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COINCIDENCE POINT RESULTS FOR WEAKLY α-ADMISSIBLE PAIRS IN EXTENDED b-METRIC SPACES

VAHID PARVANEH[1,*] ABDOLLAH DINMOHAMMADI[2] AND ZORAN KADELBURG[3]

Abstract. In this paper, we obtain some coincidence point theorems for weakly-α-admissible contractive mappings in an extended b-metric space. Examples are provided to illustrate the usability of the obtained results. Our results generalize the results of several papers from metric and b-metric framework to the setting of extended b-metric spaces.

1. Introduction

The concept of a b-metric space, as one of the useful generalizations of standard metric spaces, was firstly used by Bakhtin in [2] and Czerwik in [4]. It was further extended by Parvaneh in [12].

The concept of a weakly contractive mapping was introduced by Alber and Guerre-Delabrere [1] in the setup of Hilbert spaces. Rhoades [13] proved that every weakly contractive mapping defined on a complete metric space has a unique fixed point. This notion was extended to generalized weakly contractive pairs by Zhang [16].

Using of an auxiliary function, usually denoted as α, in (common) fixed point results started in the paper [15] by Samet *et al.*, and continued in several articles (e.g., [3, 7]).

Motivated by the works [3, 6, 8, 11, 14, 17, 18, 19, 20], we prove in this paper some coincidence point results for weakly α-admissible (ψ, φ)-contractive mappings in p-metric spaces, i.e., extended b-metric spaces. Our results extend and generalize certain recent results in the literature and provide main results in [3, 11, 14] as corollaries. We illustrate the use of these results by several examples.

2. Preliminaries

Recall (see, e.g., [2, 4]) that a b-metric d on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$d(x,z) \leq s[d(x,y) + d(y,z)], \quad x, y, z \in X, \tag{2.1}$$

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for some fixed $s \geq 1$. Parvaneh and Ghoncheh introduced in [12] the following further generalization.

Definition 2.1. [12] *Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to R^+$ is an extended b-metric (p-metric, for short) if there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ such that for all $x, y, z \in X$, the following conditions hold:*

- (p_1) $\widetilde{d}(x, y) = 0$ *iff* $x = y$,
- (p_2) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$,
- (p_3) $\widetilde{d}(x, z) \leq \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z))$.

In this case, the pair $(X, \widetilde{d})$ is called an extended b-metric space, or, briefly, a p-metric space.

It should be noted that the class of p-metric spaces is considerably larger than the class of b-metric spaces, since a b-metric is a p-metric when $\Omega(t) = st$ for fixed $s \geq 1$, while a metric is a p-metric when $\Omega(t) = t$. Here, we present an example to show that a p-metric need not be a b-metric.

Example 2.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\widetilde{d}(x, y) = \sinh d(x, y)$. We show that $\widetilde{d}$ is a p-metric with $\Omega(t) = \sinh(st)$ for all $t \geq 0$ (and $\Omega^{-1}(u) = \frac{1}{s} \sinh^{-1} u$ for $u \geq 0$).*

Obviously, conditions (p_1) and (p_2) of Definition 2.1 are satisfied. For each $x, y, z \in X$, we have

$$\begin{aligned} \widetilde{d}(x, z) &= \sinh(d(x, z)) \\ &\leq \sinh(sd(x, y) + sd(y, z)) \\ &\leq \sinh(s \sinh(d(x, y)) + s \sinh(d(y, z))) \\ &= \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z)). \end{aligned}$$

So, condition (p_3) of Definition 2.1 is also satisfied and $\widetilde{d}$ is a p-metric.

Note that $\sinh |x - y|$ *is not a metric on* $\mathbb{R}$, *as, e.g.,*

$$\sinh 5 \approx 74.203 \nleq 3.627 + 10.0179 \approx \sinh 2 + \sinh 3.$$

Similarly, although $d(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, there is no $s \geq 1$ such that $\widetilde{d}(x, y) = \sinh(x - y)^2$ is a b-metric with parameter s. Indeed, putting $z = 0$ and $y = 1$ we should have $\sinh x^2 \leq s(\sinh(x - 1)^2 + \sinh 1)$ which cannot hold for any fixed s and x sufficiently large.

More generally, several examples of p-metrics can be constructed using the following easy proposition.

Proposition 2.3. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\widetilde{d}(x, y) = \xi(d(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then $\widetilde{d}$ is a p-metric with $\Omega(t) = \xi(st)$.*

Taking various functions ξ in the previous proposition, we can obtain a lot of examples of p-metrics. We state just a few of them which we will use later in the text.

Example 2.4. (1) *If* $\xi(t) = e^t - 1$, *we get* $\widetilde{d}(x,y) = e^{d(x,y)} - 1$ *and* $\Omega(t) = e^{st} - 1$. *Note that* $\Omega^{-1}(u) = \frac{1}{s}\ln(1+u)$.

(2) *If* $\xi(t) = te^t$, *then* $\widetilde{d}(x,y) = d(x,y)e^{d(x,y)}$ *and* $\Omega(t) = ste^{st}$. *Note that in this case* $\Omega^{-1}(u) = \frac{1}{s}W(u)$, *for* $u \geq 0$, *where* W *is the Lambert* W*-function (see, e.g.,* [5]*).*

(3) *If* $\xi(t) = t + \ln(1+t)$, *then* $\widetilde{d}(x,y) = d(x,y) + \ln(1 + d(x,y))$ *and* $\Omega(t) = st + \ln(1+st)$. *Here, again* W*-function is used to express the inverse:* $\Omega^{-1}(u) = \frac{1}{s}(W(e^{u+1}) - 1)$ *for* $u \geq 0$.

Definition 2.5. [12] *Let* (X,d) *be a p-metric space. Then a sequence* $\{x_n\}$ *in* X *is called:*

(a) p-convergent if there exists $x \in X$ *such that* $\widetilde{d}(x_n, x) \to 0$, *as* $n \to \infty$. *In this case, we write* $\lim_{n\to\infty} x_n = x$*;*

(b) p-Cauchy if and only if $\widetilde{d}(x_n, x_m) \to 0$ *as* $n, m \to \infty$.

(c) The space (X,d) *is p-complete if every p-Cauchy sequence in* X *p-converges.*

We will need the following simple lemma about the p-convergent sequences.

Lemma 2.6. [12] *Let* $(X,\widetilde{d})$ *be a p-metric space with function* Ω, *and suppose that* $\{x_n\}$ *and* $\{y_n\}$ *p-converge to* x, y, *respectively. Then we have*

$$(\Omega^2)^{-1}(\widetilde{d}(x,y)) \leq \liminf_{n\to\infty} \widetilde{d}(x_n, y_n) \leq \limsup_{n\to\infty} \widetilde{d}(x_n, y_n) \leq \Omega^2(\widetilde{d}(x,y)).$$

In particular, if $x = y$, *then,* $\lim_{n\to\infty} \widetilde{d}(x_n, y_n) = 0$. *Moreover, for each* $z \in X$ *we have*

$$\Omega^{-1}(\widetilde{d}(x,z)) \leq \liminf_{n\to\infty} \widetilde{d}(x_n, z) \leq \limsup_{n\to\infty} \widetilde{d}(x_n, z) \leq \Omega(\widetilde{d}(x,z)).$$

Recall that self mappings f and g on a metric space X are called generalized weakly contractive [16], if there exists a lower semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, gy) \leq N(x,y) - \varphi(N(x,y)),$$

where,

$$N(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \tfrac{1}{2}[d(x,gy) + d(y,fx)]\},$$

for all $x, y \in X$.

Theorem 2.7. [16] *Let* (X,d) *be a complete metric space. If* $f, g : X \to X$ *are generalized weak contractions, then there exists a unique point* $u \in X$ *such that* $u = fu = gu$.

The following notions have been widely used in several recent papers, starting from [15].

Definition 2.8. *Let* (X,d) *be a (generalized) metric space,* T *be a self-mapping on* X *and let* $\alpha : X \times X \to [0,+\infty)$ *be a function.*

(1) T *is an* α*-admissible mapping* [15] *if*

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

(2) *The space* (X,d) *is said to be* α*-complete* [7] *if every Cauchy sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *converges in* X.

(3) *T is an α-continuous mapping on (X,d)* [7], *if, for given $x \in X$ and sequence $\{x_n\}$,*

$$x_n \to x \text{ as } n \to \infty \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \implies Tx_n \to Tx.$$

(4) *(X,d) is α-regular* [3] *if the following condition holds: if $x_n \to x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Let X be a non-empty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 2.9. [3] *Let X be a set, $f, g, h : X \to X$ be mappings such that $fX \cup gX \subseteq hX$ and let $\alpha : X \times X \to [0,\infty)$ be a function. The ordered pair (f,g) is said to be:*

(a) *weakly α-admissible with respect to h if for all $x \in X$, $\alpha(fx, gy) \geq 1$ for all $y \in h^{-1}(fx)$ and $\alpha(gx, fy) \geq 1$ for all $y \in h^{-1}(gx)$,*
(b) *partially weakly α-admissible with respect to h if $\alpha(fx, gy) \geq 1$ for all $y \in h^{-1}(fx)$.*
(c) *The ordered pair (f,g) is said to be triangular weakly α-admissible (triangular partially weakly α-admissible) with respect to h if it is weakly α-admissible (partially weakly α-admissible) with respect to h and if $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply $\alpha(x,y) \geq 1$ for all $x, y, z \in X$.*

If, in the previous conditions, $h = I_X$ (the identity mapping), then we omit "with respect to h" in the respective notions.

Definition 2.10. [3] *Let (X,d) be a metric space and $f, g : X \to X$. The pair (f,g) is said to be α-compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Recall [9] that the self-mappings f, g of a set X are said to be weakly compatible if f and g commute at their coincidence points (i.e., $fgx = gfx$, whenever $fx = gx$).

The function $\varphi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function [10], if the following properties hold:

(1) φ is continuous and non-decreasing,
(2) $\varphi(t) = 0$ if and only if $t = 0$.

3. Main results

Let $(X, \widetilde{d})$ be a p-metric space and let $f, g, R, S : X \to X$ be four self mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, $M(x,y)$ will denote an arbitrary element of the set

$$\Big\{\widetilde{d}(Sx,Ry), \Omega^{-1}\Big[\frac{\widetilde{d}(Sx,fx)+\widetilde{d}(Ry,gy)}{2}\Big], \Omega^{-1}\Big[\frac{\widetilde{d}(Sx,gy)+\widetilde{d}(Ry,fx)}{2}\Big]\Big\}.$$

Theorem 3.1. *Let $(X, \widetilde{d})$ be an α-complete p-metric space, $\alpha : X \times X \to [0,\infty)$ and let $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$. Suppose that for all $x, y \in X$ with $\alpha(Sx, Ry) \geq 1$,*

$$\psi\big(\Omega(\widetilde{d}(fx,gy))\big) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big), \tag{3.1}$$

where ψ, φ are some altering distance functions. Assume that f, g, R and S are α-continuous, the pairs (f,S) and (g,R) are α-compatible and the pairs (f,g) and

(g,f) are triangular partially weakly α-admissible with respect to R and S, respectively. Then, the pairs (f,S) and (g,R) have a coincidence point z in X. Moreover, if $\alpha(Sz,Rz)\ge 1$, then z is a coincidence point of f,g,R and S.

Proof. Let x_0 be an arbitrary point of X. Choose $x_1\in X$ such that $fx_0=Rx_1$ and $x_2\in X$ such that $gx_1=Sx_2$. Continuing in this way, construct a sequence $\{z_n\}$ defined by:

$$z_{2n+1}=Rx_{2n+1}=fx_{2n},\quad z_{2n+2}=Sx_{2n+2}=gx_{2n+1}$$

for all $n\ge 0$. As $x_1\in R^{-1}(fx_0)$ and $x_2\in S^{-1}(gx_1)$ and the pairs (f,g) and (g,f) are partially weakly α-admissible with respect to R and S, respectively, we have

$$\alpha(Rx_1,Sx_2)=\alpha(fx_0,gx_1)\ge 1$$

and

$$\alpha(Sx_2,Rx_3)=\alpha(gx_1,fx_2)\ge 1.$$

Repeating this process, we obtain that $\alpha(Rx_{2n+1},Sx_{2n+2})=\alpha(z_{2n+1},z_{2n+2})\ge 1$ and $\alpha(Sx_{2n+2},Rx_{2n+3})=\alpha(z_{2n+2},z_{2n+3})\ge 1$ for all $n\ge 0$.

We will complete the proof in three steps.

Step I. We will prove that $\lim\limits_{k\to\infty}\widetilde{d}(z_k,z_{k+1})=0$.

Denote $d_k=\widetilde{d}(z_k,z_{k+1})$. Suppose that $d_{k_0}=0$ for some k_0. Then, $z_{k_0}=z_{k_0+1}$. If $k_0=2n$, then $z_{2n}=z_{2n+1}$ gives $z_{2n+1}=z_{2n+2}$. Indeed,

$$\begin{aligned}\psi\big(\Omega(\widetilde{d}(z_{2n+1},z_{2n+2}))\big)&=\psi\big(\Omega(\widetilde{d}(fx_{2n},gx_{2n+1}))\big)\\&\le\psi\big(M(x_{2n},x_{2n+1})\big)-\varphi\big(M(x_{2n},x_{2n+1})\big),\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}M(x_{2n},x_{2n+1})\in\Big\{&\widetilde{d}(Sx_{2n},Rx_{2n+1}),\Omega^{-1}[\frac{\widetilde{d}(Sx_{2n},fx_{2n})+\widetilde{d}(Rx_{2n+1},gx_{2n+1})}{2}],\\&\Omega^{-1}[\frac{\widetilde{d}(Sx_{2n},gx_{2n+1})+\widetilde{d}(Rx_{2n+1},fx_{2n})}{2}]\Big\}\\=\Big\{&\widetilde{d}(z_{2n},z_{2n+1}),\Omega^{-1}[\frac{\widetilde{d}(z_{2n},z_{2n+1})+\widetilde{d}(z_{2n+1},z_{2n+2})}{2}],\\&\Omega^{-1}[\frac{\widetilde{d}(z_{2n},z_{2n+2})+\widetilde{d}(z_{2n+1},z_{2n+1})}{2}]\Big\}\\=\Big\{&0,\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}],\Omega^{-1}[\frac{\widetilde{d}(z_{2n},z_{2n+2})}{2}]\Big\}.\end{aligned}$$

If $M(x_{2n},x_{2n+1})=\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}]$, then (3.2) becomes

$$\begin{aligned}\psi\big(\Omega(\widetilde{d}(z_{2n+1},z_{2n+2}))\big)&\le\psi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}]\big)-\varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}]\big)\\&\le\psi\big(\Omega(\widetilde{d}(z_{2n+1},z_{2n+2}))\big)-\varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}]\big),\end{aligned}$$

which implies that $\varphi(\Omega^{-1}[\frac{\widetilde{d}(z_{2n+1},z_{2n+2})}{2}])=0$, that is, $z_{2n}=z_{2n+1}=z_{2n+2}$.

If $M(x_{2n}, x_{2n+1}) = \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]$, then (3.2) becomes

$$\begin{aligned}\psi\big(\Omega(\widetilde{d}(z_{2n+1}, z_{2n+2}))\big) &\leq \psi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big) \\ &\leq \psi\big(\widetilde{d}(z_{2n+1}, z_{2n+2})\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big),\end{aligned}$$

which implies that $\varphi(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]) = 0$, that is, $z_{2n} = z_{2n+1} = z_{2n+2}$.

Similarly, if $k_0 = 2n + 1$, then $z_{2n+1} = z_{2n+2}$ gives $z_{2n+2} = z_{2n+3}$. Continuing this process, we find that z_k is a constant sequence for $k \geq k_0$. Hence, $\lim_{k\to\infty} \widetilde{d}(z_k, z_{k+1}) = 0$ holds true.

Now, suppose that

$$d_k = \widetilde{d}(z_k, z_{k+1}) > 0 \tag{3.3}$$

for each k. We claim that

$$\widetilde{d}(z_{k+1}, z_{k+2}) \leq \widetilde{d}(z_k, z_{k+1}) \tag{3.4}$$

for each $k = 1, 2, 3, \cdots$.

Let $k = 2n$ and for an $n \geq 0$, $\widetilde{d}(z_{2n+1}, z_{2n+2}) \geq \widetilde{d}(z_{2n}, z_{2n+1}) > 0$. Then, as $\alpha(Sx_{2n}, Rx_{2n+1}) \geq 1$, using (3.1) we obtain that

$$\begin{aligned}\psi\big(\Omega(\widetilde{d}(z_{2n+1}, z_{2n+2}))\big) &= \psi\big(\Omega(\widetilde{d}(fx_{2n}, gx_{2n+1}))\big) \\ &\leq \psi\big(M(x_{2n}, x_{2n+1})\big) - \varphi\big(M(x_{2n}, x_{2n+1})\big),\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}&M(x_{2n}, x_{2n+1}) \\ &\in \Big\{\widetilde{d}(Sx_{2n}, Rx_{2n+1}), \Omega^{-1}[\frac{\widetilde{d}(Sx_{2n}, fx_{2n}) + \widetilde{d}(Rx_{2n+1}, gx_{2n+1})}{2}], \\ &\qquad \Omega^{-1}[\frac{\widetilde{d}(Sx_{2n}, gx_{2n+1}) + \widetilde{d}(Rx_{2n+1}, fx_{2n})}{2}]\Big\} \\ &= \Big\{\widetilde{d}(z_{2n}, z_{2n+1}), \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}], \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\Big\}.\end{aligned}$$

If

$$M(x_{2n}, x_{2n+1}) = \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}] \leq \Omega^{-1}[\widetilde{d}(z_{2n+1}, z_{2n+2})],$$

as $\widetilde{d}(z_{2n+1}, z_{2n+2}) \geq \widetilde{d}(z_{2n}, z_{2n+1})$, then from (3.5), we have

$$\begin{aligned}&\psi\big(\Omega(\widetilde{d}(z_{2n+1}, z_{2n+2}))\big) \\ &\leq \psi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}]\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}]\big) \\ &\leq \psi\big(\Omega(\widetilde{d}(z_{2n+1}, z_{2n+2}))\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}]\big),\end{aligned}$$

which implies that $\varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}]\big) \leq 0$. This is possible only if

$$\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2} = 0,$$

that is, $\widetilde{d}(z_{2n}, z_{2n+1}) = 0$, a contradiction to (3.3). Hence, $\widetilde{d}(z_{2n+1}, z_{2n+2}) \le \widetilde{d}(z_{2n}, z_{2n+1})$ for all $n \ge 0$.

Therefore, (3.4) is proved for $k = 2n$.

Similarly, it can be shown that

$$\widetilde{d}(z_{2n+2}, z_{2n+3}) \le \widetilde{d}(z_{2n+1}, z_{2n+2})$$

for all $n \ge 0$.

Analogously, for other values of $M(x_{2n}, x_{2n+1})$, we can see that $\{\widetilde{d}(z_k, z_{k+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an $r \ge 0$ such that

$$\lim_{k\to\infty} \widetilde{d}(z_k, z_{k+1}) = r. \tag{3.6}$$

We know that

$$M(x_{2n}, x_{2n+1}) \in \Big\{\widetilde{d}(z_{2n}, z_{2n+1}), \Omega^{-1}\big[\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}\big], \\ \Omega^{-1}\big[\frac{\widetilde{d}(z_{2n}, z_{2n+2}) + \widetilde{d}(z_{2n+1}, z_{2n+1})}{2}\big]\Big\}.$$

Substituting possible values of $M(x_{2n}, x_{2n+1})$ in (3.5) and then taking the limit as $n \to \infty$ in (3.5), we obtain that $r = 0$. For instance, let

$$M(x_{2n}, x_{2n+1}) = \Omega^{-1}\big[\frac{\widetilde{d}(z_{2n}, z_{2n+2}) + \widetilde{d}(z_{2n+1}, z_{2n+1})}{2}\big].$$

Then, from (3.5) we have

$$\begin{aligned} &\psi\big(\Omega(\widetilde{d}(z_{2n+1}, z_{2n+2}))\big) \\ &\le \psi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big) \\ &\le \psi\big(\frac{\widetilde{d}(z_{2n}, z_{2n+1}) + \widetilde{d}(z_{2n+1}, z_{2n+2})}{2}\big) - \varphi\big(\Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big). \end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in (3.7), using (3.6) and the continuity of ψ and φ, we have

$$\varphi\big(\lim_{n\to\infty} \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}]\big) = 0.$$

Hence, $\lim_{n\to\infty} \Omega^{-1}[\frac{\widetilde{d}(z_{2n}, z_{2n+2})}{2}] = 0$, from our assumptions about φ.

Now, taking into account (3.7) and letting $n \to \infty$, we find that $\psi\big(sr\big) \le \psi\big(0\big) - \varphi\big(0\big)$. Hence, $r = 0$. Similarly, for the other values of $M(x_{2n}, x_{2n+1})$ we can show that

$$r = \lim_{k\to\infty} \widetilde{d}(z_k, z_{k+1}) = \lim_{n\to\infty} \widetilde{d}(z_{2n}, z_{2n+1}) = 0. \tag{3.8}$$

Step II. We will show that $\{z_n\}$ is a p-Cauchy sequence in X. Assume, on contrary, that there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{2m(k)}\}$ and $\{z_{2n(k)}\}$ of $\{z_{2n}\}$ such that $n(k) > m(k) \ge k$ and

$$\widetilde{d}(z_{2m(k)}, z_{2n(k)}) \ge \varepsilon \tag{3.9}$$

and $n(k)$ is the smallest number such that the above condition holds, *i.e.*,

$$\widetilde{d}(z_{2m(k)}, z_{2n(k)-1}) < \varepsilon. \tag{3.10}$$

From the triangle inequality and (3.9) and (3.10), we have

$$\varepsilon \leq \widetilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega[\widetilde{d}(z_{2m(k)}, z_{2n(k)-1}) + \widetilde{d}(z_{2n(k)-1}, z_{2n(k)})]. \tag{3.11}$$

Taking the limit as $k \to \infty$ in (3.11), from (3.8) we obtain that

$$\varepsilon \leq \limsup_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega(\varepsilon). \tag{3.12}$$

Using the triangle inequality again we have

$$\widetilde{d}(z_{2m(k)}, z_{2n(k)}) \leq \Omega[\widetilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \widetilde{d}(z_{2m(k)+1}, z_{2n(k)})].$$

Letting $k \to \infty$ in the above inequality, we have

$$\Omega^{-1}(\varepsilon) \leq \limsup_{k\to\infty} \widetilde{d}(z_{2m(k)+1}, z_{2n(k)}). \tag{3.13}$$

Finally,

$$\widetilde{d}(z_{2m(k)+1}, z_{2n(k)-1}) \leq \Omega[\widetilde{d}(z_{2m(k)+1}, z_{2m(k)}) + \widetilde{d}(z_{2m(k)}, z_{2n(k)-1})].$$

Letting $k \to \infty$, and using (3.12), we have

$$\limsup_{k\to\infty} \widetilde{d}(z_{2m(k)+1}, z_{2n(k)-1}) \leq \Omega(\varepsilon). \tag{3.14}$$

We know that $2n(k) - 1 \geq 2m(k)$ and $\alpha(Sx_{2n+2}, Rx_{2n+1}) = \alpha(gx_{2n+1}, fx_{2n}) \geq 1$ for all $n \in \mathbb{N}$. On the other hand, the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to R and S, respectively. So, $\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-2}) \geq 1$ and $\alpha(Sx_{2n(k)-2}, Rx_{2n(k)-3}) \geq 1$ imply that $\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1$. Also, $\alpha(Rx_{2n(k)-1}, Rx_{2n(k)-3}) \geq 1$ and $\alpha(Rx_{2n(k)-3}, Sx_{2n(k)-4}) \geq 1$ imply that $\alpha(Rx_{2n(k)-1}, Sx_{2n(k)-4}) \geq 1$. Continuing in this manner, we obtain that $\alpha(Rx_{2n(k)-1}, Sx_{2m(k)}) \geq 1$. Now we can apply (3.1), to obtain that

$$\begin{aligned} \psi\big(\Omega(\widetilde{d}(z_{2m(k)+1}, z_{2n(k)}))\big) &= \psi\big(\Omega(\widetilde{d}(fx_{2m(k)}, gx_{2n(k)-1}))\big) \\ &\leq \psi\big(M(x_{2m(k)}, x_{2n(k)-1})\big) - \varphi\big(M(x_{2m(k)}, x_{2n(k)-1})\big), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} &M(x_{2m(k)}, x_{2n(k)-1}) \\ &\in \Big\{\widetilde{d}(Sx_{2m(k)}, Rx_{2n(k)-1}), \Omega^{-1}\big(\frac{\widetilde{d}(Sx_{2m(k)}, fx_{2m(k)}) + \widetilde{d}(Rx_{2n(k)-1}, gx_{2n(k)-1})}{2}\big), \\ &\quad \Omega^{-1}\big(\frac{\widetilde{d}(Sx_{2m(k)}, gx_{2n(k)-1}) + \widetilde{d}(Rx_{2n(k)-1}, fx_{2m(k)})}{2}\big)\Big\} \\ &= \Big\{\widetilde{d}(z_{2m(k)}, z_{2n(k)-1}), \Omega^{-1}\big(\frac{\widetilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \widetilde{d}(z_{2n(k)-1}, z_{2n(k)})}{2}\big), \\ &\quad \Omega^{-1}\big(\frac{\widetilde{d}(z_{2m(k)}, z_{2n(k)}) + \widetilde{d}(z_{2n(k)-1}, z_{2m(k)+1})}{2}\big)\Big\}. \end{aligned}$$

If

$$M(x_{2m(k)}, x_{2n(k)-1}) = \Omega^{-1}\big(\frac{\widetilde{d}(z_{2m(k)}, z_{2m(k)+1}) + \widetilde{d}(z_{2n(k)-1}, z_{2n(k)})}{2}\big),$$

then from (3.8), we get that $\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) = 0$. Hence, according to (3.15) we have, $\lim_{k\to\infty} \widetilde{d}(z_{2m(k)+1}, z_{2n(k)}) = 0$, which contradicts (3.13).

If

$$M(x_{2m(k)}, x_{2n(k)-1}) = \Omega^{-1}\big(\frac{\widetilde{d}(z_{2m(k)}, z_{2n(k)}) + \widetilde{d}(z_{2n(k)-1}, z_{2m(k)+1})}{2}\big),$$

then from (3.12) and (3.14), we get that

$$\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1}) \leq \Omega^{-1}\big(\frac{\Omega(\varepsilon)+\Omega(\varepsilon)}{2}\big) = \varepsilon.$$

Taking the limit as $k \to \infty$ in (3.15), we have

$$\begin{aligned}\psi(\varepsilon) &= \psi(\Omega(\Omega^{-1}(\varepsilon))) \\ &\leq \psi\big(\Omega(\limsup_{k\to\infty} \widetilde{d}(z_{2m(k)+1}, z_{2n(k)}))\big) \\ &\leq \psi\big(\limsup_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\big) - \varphi\big(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\big) \\ &\leq \psi(\varepsilon) - \varphi\big(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})\big),\end{aligned}$$

which implies that $\varphi(\liminf_{k\to\infty} M(x_{2m(k)}, x_{2n(k)-1})) \leq 0$. Hence, $\liminf_{k\to\infty} \widetilde{d}(x_{2m(k)}, x_{2n(k)}) = 0$, a contradiction to (3.12).

If

$$M(x_{2m(k)}, x_{2n(k)-1}) = \widetilde{d}(x_{2m(k)}, x_{2n(k)-1}),$$

then from (3.10), by taking the limit as $k \to \infty$ in (3.15), we have

$$\begin{aligned}\psi(\varepsilon) &= \psi(\Omega(\Omega^{-1}(\varepsilon))) \\ &\leq \psi\big(\Omega(\limsup_{k\to\infty} \widetilde{d}(z_{m(k)+1}, z_{n(k)}))\big) \\ &\leq \psi\big(\limsup_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)-1})\big) - \varphi\big(\liminf_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)-1})\big) \\ &\leq \psi(\varepsilon) - \varphi(\liminf_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)-1})),\end{aligned}$$

which implies that $\varphi(\liminf_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)-1})) \leq 0$. Hence, $\liminf_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)-1}) = 0$. Therefore, from the triangular inequality we can conclude that $\liminf_{k\to\infty} \widetilde{d}(z_{2m(k)}, z_{2n(k)}) = 0$ which contradicts (3.12).

Hence $\{z_n\}$ is a p-Cauchy sequence.

Step III. We will show that f, g, R and S have a coincidence point.

Since $\{z_n\}$ is a p-Cauchy sequence in the α-complete p-metric space X and $\alpha(z_k, z_{k+1}) \geq 1$, then there exists $z \in X$ such that

$$\lim_{n\to\infty} \widetilde{d}(z_{2n+1}, z) = \lim_{n\to\infty} \widetilde{d}(Rx_{2n+1}, z) = \lim_{n\to\infty} \widetilde{d}(fx_{2n}, z) = 0$$

and

$$\lim_{n\to\infty} \widetilde{d}(z_{2n}, z) = \lim_{n\to\infty} \widetilde{d}(Sx_{2n}, z) = \lim_{n\to\infty} \widetilde{d}(gx_{2n-1}, z) = 0.$$

Hence,

$$Sx_{2n} \to z \text{ and } fx_{2n} \to z, \text{ as } n \to \infty.$$

As (f, S) is α-compatible and $\alpha(z_{2n}, z_{2n+2}) \geq 1$, so,

$$\lim_{n\to\infty} \widetilde{d}(Sfx_{2n}, fSx_{2n}) = 0.$$

Moreover, from $\lim_{n\to\infty} \widetilde{d}(fx_{2n}, z) = 0$, $\lim_{n\to\infty} \widetilde{d}(Sx_{2n}, z) = 0$ and the α-continuity of S and f, we obtain that

$$\lim_{n\to\infty} \widetilde{d}(Sfx_{2n}, Sz) = 0 = \lim_{n\to\infty} \widetilde{d}(fSx_{2n}, fz).$$

By the triangle inequality, we have

$$\begin{aligned}\widetilde{d}(Sz, fz) &\le \Omega[\widetilde{d}(Sz, Sfx_{2n}) + \widetilde{d}(Sfx_{2n}, fz)]\\ &\le \Omega[\widetilde{d}(Sz, Sfx_{2n})] + \Omega^2[\widetilde{d}(Sfx_{2n}, fSx_{2n}) + \widetilde{d}(fSx_{2n}, fz)]. \qquad (3.16)\end{aligned}$$

Taking the limit as $n \to \infty$ in (3.16), we obtain that

$$\widetilde{d}(Sz, fz) \le 0,$$

which yields that $fz = Sz$, that is, z is a coincidence point of f and S.

Similarly, it can be proved that $gz = Rz$. Now, let $\alpha(Rz, Sz) \ge 1$. From (3.1) we have

$$\psi\big(\Omega(\widetilde{d}(fz, gz))\big) \le \psi\big(M(z, z)\big) - \varphi\big(M(z, z)\big), \qquad (3.17)$$

where

$$\begin{aligned}M(z, z) &\in \Big\{\widetilde{d}(Sz, Rz), \Omega^{-1}[\frac{\widetilde{d}(Sz, fz) + \widetilde{d}(Rz, gz)}{2}], \Omega^{-1}[\frac{\widetilde{d}(Sz, gz) + \widetilde{d}(Rz, fz)}{2}]\Big\}\\ &= \{\widetilde{d}(fz, gz), 0, \Omega^{-1}[\widetilde{d}(fz, gz)]\}.\end{aligned}$$

In all three cases, (3.17) yields that $fz = gz = Sz = Rz$. □

In the following theorem, we replace the assumption of α-continuity of f, g, R and S by an assumption on the underlying space, and replace the α-compatibility of the pairs (f, S) and (g, R) by weak compatibility of the pairs.

Theorem 3.2. *Let $(X, \widetilde{d})$ be an α-regular α-complete p-metric space, $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and $R(X)$ and $S(X)$ are p-closed subsets of X. Suppose that*

$$\psi\big(\Omega(\widetilde{d}(fx, gy))\big) \le \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big), \qquad (3.18)$$

for all x and y with $\alpha(Sx, Ry) \ge 1$. Then, the pairs (f, S) and (g, R) have a coincidence point z in X provided that the pairs (f, S) and (g, R) are weakly compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible with respect to R and S, respectively. Moreover, if $\alpha(Sz, Rz) \ge 1$, then $z \in X$ is a coincidence point of f, g, R and S.

Proof. Following the proof of Theorem 3.1, there exists $z \in X$ such that

$$\lim_{k\to\infty} \widetilde{d}(z_k, z) = 0.$$

Since $R(X)$ is p-closed and $\{z_{2n+1}\} \subseteq R(X)$, therefore $z \in R(X)$. Hence, there exists $u \in X$ such that $z = Ru$ and

$$\lim_{n\to\infty} \widetilde{d}(z_{2n+1}, Ru) = \lim_{n\to\infty} \widetilde{d}(Rx_{2n+1}, Ru) = 0.$$

Similarly, there exists $v \in X$ such that $z = Ru = Sv$ and

$$\lim_{n\to\infty} \widetilde{d}(z_{2n}, Sv) = \lim_{n\to\infty} \widetilde{d}(Sx_{2n}, Sv) = 0.$$

Now, we prove that v is a coincidence point of f and S.

Since $Rx_{2n+1} \to z = Sv$, as $n \to \infty$, from α-regularity of X, $\alpha(Rx_{2n+1}, Sv) \geq 1$. Therefore, from (3.18), we have

$$\psi\big(\Omega(\widetilde{d}(fv, gx_{2n+1}))\big) \leq \psi\big(M(v, x_{2n+1})\big) - \varphi\big(M(v, x_{2n+1})\big), \tag{3.19}$$

where

$$\begin{aligned} &M(v, x_{2n+1}) \\ &\quad \in \Big\{\widetilde{d}(Sv, Rx_{2n+1}), \Omega^{-1}[\frac{\widetilde{d}(Sv, fv) + \widetilde{d}(Rx_{2n+1}, gx_{2n+1})}{2}], \\ &\qquad \Omega^{-1}[\frac{\widetilde{d}(Sv, gx_{2n+1}) + \widetilde{d}(Rx_{2n+1}, fv)}{2}]\Big\} \\ &\quad = \Big\{\widetilde{d}(z, z_{2n+1}), \Omega^{-1}[\frac{\widetilde{d}(z, fv) + \widetilde{d}(z_{2n+1}, z_{2n})}{2}], \Omega^{-1}[\frac{\widetilde{d}(z, z_{2n}) + \widetilde{d}(z_{2n+1}, fv)}{2}]\Big\}. \end{aligned}$$

From Lemma 2.6,

$$\Omega^{-1}[\frac{\Omega^{-1}[\widetilde{d}(z, fv)]}{2}] \leq \liminf_{n\to\infty} M(v, x_{2n+1}) \leq \limsup_{n\to\infty} M(v, x_{2n+1}) \leq \frac{\widetilde{d}(z, fv)}{2}.$$

Taking the upper limit as $n \to \infty$ in (3.19), using Lemma 2.6 and the continuity of ψ and φ, we obtain that

$$\psi\big(\widetilde{d}(fv, z)\big) \leq \psi\big(\frac{\widetilde{d}(z, fv)}{2}\big) - \varphi\big(\liminf_{n\to\infty} M(v, x_{2n+1})\big),$$

which implies that $\Omega^{-1}[\frac{\Omega^{-1}[\widetilde{d}(z,fv)]}{2}] = 0$, hence, $fv = z = Sv$.

As f and S are weakly compatible, we have $fz = fSv = Sfv = Sz$. Thus, z is a coincidence point of f and S.

Similarly, it can be shown that z is a coincidence point of the pair (g, R). The rest of the proof follows from similar arguments as in Theorem 3.1. □

4. Examples and consequences

Example 4.1. *Let $X = [0, \frac{10}{57}]$, and the p-metric $\widetilde{d}$ on X be given by $\widetilde{d}(x, y) = e^{|x-y|} - 1$, for all $x, y \in X$, with $\Omega(t) = e^t - 1$ (see Example 2.4.(1)), and $\alpha : X \times X \to [0, \infty)$ be given by $\alpha(x, y) = e^{x-y}$. Define self-maps f, g, S and R on X by*

$$fx = \ln\Big(1 + \frac{x}{100}\Big), \qquad Rx = e^x - 1,$$
$$gx = \ln\Big(1 + \frac{x}{4}\Big), \qquad Sx = e^{25x} - 1.$$

Let $x, y \in X$ be such that $y \in R^{-1}fx$, that is, $Ry = fx$. By the definition of f and R, we have $e^y - 1 = \ln(1 + \frac{x}{100})$ and so, $y = \ln(1 + \ln(1 + \frac{x}{100}))$. Therefore,

$$fx = \ln(1 + x) \geq \ln\left(1 + \frac{\ln(1 + \ln(1 + \frac{x}{100}))}{2}\right) = \ln\Big(1 + \frac{y}{2}\Big) = gy.$$

Thus, $\alpha(fx, gy) \geq 1$. Hence (f, g) is partially weakly α-admissible with respect to R.

To prove that (g, f) is partially weakly α-admissible with respect to S, let $x, y \in X$ be such that $y \in S^{-1}gx$, that is, $Sy = gx$. Hence, we have $e^{25y} - 1 = \ln(1 + \dfrac{x}{4})$ and *so,* $y = \frac{\ln(1+\ln(1+\frac{x}{4}))}{25}$. *Therefore,*

$$gx = \ln\left(1+\frac{x}{4}\right) \geq \ln\left(1+\frac{\frac{\ln(1+\ln(1+\frac{x}{4}))}{25}}{100}\right) \leq \ln\left(1+\frac{y}{100}\right) = fy.$$

Thus, $\alpha(gx, fy) \geq 1$.

Furthermore,

$$fX = [0, 0.002] \subseteq [0, 0.192] = RX \text{ and } gX = [0, 0.043] \subseteq [0, 79.316] = SX.$$

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ *as* $\psi(t) = \ln(1+\ln(1+t))$ *and* $\varphi(t) = \frac{1}{100}t$ *for all* $t \in [0,\infty)$.

Using the mean value theorem, for all x *and* y *with* $\alpha(Sx, Ry) \geq 1$, *we have*

$$\begin{aligned}
\psi(\Omega[\widetilde{d}(fx, gy)]) &= \psi(\Omega[e^{|fx-gy|}-1]) = \ln(1+\ln(1+e^{e^{|fx-gy|}-1}-1)) \\
&= \left|\ln(1+\frac{x}{100}) - \ln(1+\frac{y}{4})\right| \leq \left|\frac{x}{100} - \frac{y}{4}\right| \\
&\leq \frac{|x-25y|}{100} \leq \frac{1}{100}\left|e^{x} - 1 - (e^{25y}-1)\right| \\
&\leq \frac{1}{100}|Sx - Ry| \leq \frac{1}{100}(e^{|Sx-Ry|}-1) \\
&\leq \ln(1+\ln(1+(e^{|Sx-Ry|}-1))) - \frac{1}{100}(e^{|Sx-Ry|}-1) \\
&= \psi(\widetilde{d}(Sx, Ry)) - \varphi(\widetilde{d}(Sx, Ry)).
\end{aligned}$$

Thus, (3.1) is true for all $x, y \in X$ *and* $M(x,y) = \widetilde{d}(Sx, Ry)$. *Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover,* 0 *is a coincidence point of* f, g, R *and* S. □

Example 4.2. *Let* $X = [0,1]$, *the p-metric* $\widetilde{d}$ *on* X *be given by* $\widetilde{d}(x,y) = \sinh(d(x,y))$, *for all* $x, y \in X$, *with* $\Omega(t) = \sinh t$ *(see Example 2.2), and* $\alpha : X \times X \to [0,\infty)$ *be given by* $\alpha(x,y) = e^{x-y}$. *Define self-maps* f, g, S *and* R *on* X *by*

$$fx = \arctan\frac{x}{2}, \qquad Rx = \tan\frac{2x}{3},$$
$$gx = \arctan\frac{6x}{10}, \qquad Sx = \tan\frac{7x}{9}.$$

To prove that (f, g) *is partially weakly* α*-admissible with respect to* R, *let* $x, y \in X$ *be such that* $y \in R^{-1}fx$, *that is,* $Ry = fx$. *By the definition of* f *and* R, *we have* $\arctan\frac{x}{2} = \tan\frac{2y}{3}$ *and so,* $y = \frac{3}{2}\arctan(\arctan\frac{x}{2})$. *Therefore,*

$$fx = \arctan\frac{x}{2} \geq \arctan\frac{18\arctan(\arctan(\frac{x}{2}))}{20} = \arctan\frac{6y}{10} = gy.$$

Thus, $\alpha(fx, gy) \geq 1$.

To prove that (g, f) *is partially weakly* α*-admissible with respect to* S, *let* $x, y \in X$ *be such that* $y \in S^{-1}gx$, *that is,* $Sy = gx$. *Hence, we have* $\arctan\frac{6x}{10} = \tan\frac{7y}{9}$ *and so,* $y = \frac{9}{7}\arctan(\arctan\frac{6x}{10})$. *Therefore,*

$$gx = \arctan\frac{6x}{10} \geq \arctan\left(\frac{6}{10}\cdot\frac{9}{7}\arctan(\arctan(\frac{6x}{10}))\right) \geq \arctan\frac{y}{2} = fy.$$

Thus, $\alpha(gx, fy) \geq 1$.

Furthermore,

$$fX = [0, 0.464] \subseteq [0, 0.787] = RX \text{ and } gX = [0, 0.540] \subseteq [0, 0.985] = SX.$$

Define $\psi, \varphi : [0,\infty) \to [0,\infty)$ *as* $\psi(t) = t\sinh^{-1}(\sinh^{-1} t)$ *and* $\varphi(t) = \frac{t}{1+t^2}$.

Using the mean value theorem, for all x *and* y *with* $\alpha(Sx, Ry) \geq 1$*, we have*

$$\begin{aligned}
\psi(\Omega[\widetilde{d}(fx, gy)]) &= |fx - gy| \sinh^{-1}(\sinh^{-1}(\sinh(\sinh(|fx - gy|)))) \\
&= |fx - gy|^2 = |\arctan\frac{x}{2} - \arctan\frac{6y}{10}|^2 \\
&\leq |\frac{x}{2} - \frac{6y}{10}|^2 \leq |\frac{2x}{3} - \frac{7y}{9}|^2 \\
&\leq |\tan\frac{2x}{3} - \tan\frac{7y}{9}|^2 \leq |\tan\frac{2x}{3} - \tan\frac{7y}{9}| \\
&= |Sx - Ry| \leq \sinh(|Sx - Ry|) \\
&\leq \sinh(|Sx - Ry|)\sinh^{-1}(\sinh^{-1}(\sinh(|Sx - Ry|))) - \frac{\sinh(|Sx - Ry|)}{1 + [\sinh(|Sx - Ry|)]^2} \\
&= \psi(\widetilde{d}(Sx, Ry)) - \varphi(\widetilde{d}(Sx, Ry)).
\end{aligned}$$

Thus, (3.1) is true for all $x, y \in X$ *and* $M(x, y) = \widetilde{d}(Sx, Ry)$*. Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover,* 0 *is a coincidence point of* f*,* g*,* R *and* S*.* □

We state now some special cases of Theorems 3.1 and 3.2.

Taking $S = R$ in Theorem 3.1, we obtain the following result.

Corollary 4.3. *Let* $(X, \widetilde{d})$ *be an* α*-complete* p*-metric space,* $\alpha : X \times X \to [0, \infty)$ *and let* $f, g, R : X \to X$ *be three mappings such that* $f(X) \cup g(X) \subseteq R(X)$*. Suppose that for every* $x, y \in X$ *with* $\alpha(Rx, Ry) \geq 1$*,*

$$\psi\big(\Omega(\widetilde{d}(fx, gy))\big) \leq \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big),$$

where ψ, φ *are some altering distance functions, and*

$$M(x,y) \in \left\{ \widetilde{d}(Rx, Ry), \Omega^{-1}\left[\frac{\widetilde{d}(Rx, fx) + \widetilde{d}(Ry, gy)}{2}\right], \Omega^{-1}\left[\frac{\widetilde{d}(Rx, gy) + \widetilde{d}(Ry, fx)}{2}\right] \right\}.$$

Assume that f, g *and* R *are* α*-continuous, the pairs* (f, R) *and* (g, R) *are* α*-compatible and the pair* (f, g) *is triangular weakly* α*-admissible with respect to* R*. Then, the mappings* f, g, R *have a coincidence point* z *in* X*.*

Taking $f = g$ in Corollary 4.3, we obtain the following coincidence point result.

Corollary 4.4. *Let* $(X, \widetilde{d})$ *be an* α*-complete* p*-metric space,* $\alpha : X \times X \to [0, \infty)$ *and let* $f, R : X \to X$ *be two mappings such that* $f(X) \subseteq R(X)$*. Suppose that for every* $x, y \in X$ *with* $\alpha(Rx, Ry) \geq 1$*,*

$$\psi\big(\Omega(\widetilde{d}(fx, fy))\big) \leq \psi\big(M(x, y)\big) - \varphi\big(M(x, y)\big),$$

where ψ, φ *are some altering distance functions, and*

$$M(x,y) \in \left\{ \widetilde{d}(Rx, Ry), \Omega^{-1}\left[\frac{\widetilde{d}(Rx, fx) + \widetilde{d}(Ry, fy)}{2}\right], \Omega^{-1}\left[\frac{\widetilde{d}(Rx, fy) + \widetilde{d}(Ry, fx)}{2}\right] \right\}.$$

Assume that f *and* R *are* α*-continuous and the pair* (f, R) *is* α*-compatible. Then, the mappings* f, R *have a coincidence point* z *in* X*.*

Taking $R = S = I_X$ (the identity mapping on X) in Theorems 3.1 and 3.2, we obtain the following common fixed point result.

Corollary 4.5. *Let $(X,\widetilde{d})$ be an α-complete p-metric space and let $f,g:X\to X$ be two mappings. Suppose that for all $x,y\in X$ with $\alpha(x,y)\geq 1$,*

$$\psi\big(\Omega(\widetilde{d}(fx,gy))\big)\leq \psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big),$$

where,

$$M(x,y)\in\Big\{\widetilde{d}(x,y),\Omega^{-1}\big(\frac{\widetilde{d}(x,fx)+\widetilde{d}(y,gy)}{2}\big),\Omega^{-1}\big(\frac{\widetilde{d}(x,gy)+\widetilde{d}(y,fx)}{2}\big)\Big\}.$$

Then, the pair (f,g) have a common fixed point z in X provided that the pair (f,g) is triangular weakly α-admissible and either,

a. f or g is α-continuous, or

b. X is α-regular.

Considering now the examples of p-metric spaces given in Examples 2.2 and 2.4, we obtain the following results.

Corollary 4.6. *Let (X,d) be an α-regular b-metric space with parameter s and $\alpha:X\times X\to[0,\infty)$, $f,g,R:X\to X$ be three mappings such that $f(X)\subseteq R(X)$ and $g(X)\subseteq R(X)$ and $R(X)$ is a b-closed subset of X. Suppose that for all $x,y\in X$ with $\alpha(Rx,Ry)\geq 1$, we have*

$$\psi\big(\sinh(s\sinh(d(fx,gy)))\big)\leq\psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big),$$

where ψ and φ are altering distance functions, and

$$M(x,y)\in\Big\{\sinh(d(Rx,Ry)),\frac{1}{s}\sinh^{-1}\Big[\frac{\sinh(d(Rx,fx))+\sinh(d(Ry,gy))}{2}\Big], \frac{1}{s}\sinh^{-1}\Big[\frac{\sinh(d(Rx,gy))+\sinh(d(Ry,fx))}{2}\Big]\Big\}.$$

Then, the pairs (f,R) and (g,R) have a coincidence point z in X provided that the pairs (f,R) and (g,R) are weakly compatible and the pair (f,g) is triangular weakly α-admissible with respect to R. Moreover, if $\alpha(Rz,Rz)\geq 1$, then $z\in X$ is a coincidence point of f, g and R.

Corollary 4.7. *Let (X,d) be an α-regular b-metric space with parameter s and $\alpha:X\times X\to[0,\infty)$, $f,R:X\to X$ be two mappings such that $f(X)\subseteq R(X)$ and $R(X)$ is a b-closed subset of X. Suppose that for all $x,y\in X$ with $\alpha(Rx,Ry)\geq 1$, we have*

$$\psi\big(e^{s(e^{d(fx,fy)}-1)}-1\big)\leq\psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big),$$

where

$$M(x,y)\in\Big\{e^{d(Rx,Ry)}-1,\frac{1}{s}\ln\Big(1+\frac{e^{d(Rx,fx)}+e^{d(Ry,fy)}-2}{2}\Big), \frac{1}{s}\ln\Big(1+\frac{e^{d(Rx,fy)}+e^{d(Ry,fx)}-2}{2}\Big)\Big\}.$$

Then, the pair (f,R) have a coincidence point z in X provided that the pair (f,R) is weakly compatible and f is triangular weakly α-admissible with respect to R.

Corollary 4.8. *Let (X,d) be an α-complete b-metric space with coefficient $s\geq 1$, $\alpha:X\times X\to[0,\infty)$, and let $f,g,R,S:X\to X$ be four mappings such that $f(X)\subseteq R(X)$, $g(X)\subseteq S(X)$. Suppose that for all $x,y\in X$ with $\alpha(Sx,Ry)\geq 1$,*

$$\psi\big(s\cdot d(fx,gy)e^{d(fx,gy)}e^{s\cdot d(fx,gy)e^{d(fx,gy)}}\big)\leq\psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big),$$

where

$$M(x,y) \in \Big\{ d(Sx,Ry)e^{d(Sx,Ry)}, \frac{1}{s}W\Big[\frac{d(Sx,fx)e^{d(Sx,fx)}+d(Ry,gy)e^{d(Ry,gy)}}{2}\Big], \frac{1}{s}W\Big[\frac{d(Sx,gy)e^{d(Sx,gy)}+d(Ry,fx)e^{d(Ry,fx)}}{2}\Big]\Big\},$$

and $W(x)$ *is the Lambert* W*-function. Assume that* f, g, R *and* S *are* α*-continuous, the pairs* (f,S) *and* (g,R) *are* α*-compatible and the pairs* (f,g) *and* (g,f) *are triangular partially weakly* α*-admissible with respect to* R *and* S*, respectively. Then, the pairs* (f,S) *and* (g,R) *have a coincidence point* z *in* X*. Moreover, if* $\alpha(Sz,Rz) \geq 1$*, then* z *is a coincidence point of* f, g, R *and* S.

Corollary 4.9. *Let* (X,d) *be an* α*-complete* b*-metric space with coefficient* $s \geq 1$, $\alpha : X \times X \to [0,\infty)$*, and let* $f,g,R,S : X \to X$ *be four mappings such that* $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$*. Suppose that for all* $x,y \in X$ *with* $\alpha(Sx,Ry) \geq 1$,

$$\psi\Big(s\cdot d(fx,gy)+s\cdot\ln(1+d(fx,gy))+\ln\big(1+s\cdot d(fx,gy)+s\cdot\ln(1+d(fx,gy))\big)\Big) \leq \psi\big(M(x,y)\big)-\varphi\big(M(x,y)\big),$$

where

$$M(x,y) \in \Big\{ d(Sx,Ry)+\ln(1+d(Sx,Ry)), \frac{1}{s}\Big[W\big[e^{\frac{d(Sx,fx)+\ln(1+d(Sx,fx))+d(Ry,gy)+\ln(1+d(Ry,gy))}{2}+1}\big]-1\Big], \frac{1}{s}\Big[W\big[e^{\frac{d(Sx,gy)+\ln(1+d(Sx,gy))+d(Ry,fx)+\ln(1+d(Ry,fx))}{2}+1}\big]-1\Big]\Big\}$$

and W *is the Lambert* W*-function. Assume that* f, g, R *and* S *are* α*-continuous, the pairs* (f,S) *and* (g,R) *are* α*-compatible and the pairs* (f,g) *and* (g,f) *are triangular partially weakly* α*-admissible with respect to* R *and* S*, respectively. Then, the pairs* (f,S) *and* (g,R) *have a coincidence point* z *in* X*. Moreover, if* $\alpha(Sz,Rz) \geq 1$*, then* z *is a coincidence point of* f, g, R *and* S.

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Vahid Parvaneh
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.
E-mail address: `zam.dalahoo@gmail.com`

Abdollah Dinmohammadi
Department of Mathematics, Buein Zahra Thechnical University, Buein Zahra, Qazvin, Iran
E-mail address: `mathdin@yahoo.com`

Zoran Kadelburg
Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Serbia
E-mail address: `kadelbur@matf.bg.ac.rs`