_1 (B)$ for all $B \in \mathcal{B}$, by applying the same procedure to second variable, we obtain

$$
\| f_y \|^{*}_{p_1, q_1, \mu_1} \| f_y \|^{*}_{p_2, q_2, \nu_2} \leq n^{\frac{1}{p_2}} \| f_y \|^{*}_{p_1, q_1, \mu_1} \| f_y \|^{*}_{p_2, q_2, \nu_2}. \tag{2.6}
$$
Hence we have \( \|f\|_{P,Q;\mu,\nu} \in L(P_2, Q_2)(Y, \nu_2) \). Since \( L(p_2, q_2)(Y, \nu_2) \) is a solid space, we obtain from (2.5) and (2.6)

\[
\| f \|_{P,Q;\mu,\nu} \leq \| f \|_{p_1,q_1,\mu_1} \leq m \frac{1}{n^{1/2}} \| f \|_{p_2,q_2,\nu_2} = m \frac{1}{n^{1/2}} \| f \|_{p_1,q_1,\mu_1,\nu_1}. \tag{2.7}
\]

Let \( S \) denote the set of simple functions. By Proposition 2.2 in [5], \( S = L(P, Q)(X \times Y, \mu_1 \times \nu_1) \). Let \( I \) be a unit function from \( S \) into \( L(P, Q)(X \times Y, \mu_2 \times \nu_2) \). By using (2.7), we have the inequality

\[
\| f \|_{P,Q;\mu,\nu} \leq C \| f \|_{p_1,q_1,\mu_1,\nu_1}
\]

for all \( f \in S \), where \( C = m \frac{1}{n^{1/2}} \). That means \( I \) is continuous from \( S \) into \( L(P, Q)(X \times Y, \mu_2 \times \nu_2) \). Then \( I \) is continuously extended to the space \( L(P, Q)(X \times Y, \mu_1 \times \nu_1) \).

Thus we get the inequality

\[
\| f \|_{P,Q;\mu,\nu} \leq C \| f \|_{p_1,q_1,\mu_1,\nu_1}
\]

for all \( f \in L(P, Q)(X \times Y, \mu_1 \times \nu_1) \). Hence we have the inclusion \( L(P, Q)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P, Q)(X \times Y, \mu_2 \times \nu_2) \).

In Proposition 4, if there exists the inclusion which is mentioned, we could not obtain the converse of this proposition for \( P = (p_1, p_2) \). But we can state it when \( P = (p, p) \) as in the following.

**Proposition 2.5.** Let \( 0 < P = (p, p) < \infty, 0 < Q = (q_1, q_2) \leq \infty \). If \( L(P, Q)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P, Q)(X \times Y, \mu_2 \times \nu_2) \), then there exists \( K > 0 \) such that \( (p_2 \times \nu_2)(E) \leq K (\mu_1 \times \nu_1)(E) \) for all \( E \in \mathcal{M} \).

**Proof.** Let \( L(P, Q)(X \times Y, \mu_1 \times \nu_1) \subseteq L(P, Q)(X \times Y, \mu_2 \times \nu_2) \). By Theorem 2.3 there exists \( C > 0 \) such that

\[
\| f \|_{P,Q;\mu,\nu} \leq C \| f \|_{p_1,q_1,\mu_1,\nu_1}
\]

for all \( f \in L(P, Q)(X \times Y, \mu_1 \times \nu_1) \). If it is taken \( f = \chi_E \), we have from (2.1)

\[
\mu_2(E_y) \frac{1}{p} \nu_2(E_x) \frac{1}{p} \leq C \mu_1(E_y) \frac{1}{p} \nu_1(E_x) \frac{1}{p},
\]

where \( E_y \) and \( E_x \) are the \( x \)–section and the \( y \)–section of \( E \), respectively. Hence we obtain

\[
(\mu_2(E_y) \nu_2(E_x))^\frac{1}{p} \leq C (\mu_1(E_y) \nu_1(E_x))^\frac{1}{p}
\]

and

\[
(\mu_2 \nu_2)(E) \leq K (\mu_1 \nu_1)(E),
\]

where \( K = C \frac{1}{p} \).

**Proposition 2.6.** Let \( P_1 = (p_1, p_1), P_2 = (p_2, p_2), Q_1 = (q_1, q_2), \) \( i = 1, 2, \) and \( 0 < Q_1 \leq P_1 < P_2 \leq Q_2 < \infty \). Then there is the inclusion \( L(P_1, Q_1)(X \times Y, \mu \times \nu) \subseteq L(P_2, Q_2)(X \times Y, \mu \times \nu) \) if and only if there exists a constant \( k > 0 \) such that \( (\mu \times \nu)(E) \geq k \) for every \( \mu \times \nu \)–non-null set \( E \in \mathcal{M} \).

**Proof.** Assume that \( L(P_1, Q_1)(X \times Y, \mu \times \nu) \subseteq L(P_2, Q_2)(X \times Y, \mu \times \nu) \). Then there exists \( C > 0 \) such that

\[
\| f \|_{P_2,Q_2;\mu,\nu} \leq C \| f \|_{P_1,Q_1;\mu,\nu} \tag{2.8}
\]
for all \( f \) in \( L(P_1, Q_1) (X \times Y, \mu \times \nu) \). Let \( E \in \mathcal{M} \) be a \( \mu \times \nu \)-non-null set and \( (\mu \times \nu)(E) < \infty \). Then by taking \( \chi_E \) instead of \( f \) in (2.8), we get
\[
\mu \left( E_y \right) \frac{1}{\nu} \frac{\nu \left( E_x \right)}{\pi^2} \leq C_\mu \left( E_y \right) \frac{1}{\nu} \frac{\nu \left( E_x \right)}{\pi^2} = \mu \left( E_y \right) \frac{1}{\nu} \frac{\nu \left( E_x \right)}{\pi^2} \leq (\mu \times \nu)(E),
\]

where \( k = C \frac{\nu(E)}{\pi^2} \).
Conversely, let \( f \in L(P_1, Q_1) (X \times Y, \mu \times \nu) \) and
\[
E_n = \left\{ (x, y) \right\} \mid f(x, y) > n \}
\]
for all \( n \in \mathbb{N} \). Since \( Q_1 \leq P_1 \), we have
\[
L(P_1, Q_1) (X \times Y, \mu \times \nu) \subset L(P_1, P_1) (X \times Y, \mu \times \nu) = L^{p_1,p_1} (X \times Y, \mu \times \nu),
\]
where the space \( L^{p_1,p_1} (X \times Y, \mu \times \nu) \) is the Lebesgue space with mixed norm \( L^{p_1} (X \times Y, \mu \times \nu) \). Then there exists \( C > 0 \) such that
\[
\| f \|_{p_1, \mu \times \nu} \leq C \| f \|_{P_1, Q_1, \mu \times \nu}
\]
for all \( f \in L(P_1, Q_1) (X \times Y, \mu \times \nu) \). From the definition of \( E_n \), we write
\[
n^{p_1} \mu \left( (E_n)_y \right) \leq \int_{(E_n)_y} |f_y|^{p_1} \, d\mu \leq \int_X |f_y|^{p_1} \, d\mu.
\]
Applying the same procedure to second variable, we have
\[
n^{p_1} \mu \left( (E_n)_y \right) \nu \left( (E_n)_x \right) \leq \int_X \int_Y |f_y|^{p_1} \, d\mu \, d\nu
\]
and
\[
n^{p_1} (\mu \times \nu)(E_n) \leq \| f \|_{p_1, \mu \times \nu}^{p_1} \leq \left( C \| f \|_{P_1, Q_1, \mu \times \nu} \right)^{p_1} < \infty
\]
for every \( n \in \mathbb{N} \). Thus by hypothesis, either \( (\mu \times \nu)(E_n) = 0 \) or \( (\mu \times \nu)(E_n) \geq k \). Moreover, since \( E_{n+1} \subset E_n \), \( (\mu \times \nu)(E_1) < \infty \) and \( \bigcap_{n=1}^{\infty} E_n = \emptyset \), we get
\[
(\mu \times \nu)(E_n) \to 0.
\]
Thus there exists \( n_0 \in \mathbb{N} \) such that \( |f(x, y)| \leq n_0, \mu \times \nu \text{ a.e. for all } (x, y) \in X \times Y \). Then we write
\[
\int_{X \times Y} |f|^{p_2} \, (\mu \times \nu) \leq n_0^{p_2-p_1} \int_{X \times Y} |f|^{p_1} \, (\mu \times \nu).
\]
So \( f \in L^{p_2} (X \times Y, \mu \times \nu) = L(P_2, P_2) (X \times Y, \mu \times \nu) \) and from here \( L(P_1, Q_1) (X \times Y, \mu \times \nu) \subset L(P_2, P_2) (X \times Y, \mu \times \nu) \). Also since \( P_2 \leq Q_2 \), we obtain the desired result. \( \square \)

3. Wiener-Ditkin sets for Lorentz spaces with mixed norms

In this section we investigate the Wiener-Ditkin sets for Lorentz spaces with mixed norms and Lorentz spaces. Let \( X \) and \( Y \) be locally compact Abelian groups.
In the spirit of [7] a closed set \( A \) in \( (X \times Y)^\wedge \) is said to be a Wiener-Ditkin set for \( L(P, Q) (X \times Y) \) if each \( f \in L(P, Q) (X \times Y) \) such that \( \hat{f} \) vanishes on \( A \) can be approximated in \( L(P, Q) (X \times Y) \) with functions \( f \ast F \) such that \( \hat{F} \) vanishes in some neighbourhood on \( A \).

For the next theorem, we need the following remark.
Remark. A Banach space $X$ is called a (left) Banach module over a Banach algebra $A$ if it is a (left) module over $A$ in the algebraic sense and satisfies $\|ax\|_X \leq \|a\|_A \|x\|_X$ for all $a \in A$, $x \in X$. The closed linear span of $AX = \{ax : a \in A, x \in X\}$ in $X$ is called essential part $X_e$ of $X$. $X$ is called essential if $X_e = X$. \[8\] A. SANDIKÇI

Corollary 3.3. \[8\] of Theorem 10 in \[12\], where $T_z$ is translation operator. So it is easy two show that $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$ by module factorization theorem.

Theorem 3.1. Let $1 < P < \infty$, $1 \leq Q < \infty$. If a closed set $A$ in $(X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$, then it is a Wiener-Ditkin set for $L(P,Q)(X \times Y)$.

Proof. Suppose that $A \subset (X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$. Let $f \in L(P,Q)(X \times Y)$ such that $\hat{f}$ vanishes on $A$. Since $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$ by above remark, then there exists $\alpha \in I$ such that 
\[
\|f - f \ast e_\alpha\|_{PQ} < \frac{\epsilon}{2}
\]
for any $\epsilon > 0$. Since $A \subset (X \times Y)^\wedge$ is a Wiener-Ditkin set for $L^1(X \times Y)$, then there exists $F_1 \in L^1(X \times Y)$ such that $\hat{F}_1$ vanishes on a neighbourhood of $A$. Again since $L(P,Q)(X \times Y)$ is an essential Banach module over $L^1(X \times Y)$, for the same $\epsilon > 0$, we write
\[
\|f - f \ast F_1\|_{PQ} < \frac{\epsilon}{2\|e_\alpha\|_1}
\]
Let us take $F = F_1 \ast e_\alpha$. Then $\hat{F} = \hat{F}_1 \hat{e}_\alpha$ and so $\hat{F}$ vanishes on a neighbourhood of $A$. Thus we obtain
\[
\|f - f \ast F\|_{PQ} = \|f - f \ast e_\alpha + f \ast e_\alpha - f \ast e_\alpha \ast F_1\|_{PQ}
\leq \|f - f \ast e_\alpha\|_{PQ} + \|e_\alpha \ast (f - f \ast F_1)\|_{PQ}
\leq \|f - f \ast e_\alpha\|_{PQ} + \|e_\alpha\|_1 \|f - f \ast F_1\|_{PQ}
< \frac{\epsilon}{2} + \|e_\alpha\|_1 \frac{\epsilon}{2\|e_\alpha\|_1} = \epsilon.
\]
Hence $A$ is a Wiener-Ditkin set for $L(P,Q)(X \times Y)$. \[\square\]

Since $L(P,Q) = L(p,q)$ when $P = p$ and $Q = q$, Theorem 3.1 can be stated for Lorentz space $L(p,q)$ as in the following.

Corollary 3.2. Let $1 < p < \infty$, $1 \leq q < \infty$. If a closed set $A \subset X$ is a Wiener-Ditkin set for $L^1(X)$, then it is a Wiener-Ditkin set for $L(p,q)$.

It is known that closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for weighted space $L^1_\alpha(\mathbb{R}^d)$ $(0 \leq \alpha < 1)$ from the Theorem in \[13\]. If $\alpha = 0$, then it is true for $L^1(\mathbb{R}^d)$. Namely, closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for $L^1(\mathbb{R}^d)$. So, we have the following corollaries by using Theorem 3.1 and Corollary 3.2, respectively.

Corollary 3.3. Closed subgroups of $\mathbb{R}^{2d}$ are Wiener-Ditkin sets for $L(P,Q)(\mathbb{R}^{2d})$.

Corollary 3.4. Closed subgroups of $\mathbb{R}^d$ are Wiener-Ditkin sets for $L(p,q)(\mathbb{R}^d)$.
References


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