COINCIDENCE POINT RESULTS FOR WEAKLY $\alpha$-ADMISSIBLE PAIRS IN EXTENDED $b$-METRIC SPACES

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Abstract. In this paper, we obtain some coincidence point theorems for weakly-$\alpha$-admissible contractive mappings in an extended $b$-metric space. Examples are provided to illustrate the usability of the obtained results. Our results generalize the results of several papers from metric and $b$-metric framework to the setting of extended $b$-metric spaces.

1. Introduction

The concept of a $b$-metric space, as one of the useful generalizations of standard metric spaces, was firstly used by Bakhtin in [2] and Czerwik in [4]. It was further extended by Parvaneh in [12].

The concept of a weakly contractive mapping was introduced by Alber and Guerre-Delabrere [1] in the setup of Hilbert spaces. Rhoades [13] proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point. This notion was extended to generalized weakly contractive pairs by Zhang [16].

Using of an auxiliary function, usually denoted as $\alpha$, in (common) fixed point results started in the paper [15] by Samet et al., and continued in several articles (e.g., [3, 7]).

Motivated by the works [3, 6, 8, 11, 13, 17, 18, 19, 20], we prove in this paper some coincidence point results for weakly $\alpha$-admissible ($\psi, \varphi$)-contractive mappings in $p$-metric spaces, i.e., extended $b$-metric spaces. Our results extend and generalize certain recent results in the literature and provide main results in [3, 11, 14] as corollaries. We illustrate the use of these results by several examples.

2. Preliminaries

Recall (see, e.g., [2, 4]) that a $b$-metric $d$ on a set $X$ is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x, z) \leq s[d(x, y) + d(y, z)], \quad x, y, z \in X, \quad (2.1)$$

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The corresponding author of this article, dedicated it to all the victims of the Sarpol-E-Zahab earthquake, which occurred on November 12, 2017.


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for some fixed $s \geq 1$. Parvaneh and Ghoncheh introduced in [12] the following further generalization.

**Definition 2.1.** [12] Let $X$ be a (nonempty) set. A function $\tilde{d} : X \times X \to \mathbb{R}^+$ is an extended $b$-metric (p-metric, for short) if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ such that for all $x, y, z \in X$, the following conditions hold:

1. $(p_1)$ $\tilde{d}(x, y) = 0$ iff $x = y$,
2. $(p_2)$ $\tilde{d}(x, y) = \tilde{d}(y, x)$,
3. $(p_3)$ $\tilde{d}(x, z) \leq \Omega(\tilde{d}(x, y) + \tilde{d}(y, z))$.

In this case, the pair $(X, \tilde{d})$ is called an extended $b$-metric space, or, briefly, a $p$-metric space.

It should be noted that the class of $p$-metric spaces is considerably larger than the class of $b$-metric spaces, since a $b$-metric is a $p$-metric when $\Omega(t) = st$ for fixed $s \geq 1$, while a metric is a $p$-metric when $\Omega(t) = t$. Here, we present an example to show that a $p$-metric need not be a $b$-metric.

**Example 2.2.** Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and let $\tilde{d}(x, y) = \sinh d(x, y)$. We show that $\tilde{d}$ is a $p$-metric with $\Omega(t) = \sinh(st)$ for all $t \geq 0$ (and $\Omega^{-1}(u) = \frac{1}{s} \sinh^{-1} u$ for $u \geq 0$).

Obviously, conditions $(p_1)$ and $(p_2)$ of Definition 2.1 are satisfied. For each $x, y, z \in X$, we have

$$\tilde{d}(x, z) = \sinh(d(x, z))$$
$$\leq \sinh(sd(x, y) + sd(y, z))$$
$$\leq \sinh(s \sinh(d(x, y)) + s \sinh(d(y, z)))$$
$$= \Omega(\tilde{d}(x, y) + \tilde{d}(y, z)).$$

So, condition $(p_3)$ of Definition 2.1 is also satisfied and $\tilde{d}$ is a $p$-metric.

Note that $\sinh |x - y|$ is not a metric on $\mathbb{R}$, as, e.g.,

$$\sinh 5 \approx 74.203 \nless 3.627 + 10.0179 \approx \sinh 2 + \sinh 3.$$

Similarly, although $d(x, y) = (x - y)^2$ is a $b$-metric on $\mathbb{R}$ with $s = 2$, there is no $s \geq 1$ such that $\tilde{d}(x, y) = \sinh(x - y)^2$ is a $b$-metric with parameter $s$. Indeed, putting $x = 0$ and $y = 1$ we should have $\sinh x^2 \leq s(\sinh(x - 1)^2 + \sinh 1)$ which cannot hold for any fixed $s$ and $x$ sufficiently large.

More generally, several examples of $p$-metrics can be constructed using the following easy proposition.

**Proposition 2.3.** Let $(X, d)$ be a $b$-metric space with coefficient $s \geq 1$ and let $\tilde{d}(x, y) = \xi(d(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then $\tilde{d}$ is a $p$-metric with $\Omega(t) = \xi(st)$.

Taking various functions $\xi$ in the previous proposition, we can obtain a lot of examples of $p$-metrics. We state just a few of them which we will use later in the text.
Example 2.4.  
(1) If \( \xi(t) = e^t - 1 \), we get \( \tilde{d}(x, y) = e^{d(x, y)} - 1 \) and \( \Omega(t) = e^{st} - 1 \). Note that \( \Omega^{-1}(u) = \frac{1}{t} \ln(1 + u) \).
(2) If \( \xi(t) = te^t \), then \( \tilde{d}(x, y) = d(x, y)e^{d(x, y)} \) and \( \Omega(t) = tse^{st} \). Note that in this case \( \Omega^{-1}(u) = \frac{1}{t} W(u) \), for \( u \geq 0 \), where \( W \) is the Lambert \( W \)-function (see, e.g., [5]).
(3) If \( \xi(t) = t + \ln(1 + t) \), then \( \tilde{d}(x, y) = d(x, y) + \ln(1 + d(x, y)) \) and \( \Omega(t) = st + \ln(1 + st) \). Here, again \( W \)-function is used to express the inverse: \( \Omega^{-1}(u) = \frac{1}{t}(e^{u+1} - 1) \) for \( u \geq 0 \).

Definition 2.5. [12] Let \((X, d)\) be a \( p \)-metric space. Then a sequence \( \{x_n\} \) in \( X \) is called:
(a) \( p \)-convergent if there exists \( x \in X \) such that \( \tilde{d}(x_n, x) \to 0 \), as \( n \to \infty \). In this case, we write \( \lim_{n \to \infty} x_n = x \);
(b) \( p \)-Cauchy if and only if \( \tilde{d}(x_n, x_m) \to 0 \) as \( n, m \to \infty \).
(c) The space \((X, d)\) is \( p \)-complete if every \( p \)-Cauchy sequence in \( X \) \( p \)-converges.

We will need the following simple lemma about the \( p \)-convergent sequences.

Lemma 2.6. [12] Let \((X, \tilde{d})\) be a \( p \)-metric space with function \( \Omega \), and suppose that \( \{x_n\} \) and \( \{y_n\} \) \( p \)-converge to \( x, y \), respectively. Then we have
\[
(\Omega^2)^{-1}(\tilde{d}(x, y)) \leq \liminf_{n \to \infty} \tilde{d}(x_n, y_n) \leq \limsup_{n \to \infty} \tilde{d}(x_n, y_n) \leq \Omega^2(\tilde{d}(x, y)).
\]
In particular, if \( x = y \), then \( \lim_{n \to \infty} \tilde{d}(x_n, y_n) = 0 \). Moreover, for each \( z \in X \) we have
\[
\Omega^{-1}(\tilde{d}(x, z)) \leq \liminf_{n \to \infty} \tilde{d}(x_n, z) \leq \limsup_{n \to \infty} \tilde{d}(x_n, z) \leq \Omega(\tilde{d}(x, z)).
\]

Recall that self mappings \( f \) and \( g \) on a metric space \( X \) are called generalized weakly contractive [15], if there exists a lower semicontinuous function \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(0) = 0 \) and \( \varphi(t) > 0 \) for all \( t > 0 \) such that
\[
d(fx, gy) \leq N(x, y) - \varphi(N(x, y)),
\]
where,
\[
N(x, y) = \max\{d(x, y), d(fx, x), d(y, gy), \frac{1}{2}[d(x, gy) + d(y, fx)]\},
\]
for all \( x, y \in X \).

Theorem 2.7. [16] Let \((X, d)\) be a complete metric space. If \( f, g : X \to X \) are generalized weak contractions, then there exists a unique point \( u \in X \) such that \( u = fu = gu \).

The following notions have been widely used in several recent papers, starting from [15].

Definition 2.8. Let \((X, d)\) be a (generalized) metric space, \( T \) be a self-mapping on \( X \) and let \( \alpha : X \times X \to [0, +\infty) \) be a function.

(1) \( T \) is an \( \alpha \)-admissible mapping [14] if
\[
x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.
\]
(2) The space \((X, d)\) is said to be \( \alpha \)-complete [7] if every Cauchy sequence \( \{x_n\} \) in \( X \) with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), converges in \( X \).
Theorem 3.1. \[ f, g, R, S \]

\( x_n \to x \) as \( n \to \infty \) and \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \implies Tx_n \to Tx. \)

(4) \((X, d)\) is \( \alpha \)-regular \[3\] if the following condition holds: if \( x_n \to x \), where \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \alpha(x, x) \geq 1 \) for all \( n \in \mathbb{N} \).

Let \( X \) be a non-empty set and \( f : X \to X \) be a given mapping. For every \( x \in X \), let \( f^{-1}(x) = \{ u \in X : fu = x \} \).

**Definition 2.9.** \[3\] Let \( X \) be a set, \( f, g, h : X \to X \) be mappings such that \( fX \cup gX \subseteq hX \) and let \( \alpha : X \times X \to [0, \infty) \) be a function. The ordered pair \((f, g)\) is said to be:

(a) weakly \( \alpha \)-admissible with respect to \( h \) if for all \( x \in X \), \( \alpha(fx, gy) \geq 1 \) for all \( y \in h^{-1}(fx) \) and \( \alpha(gx, fy) \geq 1 \) for all \( y \in h^{-1}(gx) \).

(b) partially weakly \( \alpha \)-admissible with respect to \( h \) if \( \alpha(fx, gy) \geq 1 \) for all \( y \in h^{-1}(fx) \).

(c) The ordered pair \((f, g)\) is said to be triangular weakly \( \alpha \)-admissible (triangular partially weakly \( \alpha \)-admissible) with respect to \( h \) if it is weakly \( \alpha \)-admissible (partially weakly \( \alpha \)-admissible) with respect to \( h \) and if \( \alpha(x, z) \geq 1 \) and \( \alpha(z, y) \geq 1 \) imply \( \alpha(x, y) \geq 1 \) for all \( x, y, z \in X \).

If, in the previous conditions, \( h = I_X \) (the identity mapping), then we omit “with respect to \( h \)” in the respective notions.

**Definition 2.10.** \[3\] Let \((X, d)\) be a metric space and \( f, g : X \to X \). The pair \((f, g)\) is said to be \( \alpha \)-compatible if \( \lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \in X \).

Recall \[9\] that the self-mappings \( f, g \) of a set \( X \) are said to be weakly compatible if \( f \) and \( g \) commute at their coincidence points (i.e., \( fgx = gfx \), whenever \( fx = gx \)).

The function \( \varphi : [0, +\infty) \to [0, +\infty) \) is called an altering distance function \[10\], if the following properties hold:

1. \( \varphi \) is continuous and non-decreasing,
2. \( \varphi(t) = 0 \) if and only if \( t = 0 \).

3. **Main Results**

Let \((X, d)\) be a \( p \)-metric space and let \( f, g, R, S : X \to X \) be four self mappings. Throughout this paper, unless otherwise stated, for all \( x, y \in X \), \( M(x, y) \) will denote an arbitrary element of the set

\[
\left\{ d(Sx, Ry), \Omega^{-1} \left[ \frac{d(Sx, fx) + d(Ry, gy)}{2} \right], \Omega^{-1} \left[ \frac{d(Sx, gy) + d(Ry, fx)}{2} \right] \right\}.
\]

**Theorem 3.1.** Let \((X, d)\) be an \( \alpha \)-complete \( p \)-metric space, \( \alpha : X \times X \to [0, \infty) \) and let \( f, g, R, S : X \to X \) be four mappings such that \( f(X) \subseteq R(X) \), \( g(X) \subseteq S(X) \). Suppose that for all \( x, y \in X \) with \( \alpha(Sx, Ry) \geq 1 \), \( \psi(\Omega(d(fx, gy))) \leq \psi(M(x, y)) - \varphi(M(x, y)) \),

\[
\text{(3.1)}
\]

where \( \psi, \varphi \) are some altering distance functions. Assume that \( f, g, R \) and \( S \) are \( \alpha \)-continuous, the pairs \((f, S)\) and \((g, R)\) are \( \alpha \)-compatible and the pairs \((f, g)\) and
\((g, f)\) are triangular partially weakly \(\alpha\)-admissible with respect to \(R\) and \(S\), respectively. Then, the pairs \((f, S)\) and \((g, R)\) have a coincidence point \(z\) in \(X\). Moreover, if \(\alpha(Sz, Rz) \geq 1\), then \(z\) is a coincidence point of \(f, g, R\) and \(S\).

Proof. Let \(x_0\) be an arbitrary point of \(X\). Choose \(x_1 \in X\) such that \(fx_0 = Rx_1\) and \(x_2 \in X\) such that \(gx_1 = Sx_2\). Continuing in this way, construct a sequence \(\{z_n\}\) defined by:

\[
z_{2n+1} = Rx_{2n+1} = fx_{2n}, \quad z_{2n+2} = Sx_{2n+2} = gx_{2n+1}
\]

for all \(n \geq 0\). As \(x_1 \in R^{-1}(fx_0)\) and \(x_2 \in S^{-1}(gx_1)\) and the pairs \((f, g)\) and \((g, f)\) are partially weakly \(\alpha\)-admissible with respect to \(R\) and \(S\), respectively, we have

\[
\alpha(Rx_1, Sx_2) = \alpha(fx_0, gx_1) \geq 1
\]

and

\[
\alpha(Sx_2, Rx_1) = \alpha(gx_1, fx_0) \geq 1.
\]

Repeating this process, we obtain that \(\alpha(Rx_{2n+1}, Sx_{2n+2}) = \alpha(z_{2n+1}, z_{2n+2}) \geq 1\) and \(\alpha(Sx_{2n+2}, Rx_{2n+3}) = \alpha(z_{2n+2}, z_{2n+3}) \geq 1\) for all \(n \geq 0\).

We will complete the proof in three steps.

Step I. We will prove that \(\lim_{k \to \infty} \bar{d}(z_k, z_k+1) = 0\).

Denote \(d_k = \bar{d}(z_k, z_k+1)\). Suppose that \(d_{k_0} = 0\) for some \(k_0\). Then, \(z_{k_0} = z_{k_0+1}\). If \(k_0 = 2n\), then \(z_{2n} = z_{2n+1}\) gives \(z_{2n+1} = z_{2n+2}\). Indeed,

\[
\psi(\Omega(\bar{d}(z_{2n+1}, z_{2n+2}))) = \psi(\Omega(\bar{d}(fx_{2n}, gx_{2n+1})))
\]

\[
\leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})), \tag{3.2}
\]

where

\[
M(x_{2n}, x_{2n+1}) = \frac{\bar{d}(Sx_{2n}, Rx_{2n+1})}{2}, \quad \frac{\bar{d}(Sx_{2n}, gx_{2n+1}) + \bar{d}(Rx_{2n+1}, fx_{2n})}{2}
\]

\[
\frac{\bar{d}(Sx_{2n}, fx_{2n}) + \bar{d}(Rx_{2n+1}, gx_{2n+1})}{2}
\]

\[
\frac{\bar{d}(Sx_{2n}, gx_{2n+1}) + \bar{d}(Rx_{2n+1}, fx_{2n})}{2}
\]

\[
\frac{\bar{d}(z_{2n}, z_{2n+1})}{2}, \quad \frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2}
\]

\[
\frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2}
\]

If \(M(x_{2n}, x_{2n+1}) = \frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2}\), then (3.2) becomes

\[
\psi(\Omega(\bar{d}(z_{2n+1}, z_{2n+2}))) \leq \psi(\Omega(\frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2})) - \varphi(\Omega(\frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2}))
\]

\[
\leq \psi(\Omega(\bar{d}(z_{2n+1}, z_{2n+2}))) - \varphi(\Omega(\frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2}))
\]

which implies that \(\varphi(\Omega(\frac{\bar{d}(z_{2n+1}, z_{2n+2})}{2})) = 0\), that is, \(z_{2n} = z_{2n+1} = z_{2n+2}\).
If $M(x_{2n}, x_{2n+1}) = \Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1})}{2}\right]$, then (3.2) becomes

$$\psi(\Omega(\overline{d}(z_{2n+1}, z_{2n+2}))) \leq \psi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+2})}{2}\right]) - \varphi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+2})}{2}\right]),$$

which implies that $\varphi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+2})}{2}\right]) = 0$, that is, $z_{2n} = z_{2n+1} = z_{2n+2}$.

Similarly, if $k_0 = 2n + 1$, then $z_{2n+1} = z_{2n+2}$ gives $z_{2n+2} = z_{2n+3}$. Continuing this process, we find that $z_k$ is a constant sequence for $k \geq k_0$. Hence, $\lim_{k \to \infty} d(z_k, z_{k+1}) = 0$ holds true.

Now, suppose that

$$d_k = d(z_k, z_{k+1}) > 0$$

for each $k$. We claim that

$$d(z_{k+1}, z_{k+2}) \leq d(z_k, z_{k+1})$$

for each $k = 1, 2, 3, \ldots$.

Let $k = 2n$ and for an $n \geq 0$, $d(z_{2n+1}, z_{2n+2}) \geq d(z_{2n}, z_{2n+1}) > 0$. Then, as $\alpha(Sx_{2n}, Rx_{2n+1}) \geq 1$, using (3.1) we obtain that

$$\psi(\Omega(\overline{d}(z_{2n+1}, z_{2n+2}))) = \psi(\Omega(\overline{d}(f x_{2n}, g x_{2n+1}))) \leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})), \tag{3.5}$$

where

$$M(x_{2n}, x_{2n+1})$$

$$\in \left\{ \overline{d}(Sx_{2n}, Rx_{2n+1}), \Omega^{-1}\left[\frac{d(Sx_{2n}, f x_{2n}) + d(Rx_{2n+1}, g x_{2n+1})}{2}\right], \right\}$$

$$\Omega^{-1}\left[\frac{d(Sx_{2n}, g x_{2n+1}) + d(Rx_{2n+1}, f x_{2n})}{2}\right] \right\}$$

$$= \left\{ \overline{d}(z_{2n}, z_{2n+1}), \Omega^{-1}\left[\frac{\overline{d}(z_{2n}, z_{2n+1}) + \overline{d}(z_{2n+1}, z_{2n+2})}{2}\right], \Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+2})}{2}\right] \right\}. \right.$$

If

$$M(x_{2n}, x_{2n+1}) = \Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right] \leq \Omega^{-1}\left[\frac{d(z_{2n+1}, z_{2n+2})}{2}\right],$$

as $\overline{d}(z_{2n+1}, z_{2n+2}) \geq d(z_{2n}, z_{2n+1})$, then from (3.3), we have

$$\psi(\Omega(\overline{d}(z_{2n+1}, z_{2n+2})))$$

$$\leq \psi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right]) - \varphi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right])$$

$$\leq \psi(\Omega(\overline{d}(z_{2n+1}, z_{2n+2}))) - \varphi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right]),$$

which implies that $\varphi(\Omega^{-1}\left[\frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right]) \leq 0$. This is possible only if

$$d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2}) = 0,$$
that is, \( \bar{d}(z_{2n}, z_{2n+1}) = 0 \), a contradiction to (3.3). Hence, \( \bar{d}(z_{2n+1}, z_{2n+2}) \leq \bar{d}(z_{2n}, z_{2n+1}) \) for all \( n \geq 0 \).

Therefore, (3.4) is proved for \( k = 2n \).

Similarly, it can be shown that

\[
\bar{d}(z_{2n+2}, z_{2n+3}) \leq \bar{d}(z_{2n+1}, z_{2n+2})
\]

for all \( n \geq 0 \).

Analogously, for other values of \( M(x_{2n}, x_{2n+1}) \), we can see that \( \{\bar{d}(z_k, z_{k+1})\} \) is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an \( r \geq 0 \) such that

\[
\lim_{k \to \infty} \bar{d}(z_k, z_{k+1}) = r.
\]

(3.6)

We know that

\[
M(x_{2n}, x_{2n+1}) \in \{\bar{d}(z_{2n}, z_{2n+1}), \Omega^{-1}\left[\bar{d}(z_{2n}, z_{2n+1}) + \bar{d}(z_{2n+1}, z_{2n+2})\right]\}
\]

Substituting possible values of \( M(x_{2n}, x_{2n+1}) \) in (3.5) and then taking the limit as \( n \to \infty \) in (3.5), we obtain that \( r = 0 \). For instance, let

\[
M(x_{2n}, x_{2n+1}) = \Omega^{-1}\left[\frac{\bar{d}(z_{2n}, z_{2n+2}) + \bar{d}(z_{2n+1}, z_{2n+1})}{2}\right].
\]

Then, from (3.5) we have

\[
\psi(\Omega(\bar{d}(z_{2n+1}, z_{2n+2}))) \leq \psi(\Omega^{-1}\left[\frac{\bar{d}(z_{2n}, z_{2n+2})}{2}\right]) - \varphi(\Omega^{-1}\left[\frac{\bar{d}(z_{2n}, z_{2n+2})}{2}\right])
\]

\[
\leq \psi\left(\frac{\bar{d}(z_{2n}, z_{2n+1}) + \bar{d}(z_{2n+1}, z_{2n+2})}{2}\right) - \varphi\left(\frac{\bar{d}(z_{2n}, z_{2n+2})}{2}\right).
\]

(3.7)

Letting \( n \to \infty \) in (3.7), using (3.6) and the continuity of \( \psi \) and \( \varphi \), we have

\[
\varphi\left(\lim_{n \to \infty} \Omega^{-1}\left[\frac{\bar{d}(z_{2n}, z_{2n+2})}{2}\right]\right) = 0.
\]

Hence, \( \lim_{n \to \infty} \Omega^{-1}\left[\frac{\bar{d}(z_{2n}, z_{2n+2})}{2}\right] = 0 \), from our assumptions about \( \varphi \).

Now, taking into account (3.7) and letting \( n \to \infty \), we find that \( \psi(sr) \leq \psi(0) - \varphi(0) \). Hence, \( r = 0 \). Similarly, for the other values of \( M(x_{2n}, x_{2n+1}) \) we can show that

\[
r = \lim_{k \to \infty} \bar{d}(z_k, z_{k+1}) = \lim_{n \to \infty} \bar{d}(z_{2n}, z_{2n+1}) = 0.
\]

(3.8)

Step II. We will show that \( \{z_n\} \) is a \( p \)-Cauchy sequence in \( X \). Assume, on contrary, that there exists \( \varepsilon > 0 \) for which we can find subsequences \( \{z_{2m(k)}\} \) and \( \{z_{2n(k)}\} \) of \( \{z_n\} \) such that \( n(k) > m(k) \geq k \) and

\[
\bar{d}(z_{2m(k)}, z_{2n(k)}) \geq \varepsilon
\]

and \( n(k) \) is the smallest number such that the above condition holds, i.e.,

\[
\bar{d}(z_{2m(k)}, z_{2n(k)}-1) < \varepsilon.
\]

(3.9)
From the triangle inequality and (3.9) and (3.10), we have
\[
\varepsilon \leq \tilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega\left[\tilde{d}(z_{2m(k)}, z_{2n(k)}) + \tilde{d}(z_{2n(k)-1}, z_{2m(k)})\right].
\] (3.11)

Taking the limit as \(k \to \infty\) in (3.11), from (3.8) we obtain that
\[
\varepsilon \leq \limsup_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega(\varepsilon).
\] (3.12)

Using the triangle inequality again we have
\[
\tilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega\left[\tilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \tilde{d}(z_{2m(k)+1}, z_{2n(k)})\right].
\]

Letting \(k \to \infty\) in the above inequality, we have
\[
\Omega^{-1}(\varepsilon) \leq \limsup_{k \to \infty} \tilde{d}(z_{2m(k)+1}, z_{2n(k)}).
\] (3.13)

Finally,
\[
\tilde{d}(z_{2m(k)+1}, z_{2n(k)-1}) \leq \Omega\left[\tilde{d}(z_{2m(k)+1}, z_{2m(k)}) + \tilde{d}(z_{2m(k)}, z_{2n(k)-1})\right].
\]

Letting \(k \to \infty\), and using (3.12), we have
\[
\limsup_{k \to \infty} \tilde{d}(z_{2m(k)+1}, z_{2n(k)-1}) \leq \Omega(\varepsilon).
\] (3.14)

We know that \(2n(k) - 1 \geq 2m(k)\) and \(\alpha(Sx_{2n+2}, Rx_{2n+1}) = \alpha(gx_{2n+1}, fx_{2n}) \geq 1\) for all \(n \in \mathbb{N}\). On the other hand, the pairs \((f, g)\) and \((g, f)\) are triangular partially weakly \(\alpha\)-admissible with respect to \(R\) and \(S\), respectively. So, \(\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-2}) \geq 1\) and \(\alpha(Sx_{2n(k)-2}, Rx_{2n(k)-3}) \geq 1\) imply that \(\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1\). Also, \(\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1\) and \(\alpha(Rx_{2n(k)-3}, Sx_{2n(k)-4}) \geq 1\) imply that \(\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-4}) \geq 1\). Continuing in this manner, we obtain that \(\alpha(Rx_{2n(k)-1}, Sx_{2n(k)}) \geq 1\). Now we can apply (3.1), to obtain that
\[
\psi\left(\Omega(\tilde{d}(z_{2m(k)+1}, z_{2n(k)}))\right) = \psi\left(\Omega(\tilde{d}(fx_{2m(k)}), gx_{2n(k)-1})\right)
\leq \psi\left(M(x_{2m(k)}, x_{2n(k)-1})\right) - \varphi\left(M(x_{2m(k)}, x_{2n(k)-1})\right),
\] (3.15)

where
\[
M(x_{2m(k)}, x_{2n(k)-1})
\]
\[
\in \left\{ \tilde{d}(Sx_{2m(k)}, Rx_{2n(k)-1}), \Omega^{-1}(\tilde{d}(Sx_{2m(k)}, fx_{2m(k)}) + \tilde{d}(Rx_{2n(k)-1}, gx_{2n(k)-1}) / 2), \right. \\
\left. \Omega^{-1}(\tilde{d}(Sx_{2m(k)}, gx_{2n(k)-1}) + \tilde{d}(Rx_{2n(k)-1}, fx_{2m(k)}) / 2) \right\}
\]
\[
= \left\{ \tilde{d}(z_{2m(k)}, z_{2n(k)-1}), \Omega^{-1}(\tilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \tilde{d}(z_{2n(k)-1}, z_{2n(k)}) / 2), \right. \\
\left. \Omega^{-1}(\tilde{d}(z_{2m(k)}, z_{2n(k)}) + \tilde{d}(z_{2n(k)-1}, z_{2m(k)+1}) / 2) \right\}.
\]

If
\[
M(x_{2m(k)}, x_{2n(k)-1}) = \Omega^{-1}(\tilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \tilde{d}(z_{2n(k)-1}, z_{2n(k)}) / 2),
\]
then from (3.8), we get that \(\lim_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1}) = 0\). Hence, according to (3.15) we have, \(\lim_{k \to \infty} \tilde{d}(z_{2m(k)+1}, z_{2n(k)}) = 0\), which contradicts (3.13).
If
\[ M(x_{2m(k)}, x_{2n(k)-1}) = \Omega^{-1}\left(\tilde{d}(z_{2m(k)}, z_{2n(k)}) + \tilde{d}(z_{2n(k)-1}, z_{2m(k)+1})\right), \]
then from (3.12) and (3.14), we get that
\[ \limsup_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1}) \leq \Omega^{-1}\left(\Omega(\varepsilon) + \Omega(\varepsilon)\right) = \varepsilon. \]
Taking the limit as \( k \to \infty \) in (3.15), we have
\[ \psi(\varepsilon) = \psi(\Omega(\Omega^{-1}(\varepsilon))) \leq \psi(\limsup_{k \to \infty} \tilde{d}(z_{2m(k)+1}, z_{2n(k)})) \]
\[ \leq \psi\left(\limsup_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1})\right) - \varphi\left(\liminf_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1})\right) \]
\[ \leq \psi(\varepsilon) - \varphi\left(\liminf_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1})\right), \]
which implies that \( \varphi\left(\liminf_{k \to \infty} M(x_{2m(k)}, x_{2n(k)-1})\right) \leq 0. \) Hence, \( \liminf_{k \to \infty} \tilde{d}(x_{2m(k)}, x_{2n(k)}) = 0, \) a contradiction to (3.12).
If
\[ M(x_{2m(k)}, x_{2n(k)-1}) = \tilde{d}(x_{2m(k)}, x_{2n(k)-1}), \]
then from (3.10), by taking the limit as \( k \to \infty \) in (3.15), we have
\[ \psi(\varepsilon) = \psi(\Omega(\Omega^{-1}(\varepsilon))) \leq \psi(\limsup_{k \to \infty} \tilde{d}(z_{2m(k)+1}, z_{2n(k)})) \]
\[ \leq \psi\left(\limsup_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)-1})\right) - \varphi\left(\liminf_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)-1})\right) \]
\[ \leq \psi(\varepsilon) - \varphi\left(\liminf_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)-1})\right), \]
which implies that \( \varphi\left(\liminf_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)-1})\right) \leq 0. \) Hence, \( \liminf_{k \to \infty} \tilde{d}(z_{2m(k)}, z_{2n(k)-1}) = 0 \) which contradicts (3.12).
Hence \( \{z_n\} \) is a \( p \)-Cauchy sequence.

**Step III.** We will show that \( f, g, R \) and \( S \) have a coincidence point.
Since \( \{z_n\} \) is a \( p \)-Cauchy sequence in the \( \alpha \)-complete \( p \)-metric space \( X \) and \( \alpha(z_k, z_{k+1}) \geq 1 \), then there exists \( z \in X \) such that
\[ \lim_{n \to \infty} \tilde{d}(z_{2n+1}, z) = \lim_{n \to \infty} \tilde{d}(Rx_{2n+1}, z) = \lim_{n \to \infty} \tilde{d}(fx_{2n}, z) = 0 \]
and
\[ \lim_{n \to \infty} \tilde{d}(z_{2n}, z) = \lim_{n \to \infty} \tilde{d}(Sx_{2n}, z) = \lim_{n \to \infty} \tilde{d}(gx_{2n-1}, z) = 0. \]
Hence,
\[ Sx_{2n} \to z \text{ and } fx_{2n} \to z, \text{ as } n \to \infty. \]
As \( (f, S) \) is \( \alpha \)-compatible and \( \alpha(z_{2n}, z_{2n+2}) \geq 1 \), so,
\[ \lim_{n \to \infty} \tilde{d}(Sfx_{2n}, fSx_{2n}) = 0. \]
Moreover, from \( \lim_{n \to \infty} \tilde{d}(fx_{2n}, z) = 0 \) and \( \lim_{n \to \infty} \tilde{d}(Sx_{2n}, z) = 0 \), we obtain that
\[
\lim_{n \to \infty} \tilde{d}(Sfx_{2n}, Sz) = 0 = \lim_{n \to \infty} \tilde{d}(Sx_{2n}, fz).
\]

By the triangle inequality, we have
\[
\tilde{d}(Sz, fz) \leq \Omega[\tilde{d}(Sz, Sfx_{2n}) + \tilde{d}(Sfx_{2n}, fz)]
\]
\[
\leq \Omega[\tilde{d}(Sz, Sfx_{2n})] + \Omega^2[\tilde{d}(Sfx_{2n}, ffx_{2n}) + \tilde{d}(ffx_{2n}, fz)].
\]
Taking the limit as \( n \to \infty \) in (3.16), we obtain that
\[
\tilde{d}(Sz, fz) \leq 0,
\]
which yields that \( fz = Sz \), that is, \( z \) is a coincidence point of \( f \) and \( S \).

Similarly, it can be proved that \( gz = Rz \). Now, let \( \alpha(Rz, Sz) \geq 1 \). From (3.1) we have
\[
\psi(\Omega(\tilde{d}(fz, gz))) \leq \psi(M(z, z)) - \varphi(M(z, z)),
\]
where
\[
M(z, z) \in \left\{ \tilde{d}(Sz, Rz), \Omega^{-1}\left[ \tilde{d}(Sfx_{2n}, fz) + \Omega(\tilde{d}(Fx_{2n}, fz)) \right], \Omega^{-1}\left[ \tilde{d}(Sfx_{2n}, fz) + \tilde{d}(fx_{2n}, fz) \right] \right\}
\]
\[
= \{ \tilde{d}(fz, gz), 0, \Omega^{-1}(\tilde{d}(fz, gz)) \}.
\]
In all three cases, (3.17) yields that \( fz = gz = Sz = Rz \). \( \square \)

In the following theorem, we replace the assumption of \( \alpha \)-continuity of \( f, g, R \) and \( S \) by an assumption on the underlying space, and replace the \( \alpha \)-compatibility of the pairs \( (f, S) \) and \( (g, R) \) by weak compatibility of the pairs.

**Theorem 3.2.** Let \( (X, \tilde{d}) \) be an \( \alpha \)-regular \( \alpha \)-complete \( p \)-metric space, \( f, g, R, S : X \to X \) be four mappings such that \( f(X) \subseteq R(X) \) and \( g(X) \subseteq S(X) \) and \( R(X) \) and \( S(X) \) are \( p \)-closed subsets of \( X \). Suppose that
\[
\psi(\Omega(\tilde{d}(fx, gy))) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]
for all \( x \) and \( y \) with \( \alpha(Sx, Ry) \geq 1 \). Then, the pairs \( (f, S) \) and \( (g, R) \) have a coincidence point \( z \) in \( X \) provided that the pairs \( (f, S) \) and \( (g, R) \) are weakly compatible and the pairs \( (f, g) \) and \( (g, f) \) are triangular partially weakly \( \alpha \)-admissible with respect to \( R \) and \( S \), respectively. Moreover, if \( \alpha(Sz, Rz) \geq 1 \), then \( z \in X \) is a coincidence point of \( f, g, R \) and \( S \).

**Proof.** Following the proof of Theorem 3.1, there exists \( z \in X \) such that
\[
\lim_{k \to \infty} \tilde{d}(z_k, z) = 0.
\]
Since \( R(X) \) is \( p \)-closed and \( \{z_{2n+1}\} \subseteq R(X) \), therefore \( z \in R(X) \). Hence, there exists \( u \in X \) such that \( z = Ru \) and
\[
\lim_{n \to \infty} \tilde{d}(z_{2n+1}, Ru) = \lim_{n \to \infty} \tilde{d}(fx_{2n+1}, Ru) = 0.
\]
Similarly, there exists \( v \in X \) such that \( z = Rv = Sv \) and
\[
\lim_{n \to \infty} \tilde{d}(z_{2n}, Sv) = \lim_{n \to \infty} \tilde{d}(sx_{2n}, Sv) = 0.
\]
Now, we prove that \( v \) is a coincidence point of \( f \) and \( S \).
Since $Rx_{2n+1} \rightarrow z = Sv$, as $n \rightarrow \infty$, from $\alpha$-regularity of $X$, $\alpha(Rx_{2n+1}, Sv) \geq 1$. Therefore, from (3.18), we have

$$\psi(\tilde{d}(fv, gx_{2n+1})) \leq \psi(M(v, x_{2n+1})) - \varphi(M(v, x_{2n+1}))$$

(3.19)

where

$$M(v, x_{2n+1})$$

$$\in \left\{ \tilde{d}(Sv, rx_{2n+1}), \Omega^{-1}\left[ \frac{\tilde{d}(Sv, fv) + \tilde{d}(rx_{2n+1}, gx_{2n+1})}{2} \right] \right\}$$

$$= \left\{ \tilde{d}(z, z_{2n+1}), \Omega^{-1}\left[ \frac{\tilde{d}(z, fv) + \tilde{d}(z_{2n+1}, z_{2n})}{2} \right], \Omega^{-1}\left[ \frac{\tilde{d}(z, z_{2n}) + \tilde{d}(z_{2n+1}, fv)}{2} \right] \right\}.$$ 

From Lemma 2.6

$$\Omega^{-1}\left[ \frac{\tilde{d}(z, fv)}{2} \right] \leq \liminf_{n \rightarrow \infty} M(v, x_{2n+1}) \leq \limsup_{n \rightarrow \infty} M(v, x_{2n+1}) \leq \frac{\tilde{d}(z, fv)}{2}.$$ 

Taking the upper limit as $n \rightarrow \infty$ in (3.19), using Lemma 2.6 and the continuity of $\psi$ and $\varphi$, we obtain that

$$\psi(\tilde{d}(fv, z)) \leq \psi\left( \frac{\tilde{d}(z, fv)}{2} \right) - \varphi\left( \liminf_{n \rightarrow \infty} M(v, x_{2n+1}) \right),$$

which implies that $\Omega^{-1}\left[ \frac{\tilde{d}(z, fv)}{2} \right] = 0$, hence, $fv = z = Sv$.

As $f$ and $S$ are weakly compatible, we have $fz = fSv = Sfv = Sz$. Thus, $z$ is a coincidence point of $f$ and $S$.

Similarly, it can be shown that $z$ is a coincidence point of the pair $(g, R)$. The rest of the proof follows from similar arguments as in Theorem 3.1. $\square$

4. Examples and Consequences

**Example 4.1.** Let $X = [0, \frac{10}{37}]$, and the $p$-metric $\tilde{d}$ on $X$ be given by $\tilde{d}(x, y) = e^{|x-y|} - 1$, for all $x, y \in X$, with $\Omega(t) = e^t - 1$ (see Example 2.4.(1)), and $\alpha : X \times X \rightarrow [0, \infty)$ be given by $\alpha(x, y) = e^{1 - y}$. Define self-maps $f, g, S$ and $R$ on $X$ by

$$fx = \ln\left( 1 + \frac{x}{100} \right), \quad Rx = e^{x} - 1,$$

$$gx = \ln\left( 1 + \frac{x}{4} \right), \quad Sx = e^{25x} - 1.$$ 

Let $x, y \in X$ be such that $y \in R^{-1}fx$, that is, $Ry = fx$. By the definition of $f$ and $R$, we have $e^y - 1 = \ln(1 + (y - 25))$ and so, $y = \ln(1 + \ln(1 + \frac{y}{100}))$. Therefore,

$$fx = \ln(1 + x) \geq \ln\left( 1 + \frac{\ln(1 + \frac{1}{100})}{2} \right) = \ln\left( 1 + \frac{y}{2} \right) = gy.$$ 

Thus, $\alpha(fx, gy) \geq 1$. Hence $(f, g)$ is partially weakly $\alpha$-admissible with respect to $R$.

To prove that $(g, f)$ is partially weakly $\alpha$-admissible with respect to $S$, let $x, y \in X$ be such that $y \in S^{-1}gx$, that is, $Sy = gx$. Hence, we have $e^{25y} - 1 = \ln(1 + \frac{y}{4})$ and
so, \( y = \frac{\ln(1 + \ln(1 + \frac{x}{4}))}{25} \). Therefore,
\[
gx = \ln \left( 1 + \frac{x}{4} \right) \geq \ln \left( 1 + \frac{\ln(1 + \ln(1 + \frac{x}{4}))}{25} \right) \leq \ln \left( 1 + \frac{y}{100} \right) = fy.
\]
Thus, \( \alpha(gx, fy) \geq 1 \).

Furthermore,
\[
fX = [0, 0.002] \subseteq [0, 0.192] = RX \text{ and } gX = [0, 0.043] \subseteq [0, 79.316] = SX.
\]

Define \( \psi, \varphi : [0, \infty) \to [0, \infty) \) as \( \psi(t) = \ln(1 + \ln(1 + t)) \) and \( \varphi(t) = \frac{1}{100}t \) for all \( t \in [0, \infty) \).

Using the mean value theorem, for all \( x \) and \( y \) with \( \alpha(Sx, Ry) \geq 1 \), we have
\[
\psi(\Omega(d(fx, gy))) = \psi(\Omega(e^{[fx-gy]} - 1)) = \ln(1 + \ln(1 + e^{[fx-gy]} - 1))
\]
\[
= \left| \ln(1 + \frac{x}{100}) - \ln(1 + \frac{y}{4}) \right| \leq \left| \frac{x}{100} - \frac{y}{4} \right|
\]
\[
\leq \frac{1}{100} |x - 25y| \leq \frac{1}{100} \left| e^x - 1 - (e^{25y} - 1) \right|
\]
\[
\leq \frac{1}{100} |Sx - Ry| \leq \frac{1}{100} (e^{|Sx-Ry|} - 1)
\]
\[
\leq \ln(1 + \ln(1 + (e^{|Sx-Ry|} - 1))) \leq \frac{1}{100} (e^{|Sx-Ry|} - 1)
\]
\[
= \psi(d(Sx, Ry)) - \varphi(d(Sx, Ry)).
\]
Thus, (3.1) is true for all \( x, y \in X \) and \( M(x, y) = d(Sx, Ry) \). Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, \( 0 \) is a coincidence point of \( f, g, R \) and \( S \). \( \Box \)

**Example 4.2.** Let \( X = [0, 1] \), the p-metric \( \widetilde{d} \) on \( X \) be given by \( \widetilde{d}(x, y) = \sinh(d(x, y)) \), for all \( x, y \in X \), with \( \Omega(t) = \sinh t \) (see Example 2.2), and \( \alpha : X \times X \to [0, \infty) \) be given by \( \alpha(x, y) = e^{x-y} \). Define self-maps \( f, g, S \) and \( R \) on \( X \) by
\[
fX = \arctan \frac{x}{2}, \quad Rx = \tan \frac{2x}{3},
\]
\[
gx = \arctan \frac{6x}{10}, \quad Sx = \tan \frac{7x}{9}.
\]

To prove that \( (f, g) \) is partially weakly \( \alpha \)-admissible with respect to \( R \), let \( x, y \in X \) be such that \( y \in R^{-1}fx \), that is, \( Ry = fx \). By the definition of \( f \) and \( R \), we have
\[
\arctan \frac{x}{2} = \tan \frac{2y}{3} \text{ and so, } y = \frac{3}{2} \arctan(\arctan \frac{x}{2}).
\]
Therefore,
\[
fX = \arctan \frac{x}{2} \geq \arctan \frac{18 \arctan(\arctan(\frac{x}{2}))}{20} = \arctan \frac{6y}{10} = gy.
\]
Thus, \( \alpha(fx, gy) \geq 1 \).

To prove that \( (g, f) \) is partially weakly \( \alpha \)-admissible with respect to \( S \), let \( x, y \in X \) be such that \( y \in S^{-1}gx \), that is, \( Sy = gx \). Hence, we have \( \arctan \frac{6x}{10} = tan \frac{7y}{9} \) and so, \( y = \frac{9}{7} \arctan(\arctan \frac{6x}{10}) \). Therefore,
\[
gx = \arctan \frac{6x}{10} \geq \arctan \left( \frac{6 \cdot 9}{10} \arctan(\arctan \frac{6x}{10}) \right) \geq \arctan \frac{y}{2} = fy.
\]
Thus, \( \alpha(gx, fy) \geq 1 \).

Furthermore,
\[
fX = [0, 0.464] \subseteq [0, 0.787] = RX \text{ and } gX = [0, 0.540] \subseteq [0, 0.985] = SX.
\]
Define \( \psi, \varphi : [0, \infty) \to [0, \infty) \) as \( \psi(t) = t \sinh^{-1}(\sinh^{-1} t) \) and \( \varphi(t) = \frac{t}{1 + t^2} \).

Using the mean value theorem, for all \( x, y \) with \( \alpha(Sx, Ry) \geq 1 \), we have

\[
\psi(\Omega[\tilde{d}(fx, gy)]) = |fx - gy| \sinh^{-1}(\sinh(\sinh(|fx - gy|)))
\]

In Theorem 3.1, we obtain the following common fixed point result.

Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, 0 is a coincidence point of \( f, g, R \) and \( S \).

We state now some special cases of Theorems 3.1 and 3.2.

Taking \( S = R \) in Theorem 3.1, we obtain the following result.

**Corollary 4.3.** Let \( (X, \tilde{d}) \) be an \( \alpha \)-complete \( p \)-metric space, \( \alpha : X \times X \to [0, \infty) \) and let \( f, g, R : X \to X \) be three mappings such that \( f(X) \cup g(X) \subseteq R(X) \). Suppose that for every \( x, y \in X \) with \( \alpha(Rx, Ry) \geq 1 \),

\[
\psi(\Omega(\tilde{d}(fx, gy))) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where \( \psi, \varphi \) are some altering distance functions, and

\[
M(x, y) = \left\{ \tilde{d}(Rx, Ry), \Omega^{-1} \left[ \frac{\tilde{d}(Rx, fx) + \tilde{d}(Ry, gy)}{2} \right], \Omega^{-1} \left[ \frac{\tilde{d}(Rx, fy) + \tilde{d}(Ry, fx)}{2} \right] \right\}.
\]

Assume that \( f, g \) and \( R \) are \( \alpha \)-continuous, the pairs \( (f, R) \) and \( (g, R) \) are \( \alpha \)-compatible and the pair \( (f, g) \) is triangular weakly \( \alpha \)-admissible with respect to \( R \). Then, the mappings \( f, g, R \) have a coincidence point \( z \) in \( X \).

Taking \( f = g \) in Corollary 4.3, we obtain the following coincidence point result.

**Corollary 4.4.** Let \( (X, \tilde{d}) \) be an \( \alpha \)-complete \( p \)-metric space, \( \alpha : X \times X \to [0, \infty) \) and let \( f, R : X \to X \) be two mappings such that \( f(X) \subseteq R(X) \). Suppose that for every \( x, y \in X \) with \( \alpha(Rx, Ry) \geq 1 \),

\[
\psi(\Omega(\tilde{d}(fx, fy))) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where \( \psi, \varphi \) are some altering distance functions, and

\[
M(x, y) = \left\{ \tilde{d}(Rx, Ry), \Omega^{-1} \left[ \frac{\tilde{d}(Rx, fx) + \tilde{d}(Ry, fy)}{2} \right], \Omega^{-1} \left[ \frac{\tilde{d}(Rx, fy) + \tilde{d}(Ry, fx)}{2} \right] \right\}.
\]

Assume that \( f \) and \( R \) are \( \alpha \)-continuous and the pair \( (f, R) \) is \( \alpha \)-compatible. Then, the mappings \( f, R \) have a coincidence point \( z \) in \( X \).

Taking \( R = S = I_X \) (the identity mapping on \( X \)) in Theorems 3.1 and 3.2, we obtain the following common fixed point result.
Corollary 4.5. Let \( (X, d) \) be an \( \alpha \)-complete \( p \)-metric space and let \( f, g : X \to X \) be two mappings. Suppose that for all \( x, y \in X \) with \( \alpha(x, y) \geq 1 \),

\[
\psi\left(\Omega(d(fy, gy))\right) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where,

\[
M(x, y) \in \left\{ \frac{1}{2}\left(\frac{\Omega^{-1}(\frac{d(x, fx) + d(y, gy)}{2})}{\Omega^{-1}(\frac{d(x, fy) + d(y, fx)}{2})}\right) \right\}.
\]

Then, the pair \( (f, g) \) have a common fixed point \( z \) in \( X \) provided that the pair \( (f, g) \) is triangular weakly \( \alpha \)-admissible and either,

a. \( f \) or \( g \) is \( \alpha \)-continuous, or

b. \( X \) is \( \alpha \)-regular.

Considering now the examples of \( p \)-metric spaces given in Examples 2.2 and 2.4, we obtain the following results.

Corollary 4.6. Let \( (X, d) \) be an \( \alpha \)-regular \( b \)-metric space with parameter \( s \) and \( \alpha : X \times X \to [0, \infty) \), \( f, g, R : X \to X \) be three mappings such that \( f(X) \subseteq R(X) \) and \( g(X) \subseteq R(X) \) and \( R(X) \) is a \( b \)-closed subset of \( X \). Suppose that for all \( x, y \in X \) with \( \alpha(Rx, Ry) \geq 1 \), we have

\[
\psi\left(\sinh(s \sinh(d(fx, gy)))\right) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where \( \psi \) and \( \varphi \) are altering distance functions, and

\[
M(x, y) \in \left\{ \sinh(d(Rx, Ry)), \frac{1}{s} \sinh^{-1}\left(\frac{\sinh(d(Rx, fx)) + \sinh(d(Ry, fy))}{2}\right), \frac{1}{s} \sinh^{-1}\left(\frac{\sinh(d(Rx, fy)) + \sinh(d(Ry, fx))}{2}\right) \right\}.
\]

Then, the pairs \( (f, R) \) and \( (g, R) \) have a coincidence point \( z \) in \( X \) provided that the pairs \( (f, R) \) and \( (g, R) \) are weakly compatible and the pair \( (f, g) \) is triangular weakly \( \alpha \)-admissible with respect to \( R \). Moreover, if \( \alpha(Rx, Rz) \geq 1 \), then \( z \in X \) is a coincidence point of \( f \), \( g \) and \( R \).

Corollary 4.7. Let \( (X, d) \) be an \( \alpha \)-regular \( b \)-metric space with parameter \( s \) and \( \alpha : X \times X \to [0, \infty) \), \( f, R : X \to X \) be two mappings such that \( f(X) \subseteq R(X) \) and \( R(X) \) is a \( b \)-closed subset of \( X \). Suppose that for all \( x, y \in X \) with \( \alpha(Rx, Ry) \geq 1 \), we have

\[
\psi\left(\frac{e^{\alpha(\alpha(\alpha(fx, fy) - 1))}}{2} - \psi(M(x, y)) - \varphi(M(x, y))\right),
\]

where

\[
M(x, y) \in \left\{ \frac{1}{s} \ln\left(1 + \frac{\psi(\frac{e^{\alpha(\alpha(\alpha(fx, fy) - 1))}}{2} - \psi(M(x, y)) - \varphi(M(x, y)))}{2}\right), \right\}.
\]

Then, the pair \( (f, R) \) have a coincidence point \( z \) in \( X \) provided that the pair \( (f, R) \) is weakly compatible and \( f \) is triangular weakly \( \alpha \)-admissible with respect to \( R \).

Corollary 4.8. Let \( (X, d) \) be an \( \alpha \)-complete \( b \)-metric space with coefficient \( s \geq 1 \), \( \alpha : X \times X \to [0, \infty) \), and let \( f, g, R, S : X \to X \) be four mappings such that \( f(X) \subseteq R(X) \), \( g(X) \subseteq S(X) \). Suppose that for all \( x, y \in X \) with \( \alpha(Sx, Ry) \geq 1 \),

\[
\psi\left(s \cdot d(fx, gy) e^{\alpha(\alpha(\alpha(fx, fy) - 1))}, e^{\alpha(\alpha(\alpha(fx, fy) - 1))}\right) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]
where

\[
M(x, y) \in \left\{ d(Sx, Ry) + \ln(1 + d(Sx, Ry)), \right.
\]

\[
\frac{1}{s} \left[ \frac{d(Sx, f(x))e^{d(Sx, f(x))} + d(Ry, gy)e^{d(Ry, gy)}}{2} \right],
\]

\[
\frac{1}{s} \left[ \frac{d(Sx, gy)e^{d(Sx, gy)} + d(Ry, fx)e^{d(Ry, fx)}}{2} \right],
\]

and \(W(x)\) is the Lambert \(W\)-function. Assume that \(f, g, R, S\) are \(\alpha\)-continuous, the pairs \((f, S)\) and \((g, R)\) are \(\alpha\)-compatible and the pairs \((f, g)\) and \((g, f)\) are triangular partially weakly \(\alpha\)-admissible with respect to \(R\) and \(S\), respectively. Then, the pairs \((f, S)\) and \((g, R)\) have a coincidence point \(z\) in \(X\). Moreover, if \(\alpha(Sz, Rz) \geq 1\), then \(z\) is a coincidence point of \(f, g, R\) and \(S\).

**Corollary 4.9.** Let \((X, d)\) be an \(\alpha\)-complete \(b\)-metric space with coefficient \(s \geq 1\), \(\alpha : X \times X \to [0, \infty)\), and let \(f, g, R, S : X \to X\) be four mappings such that \(f(X) \subseteq R(X)\), \(g(X) \subseteq S(X)\). Suppose that for all \(x, y \in X\) with \(\alpha(Sx, Ry) \geq 1\),

\[
\psi \left( s \cdot d(f(x), gy) + s \cdot \ln(1 + d(f(x), gy)) + \ln(1 + s \cdot d(fx, gy) + s \cdot \ln(1 + d(fx, gy))) \right) \leq \psi(M(x, y)) - \varphi(M(x, y)),
\]

where

\[
M(x, y) \in \left\{ d(Sx, Ry) + \ln(1 + d(Sx, Ry)), \right.
\]

\[
\frac{1}{s} \left[ e^{d(Sx, f(x)) + \ln(1 + d(Sx, f(x))) + d(Ry, gy) + \ln(1 + d(Ry, gy))} \right] - 1,\]

\[
\frac{1}{s} \left[ e^{d(Sx, gy) + \ln(1 + d(Sx, gy)) + d(Ry, fx) + \ln(1 + d(Ry, fx))} \right] - 1,\]

and \(W\) is the Lambert \(W\)-function. Assume that \(f, g, R, S\) are \(\alpha\)-continuous, the pairs \((f, S)\) and \((g, R)\) are \(\alpha\)-compatible and the pairs \((f, g)\) and \((g, f)\) are triangular partially weakly \(\alpha\)-admissible with respect to \(R\) and \(S\), respectively. Then, the pairs \((f, S)\) and \((g, R)\) have a coincidence point \(z\) in \(X\). Moreover, if \(\alpha(Sz, Rz) \geq 1\), then \(z\) is a coincidence point of \(f, g, R\) and \(S\).

**References**


COINCIDENCE POINT RESULTS IN EXTENDED b-METRIC SPACES 89


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