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DOUBLE CONVECTION OF HEAT AND MASS TRANSFER FLOW OF MHD GENERALIZED SECOND GRADE FLUID OVER AN EXPONENTIALLY ACCELERATED INFINITE VERTICAL PLATE WITH HEAT ABSORPTION

M. NAZAR, M.AHMAD, M. A. IMRAN, N. A. SHAH

Abstract. Influence of non-integer order fractional parameter and magnetic field is studied on a generalized second grade fluid model with double convection, caused due to simultaneous effects of heat and mass transfer induced by temperature and concentration gradients. Additional effects of heat generation and chemical reaction are also considered. A generalized model of second grade fluid consists of three partial differential equations of momentum, heat and mass transfer with corresponding initial and boundary condition is formed with non-integer order Caputo time fractional derivative. Exact solutions for temperature, concentration and velocity fields in terms of special functions are developed by means of Laplace transform method. A comparison of results is plotted graphically for second grade and Newtonian fluids and interesting behavior of the flow was seen. It shows how fractional derivative controls the fluid flow for small and large time.

Nomenclature

B_0- uniform applied magnetic field,
c_w- concentration of the plate,
$c_\infty-$concentration of the fluid far away from the plate,
c_p- specific heat at constant pressure,
$d-$ solute mass diffusivity,
$g-$ acceleration due to gravity,
$gr-$ Grashof number for heat transfer,
$gm-$ Grashof number for mass transfer,
k_r-dimensional chemical reaction parameter,
$\lambda-$dimensionless chemical reaction parameter,
$m-$ magnetic field parameter,
$pr-$ Prandtl number,
$Q-$ Dimensional Heat absorption parameter,
$S-$ Non-Dimensional Heat absorption parameter,

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$Sc-$ Schmidt number,
$P-$ Pressure,
T_W- Temperature of the plate,
$T_\infty-$ Temperature of the fluid far away from the plate,
$k-$ Thermal conductivity of the fluid,
β_T- Thermal expansion coefficient,
β_C- Volumetric coefficient of expansion with species concentration,
$\nu-$ Kinematic viscosity,
$\rho-$ Density of the fluid,
$\sigma-$ Electric conductivity,
α_2- Second grade parameter,
$Nu-$ Nusselt number,

1. Introduction

Natural convection over an infinite vertical plate, resulting from buoyancy forces, has useful applications in industrial and technological areas such as heat exchangers, electronic cooling instruments, aeronautics, and nuclear reactors. Free convection flow occurs not only due to temperature gradient, but also due to concentration difference or both. Several transports phenomena exist in nature and industry, in which the simultaneous heat and mass transfer occur as a result of combined buoyancy effects and diffusion of chemical species. Fractional calculus technique is an old one and has been extended in various fields such as fractional-order multipoles in electromagnetism, electrochemistry, tracer in fluid flows, neurons model in biology, finance, signal processing, applied mathematics, bio-engineering, viscoelasticity, fluid mechanics, and fluid dynamics [1]. Fractional approach is useful when it comes to generalization of complex dynamics of fluid motion. Increasing trends in modeling with the help of fractional derivative has been seen in several existing fluid models. Exact solutions of various constitutive models with fractional derivatives approach to the natural convection flows are not common in present literature. Fractional model with non-integer order parameter has an advantage over ordinary constitutive relationship as it readily assesses the properties of ordinary derivative models. Recently, Vieru *et al.* [2] used Caputo time fractional derivative to find the exact solutions for temperature, concentration, and velocity field of viscous fluid by considering the Newtonian heating and constant mass diffusion with first order chemical reaction on the boundary. Imran *et al.* [3] extended [2] to some large class namely the second grade fluid for the similar motion. Khan *et al.* [4] has used time fractional derivative technique for the oscillating motion of infinite vertical plate with constant temperature and in the absence of mass diffusion. A latest survey has been captured in the reference list in which many researchers had used the fractional derivative approach to solve different problems of non- Newtonian fluids under different thermal and mechanical conditions [5-24]. To study the effect of non-integer order fractional parameter and magnetic field on the coupled heat and mass transfer for free convective flows over an infinite vertical plate with radiation and chemical reaction using fractional calculus tool is a motivation for present investigation. The novelty of the present paper consists in identifying solution that has never been reported in the existing block of literature. Therefore, the aim of the present study is to analyze the influence of non-integer order fractional parameter and magnetic field on a generalized second grade fluid model with double

convection, caused due to simultaneous effects of heat and mass transfer induced by temperature and concentration gradients. Additional effects of heat generation and chemical reaction are also considered. A generalized model of second grade fluid consists of three partial differential equations of momentum, heat and mass transfer with corresponding initial and boundary condition is formed with non-integer order Caputo time fractional derivative. Exact solutions for temperature, concentration, and velocity fields in terms of special functions are developed by means of Laplace transform method. A comparison of results is plotted graphically for second grade and Newtonian fluids and interesting observations were seen that how fractional parameter controls the fluid flow for small and large time.

2. Mathematical formulation of the problem

The unsteady magneto-hydrodynamic flow of second grade incompressible fluid past an exponentially accelerated isothermal infinite vertical plate with variable temperature and variable mass diffusion in the presence of heat absorption has been studied. Initially, the plate and the fluid are at the same temperature T_∞ in the stationary condition with concentration level C_∞ at all the points. At time $t > 0$, the plate is exponentially accelerated with a velocity $U_0 H(t)\exp(at)$ in its own plane and the temperature of the plate is raised linearly with time and species concentration level near the plate is also raised linearly with time t. The temperature of the plate and the concentration level are also raised or lowered to $T_\infty + (T_w - T_\infty)t$ and $C_\infty + (C_w - C_\infty)t$ respectively. We made the following assumptions: All the fluid physical properties are considered to be constant except the influence of the body force term. Applied transverse magnetic field of uniform strength B_0 is normal to the plate. The fluid's conducting property is supposed to be slight and hence the magnetic Reynolds number is lesser than unity and the induced magnetic field is small in comparison with the transverse magnetic field. It is further supposed that there is no applied voltage, as the electric field is absent. Viscous dissipation and Joule heating in energy equation are neglected. According to Boussinesq's approximation, the unsteady flow is governed by the following set of equations

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\alpha_1}{\rho}\frac{\partial^3 u}{\partial y^2 \partial t} + g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty) - \frac{\sigma B_0^2}{\rho}u, \tag{2.1}$$

$$C_P \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial y^2} - Q(T - T_\infty), \tag{2.2}$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial y^2} - K_r(C - C_\infty), \tag{2.3}$$

with initial and boundary conditions

$$u = 0,\ T = 0,\ C = 0;\ t = 0,\ y > 0, \tag{2.4}$$

$$u = U_0 H(t)\exp(at), T = T_\infty + (T - T_\infty)t, C = C_\infty + (C - C_\infty)t; y = 0,\ \ t > 0, \tag{2.5}$$

$$u \to 0,\ T \to 0,\ C \to 0 \quad \text{as} \ \ y \to \infty,\ t > 0. \tag{2.6}$$

Introducing the following dimensionless variables and parameters

$$y^* = \frac{U_0 y}{\nu}, t^* = \frac{U_0^2 t}{\nu}, u^* = \frac{u}{U_0}, T^* = \frac{T - T_\infty}{T_w - T_\infty}, C^* = \frac{C - C_\infty}{C_w - C_\infty},$$
$$Gr = \frac{g\beta_T \nu (T_w - T_\infty)}{U_0^3}, Gm = \frac{g\beta_C \nu (C_w - C_\infty)}{U_0^3}, M = \frac{\sigma B_0^2 \nu}{\rho U_0^2}, \tag{2.7}$$
$$Pr = \frac{\nu C_P}{k}, Sc = \frac{\nu}{D}, N = \frac{16\sigma T_\infty^3}{3kk'}, \alpha_2 = \frac{\alpha_1 U_0^2}{\mu\nu}, S = \frac{Q\nu^2}{ku_o^2},\ \lambda = \frac{D\nu^2}{u_o^2}$$

into Eqs. (2.1)-(2.6) and dropping the star notation, we have the following initial-boundary problem

$$\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2} + \alpha_2 \frac{\partial^3 u(y,t)}{\partial y^2 \partial t} + GrT(y,t) + GmC(y,t) - Mu(y,t), \tag{2.8}$$

$$Pr\frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 T}{\partial y^2} - ST(y,t), \tag{2.9}$$

$$Sc\frac{\partial C(y,t)}{\partial t} = \frac{\partial^2 C(y,t)}{\partial y^2} - \lambda C(y,t), \tag{2.10}$$

with dimensionless initial and boundary conditions

$$u(y,0) = 0,\ T(y,0) = 0,\ C(y,0) = 0;\ y > 0, \tag{2.11}$$

$$u(0,t) = H(t)\exp(at),\ T(0,t) = t,\ C(0,t) = t, t > 0, \tag{2.12}$$

$$u(y,t) \to 0,\ T(y,t) \to 0,\ C(y,t) \to 0,\ \text{as } y \to \infty,\ t > 0. \tag{2.13}$$

Here, we have to develop a fractional model, replacing the time derivative in Eqs. (2.8), (2.9) and (2.10), with time-fractional derivatives, we obtain the following fractional differential equations

$$D_t^\alpha u(y,t) = \frac{\partial^2 u(y,t)}{\partial y^2} + \alpha_2 D_t^\alpha \frac{\partial^2 u(y,t)}{\partial y^2} + GrT(y,t) + GmC(y,t) - Mu(y,t), \tag{2.14}$$

$$PrD_t^\alpha T(y,t) = \frac{\partial^2 T}{\partial y^2} - ST(y,t), \tag{2.15}$$

$$ScD_t^\alpha C(y,t) = \frac{\partial^2 C}{\partial y^2} - \lambda C(y,t), \tag{2.16}$$

where $D_t^\alpha u(y,t)$ represent the Caputo time-fractional derivative of $u(y,t)$ defined as

$$D_t^\alpha u(y,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{\partial u(y,\tau)}{\partial \tau} d\tau, & 0 \le \alpha < 1; \\ \frac{\partial u(y,t)}{\partial t}, & \alpha = 1. \end{cases} \tag{2.17}$$

3. Solution of the problem

3.1. **Calculation for temperature.** Applying the Laplace transform to Eqs. (2.15), $(2.12)_2, (2.13)_2$, and using initial condition, we obtain

$$Prq^\alpha \bar{T}(y,q) = \frac{\partial^2 \bar{T}(y,q)}{\partial y^2} - S\bar{T}(y,q), \tag{3.1}$$

$$\bar{T}(0,q) = \frac{1}{q^2}, \quad \bar{T}(y,q) \to 0 \ \text{ as } \ y \to \infty. \tag{3.2}$$

The solution of the partial differential equation (3.1), by using conditions in Eq. (3.2) is

$$\bar{T}(y,q) = \frac{1}{q^2}\exp\Big(- y\sqrt{Pr}\sqrt{\Big(q^\alpha + \frac{S}{Pr}\Big)}\Big). \tag{3.3}$$

Taking the inverse Laplace transform of Eq. (3.3), we obtain

$$\begin{aligned} T(y,t) = \int_0^\infty \int_0^t \operatorname{erfc}\Big(\frac{y\sqrt{Pr}}{2\sqrt{x}}\Big)\exp\Big(- \frac{S}{Pr}x\Big)\frac{1}{\tau}\phi(0,\alpha;-x\tau^{-\alpha})\times \\ \times\Big[\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{S}{Pr}(t-\tau)\Big]d\tau dx, \end{aligned} \tag{3.4}$$

where $\phi(\beta,-\sigma;z)$ is a Wright function defined in Appendix.

The heat transfer rate at the plate in terms of Nusselt number Nu can be obtained using any of the above expressions for temperature, namely

$$Nu = -\frac{\partial T(y,t)}{\partial y}|_{y=0} = -\frac{\partial}{\partial y}L^{-1}\{\overline{T}(y,q)\}|_{y=0} = L^{-1}\Big\{\frac{\partial\overline{T}(y,q)}{\partial y}\Big|_{y=0}\Big\}, \tag{3.5}$$

$$Nu = -\sqrt{\mathrm{Pr}}\Big[\frac{t^{-\frac{\alpha}{2}}}{\Gamma(1-\frac{\alpha}{2})} - aG_{1,\frac{\alpha}{2},1}(-b,t)\Big], \alpha\in(0,1). \tag{3.6}$$

3.2. **Temperature field for the ordinary case** ($\alpha = 1$)**.** By taking $\alpha \to 1$ into Eq. (3.3), we obtain the

$$\overline{T}(y,q) = \frac{1}{q^2}\exp\Big(- y\sqrt{Pr}\sqrt{\Big(q + \frac{S}{Pr}\Big)}\Big), \tag{3.7}$$

with inverse Laplace transform

$$\begin{aligned} T(y,t) = \frac{1}{2}\Big[\Big(t - \frac{y}{2\sqrt{S}}\Big)\exp\Big(- \frac{y\sqrt{S}}{Pr}\Big)\operatorname{erfc}\Big(\frac{y\sqrt{Pr}}{2\sqrt{t}} - \sqrt{\frac{S}{Pr}t}\Big) + \\ +\Big(t + \frac{y}{2\sqrt{S}}\Big)\exp\Big(\frac{y\sqrt{S}}{Pr}\Big)\operatorname{erfc}\Big(\frac{y\sqrt{Pr}}{2\sqrt{t}} + \sqrt{\frac{S}{Pr}t}\Big)\Big], \end{aligned} \tag{3.8}$$

and Nusselt number

$$Nu = -\sqrt{\mathrm{Pr}}\Big[\frac{1}{\sqrt{\pi t}} - aG_{1,\frac{1}{2},1}(-b,t)\Big],\ \alpha = 1, \tag{3.9}$$

where $\mathrm{G}_{a,b,c}(d,t)$ is the G-function of Lorenzo-Hartley.

3.3. **Calculation for concentration.** It is observed that, the initial-boundary value problem for concentration $C(y,t)$ has the same form as the problem for temperature field, i.e.

$$\overline{C}(y,q) = \frac{1}{q^2}\exp\Big(- y\sqrt{Sc}\sqrt{\Big(q^\alpha + \frac{\lambda}{Sc}\Big)}\Big), \tag{3.10}$$

so, using the results from previous section, we have

$$C(y,t)=\int_0^\infty\int_0^t \operatorname{erfc}\Big(\frac{y\sqrt{Sc}}{2\sqrt{x}}\Big)\exp\Big(-\frac{\lambda}{Sc}x\Big)\frac{1}{\tau}\phi(0,\alpha,-x\tau^{-\alpha})\times \\ \times\Big[\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}+\frac{\lambda}{Sc}(t-\tau)\Big]d\tau dx, \tag{3.11}$$

respectively

$$C(y,t)=\frac{1}{2}\Big[\Big(t-\frac{y}{2\sqrt{\lambda}}\Big)\exp\Big(-\frac{y\sqrt{\lambda}}{Sc}\Big)\operatorname{erfc}\Big(\frac{y\sqrt{Sc}}{2\sqrt{t}}-\sqrt{\frac{\lambda}{Sc}t}\Big)+ \\ +\Big(t+\frac{y}{2\sqrt{\lambda}}\Big)\exp\Big(\frac{y\sqrt{\lambda}}{Sc}\Big)\operatorname{erfc}\Big(\frac{y\sqrt{Sc}}{2\sqrt{t}}+\sqrt{\frac{\lambda}{Sc}t}\Big)\Big],\alpha=1. \tag{3.12}$$

3.4. **Calculation for velocity.** Applying the Laplace transform to Eqs. (2.8), $(2.12)_1, (2.13)_1$, and using initial condition and Eqs. (3.3) and (3.10), we obtain

$$(q^\alpha+M)\bar{u}(y,q)=\frac{\partial^2\bar{u}(y,q)}{\partial y^2}+\alpha_2 q^\alpha\frac{\partial^2\bar{u}(y,q)}{\partial y^2}+Gr\frac{1}{q^2}\exp\Big(-y\sqrt{Pr}\sqrt{\Big(q^\alpha+\frac{S}{Pr}\Big)}\Big) \\ +Gm\frac{1}{q^2}\exp\Big(-y\sqrt{Sc}\sqrt{\Big(q^\alpha+\frac{\lambda}{Sc}\Big)}\Big), \tag{3.13}$$

$$\bar{u}(0,q)=\frac{1}{q-a},\overline{u}(y,q)\to 0 \text{ as } y\to\infty, \tag{3.14}$$

where $\bar{u}(y,q)$ is the Laplace transform of the function $u(y,t)$ and q is the transform variable. The solution of the partial differential equation (3.13), with conditions (3.14), is

$$\bar{u}(y,q) = \frac{1}{q-a}\exp\Big(-y\sqrt{q^\alpha+M}\Big)+ \\ \frac{Gr}{[(\alpha_2 q^\alpha+1)(Prq^\alpha+S)-(q^\alpha+M)]q^2}\Big[\exp\Big(-y\sqrt{\frac{q^\alpha+M}{\alpha_2 q^\alpha+1}}\Big)-\exp\Big(-y\sqrt{Prq^\alpha+S}\Big)\Big]+ \\ \frac{Gm}{[(\alpha_2 q^\alpha+1)(Scq^\alpha+\lambda)-(q^\alpha+M)]q^2}\Big[\exp\Big(-y\sqrt{\frac{q^\alpha+M}{\alpha_2 q^\alpha+1}}\Big)-\exp\Big(-y\sqrt{Scq^\alpha+\lambda}\Big)\Big]. \tag{3.15}$$

Eq. (3.15) can also be written in suitable form

$$\begin{aligned}
\bar{u}(y,q) &= \frac{1}{q-a}\exp\Big(-y\sqrt{q^\alpha+M}\Big)+ \\
&\frac{Gr}{m_1-m_2}\left[m_1\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-m_1}-m_2\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-m_2}\right]\frac{1}{q^\alpha}.\exp\Big(-y\sqrt{\frac{q^\alpha+M}{\alpha_2 q^\alpha+1}}\Big) \\
&-\frac{Gm}{m_1-m_2}\left[(Prm_1+S)\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-m_1}-(Prm_2+S)\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-m_2}\right] \\
&\times\frac{1}{(Prq^\alpha+S)}.\exp\Big(-y\sqrt{Prq^\alpha+S}\Big)+ \\
&\frac{Gm}{n_1-n_2}\left[n_1\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-n_1}-n_2\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-n_2}\right]\frac{1}{q^\alpha}.\exp\Big(-y\sqrt{\frac{q^\alpha+M}{\alpha_2 q^\alpha+1}}\Big)- \\
&\frac{Gm}{n_1-n_2}\left[(Scn_1+\lambda)\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-n_1}-(Scn_2+\lambda)\frac{q^{\alpha-(\alpha+2)}}{q^\alpha-n_2}\right] \\
&\times\frac{1}{(Scq^\alpha+\lambda)}.\exp\Big(-y\sqrt{Scq^\alpha+\lambda}\Big). \qquad (3.16)
\end{aligned}$$

Applying the inverse Laplace transform to Eq. (3.16), using the Appendix, we have

$$\begin{aligned}
u(y,t) &= [t^{-\alpha}E_{1,1-\alpha}(at)+Mexp(at)]*\left[\int_0^\infty erfc\Big(\frac{y}{2\sqrt{u}}\Big)exp(-Mu)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right] \\
&+\frac{Gr}{m_1-m_2}\Big[m_1t_{\alpha-1}E_{\alpha,\alpha+2}(m_1t_\alpha)-m_2t_{\alpha-1}E_{\alpha,\alpha+2}(m_2t_\alpha)\Big]*\int_0^\infty a(y,u)\frac{1}{t}\phi(0,-\alpha;-ut^\alpha)du \\
&-\frac{Gr}{m_1-m_2}\Big[(Prm_1+S)t_{\alpha-1}E_{\alpha,\alpha+2}(m_1t_\alpha)-(Prm_2+S)t^{\alpha-1}E_{\alpha,\alpha+2}(m_2t_\alpha)\Big]* \\
&\frac{1}{Pr}\int_0^\infty e^{-\frac{Su}{Pr}}erfc\Big(\frac{-y\sqrt{Pr}}{2\sqrt{u}}\Big)\frac{1}{t}\phi(0,-\alpha;-ut^{-\alpha})du \\
&+\frac{Gm}{n_1-n_2}\Big[n_1t_{\alpha-1}E_{\alpha,\alpha+2}(n_1t_\alpha)-n_2t_{\alpha-1}E_{\alpha,\alpha+2}(n_2t_\alpha)\Big]*\int_0^\infty a(y,u)\frac{1}{t}\phi(0,-\alpha;-ut^\alpha)du \\
&-\frac{Gm}{n_1-n_2}\Big[(Scm_1+\lambda)t_{\alpha-1}E_{\alpha,\alpha+2}(n_1t_\alpha)-(Scm_2+\lambda)t^{\alpha-1}E_{\alpha,\alpha+2}(n_2t_\alpha)\Big]* \\
&\frac{1}{Sc}\int_0^\infty e^{-\frac{\lambda u}{Sc}}erfc\Big(\frac{-y\sqrt{Sc}}{2\sqrt{u}}\Big)\frac{1}{t}\phi(0,-\alpha;-ut^{-\alpha})du, \qquad (3.17)
\end{aligned}$$

where $*$ denotes the convolution product, $\phi(y,t;c)$ is defined in Appendix and

$$(m_1,m_2)=\frac{-(\alpha_2S+Pr-1)\pm\sqrt{(\alpha_2S+Pr-1)-4\alpha_2Pr(S-M)}}{2\alpha_2Pr},$$

$$(n_1,n_2)=\frac{-(\alpha_2\lambda+Sc-1)\pm\sqrt{(\alpha_2\lambda+Sc-1)-4\alpha_2Sc(\lambda-M)}}{2\alpha_2Sc},$$

$$\begin{aligned}
a(y,t) &= \frac{1}{\alpha_2}e^{-\frac{t}{\alpha_2}}\int_0^t erfc\Big(\frac{y}{2\sqrt{Z}}\Big)e^{\frac{-Z}{\alpha_2}}I_0\Big(\frac{2}{\alpha_2}\sqrt{(1-\alpha_2M)Zt}\Big)dZ+\frac{M}{\alpha_2}\int_0^\infty\int_0^t erfc\Big(\frac{y}{2\sqrt{Z}}\Big)e^{\frac{-(Z+s)}{\alpha_2}} \\
&\times I_0\Big(\frac{2}{\alpha_2}\sqrt{(1-\alpha_2M)Zs}\Big)dZ.
\end{aligned}$$

3.5. **Velocity field for fractional viscous fluid when** $(\alpha_2 = 0)$**.**

$$
\begin{aligned}
u(y,t) &= [t^{-\alpha}E_{1,1-\alpha}(at) + Mexp(at)] \\
&*\left[\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)exp(-Mu)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right] \\
&\quad -\frac{GrPr}{Pr-1}\left[tE_{\alpha,2}(-m_1t) + Mt^{\alpha+1}E_{\alpha,\alpha+2}(-m_1t)\right] \\
&*\left[\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)exp(-Mu)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right] \\
&\quad -\frac{GmSc}{Sc-1}\left[tE_{\alpha,2}(-m_2t) + Mt^{\alpha+1}E_{\alpha,\alpha+2}(-m_2t)\right] \\
&*\left[\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)exp(-Mu)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right] \\
&\quad +\frac{GrPr}{Pr-1}\left[tE_{\alpha,2}(-m_1t) + \frac{S}{Pr}t^{\alpha+1}E_{\alpha,\alpha+2}(-m_1t)\right] \\
&*\left[\int_0^\infty erfc\left(\frac{y\sqrt{Pr}}{2\sqrt{u}}\right)exp(-\frac{S}{Pr}u)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right] \\
&\quad +\frac{GmSc}{Sc-1}\left[tE_{\alpha,2}(-m_2t) + \frac{\lambda}{Sc}t^{\alpha+1}E_{\alpha,\alpha+2}(-m_2t)\right] \\
&*\left[\int_0^\infty erfc\left(\frac{y\sqrt{Sc}}{2\sqrt{u}}\right)exp(-\frac{\lambda}{Sc}u)\frac{1}{t}\phi(0,-\alpha;ut^{-\alpha})du\right]
\end{aligned}
$$

where $*$ denotes the convolution product and $\mathrm{Pr} \neq 1, \mathrm{Sc} \neq 1$.

3.6. **Velocity field for ordinary viscous fluid when**$(\alpha \to 1, \alpha_2 = 0)$**.**

$$
\begin{aligned}
u(y,t) = &\int_0^t exp(a(t-\tau))\phi(y,\tau,a+M)d\tau - \\
&\frac{GrPr}{Pr-1}\int_0^t\left[(t-\tau)exp(m_1\tau)\phi(y,\tau,m_1+M)\right]d\tau \\
&-\frac{GmSc}{Sc-1}\int_0^t\left[(t-\tau)exp(m_2\tau)\phi(y,\tau,m_2+M)\right]d\tau \\
&+\frac{GrPr}{Pr-1}\int_0^t\left[(t-\tau)exp(m_1\tau)\phi(y\sqrt{Pr},\tau,m_1+\frac{S}{Pr})\right]d\tau \\
&+\frac{GmSc}{Sc-1}\int_0^t\left[(t-\tau)exp(m_2\tau)\phi(y\sqrt{Sc},\tau,m_2+\frac{\lambda}{Sc})\right]d\tau,
\end{aligned}
$$

$Pr \neq 1, Sc \neq 1$.

4. Numerical results and discussions

In this section, we discussed some physical aspects of the studied problem. Numerical computations were made for the unsteady magneto-hydrodynamic flow of an incompressible second grade fluid past an exponentially accelerated infinite vertical plate with variable temperature and variable mass diffusion in the presence of heat absorption by using Mathcad software. All the computations were made due to negative exponent of accelerated vertical plate. Here, we consider all parameters

and profile dimensionless and the flow pattern of fluid flow presented in Fig 1-5. The influence of fractional parameter , presented in Fig. 1 and 2, respectively, for small and large values of time. It is clear that for small values of time, the ordinary model fluid is slower than the fractional model. Due to the negative exponent the fluid velocity on the plate decreases while has maximum influence near the plate. For large values of time, the ordinary model fluid becomes faster than the fractional model. This is the advantage of non-integer time fractional derivative. By increasing the time with the variation of fractional parameter flow becomes faster. In many industry processes by this reason fractional model can be used.

The effect of magnetic field parameter is discussed in Fig. 3 and 4, for small and large value of time, for different values of fractional parameter . As usual, by increasing the amount of magnetic field, the fluid velocity decreases due to the fact that magnetic field produces the resistive forces. But we are interested to see the influence of fractional parameter, with different values of magnetic field parameter. It is observed that for small time the effect of magnetic field reduces the momentum boundary layer thickness by increasing the values of non-integer order fractional parameter. It shows the same phenomena for large values of time with different starting point.

Interesting factor of the present problem can be seen in the comparison between the models of fractional and ordinary fluids. In this comparison, we will see how the non-integer order time fractional parameter controls the fluid flow presented in Fig. 5. It is noted that near the plate viscous fluid (fractional and ordinary) models has maximum velocity than second grade (fractional and ordinary) models. But after critical points the vicinity changes for both cases. Further, it is important to make a remark that the viscous fluid is swiftest than second grade fluid for small time and for large values of time opposite behavior has been observed.

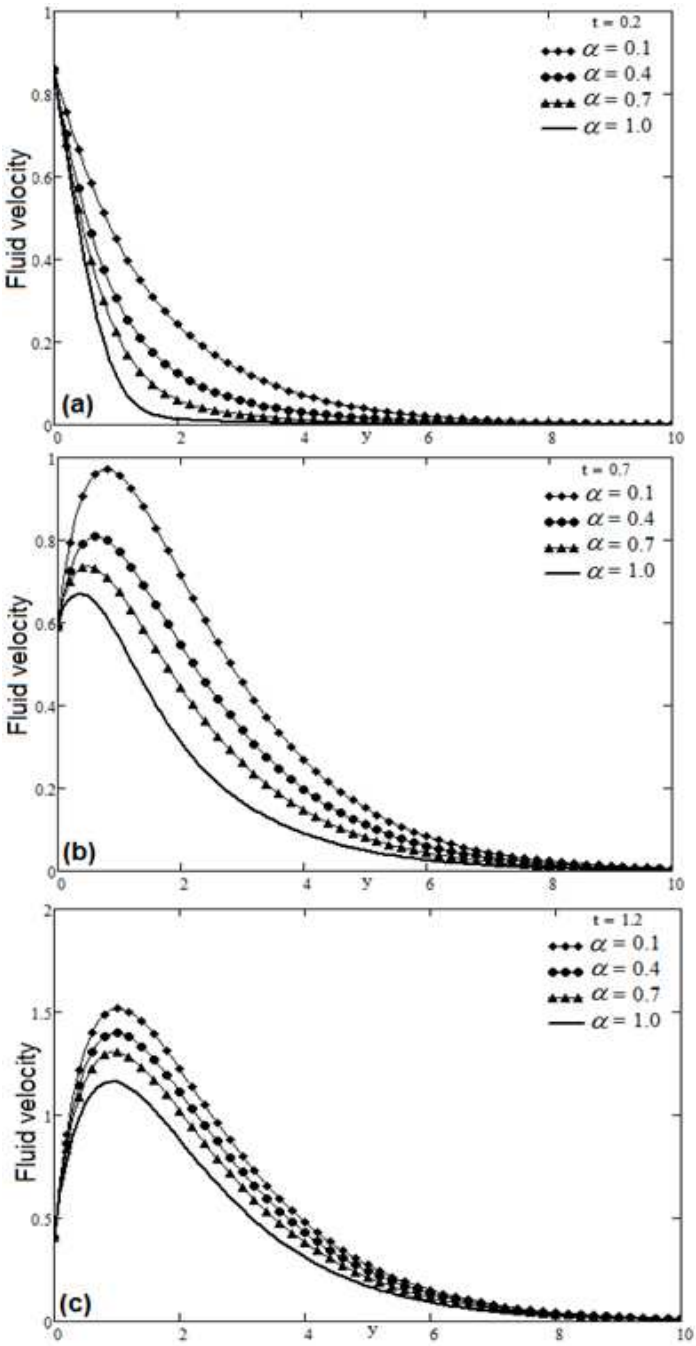

FIGURE 1. Profiles of fluid velocity for α variation versus y for $Pr = 5$. $Gr = 8$, $Gm = 10$, $Sc = 0.1$, $\alpha_2 = 2$, $S = 0.5$, $\lambda = 0.7$, $M = 0.4$, $a = -0.75$, for different small values of time

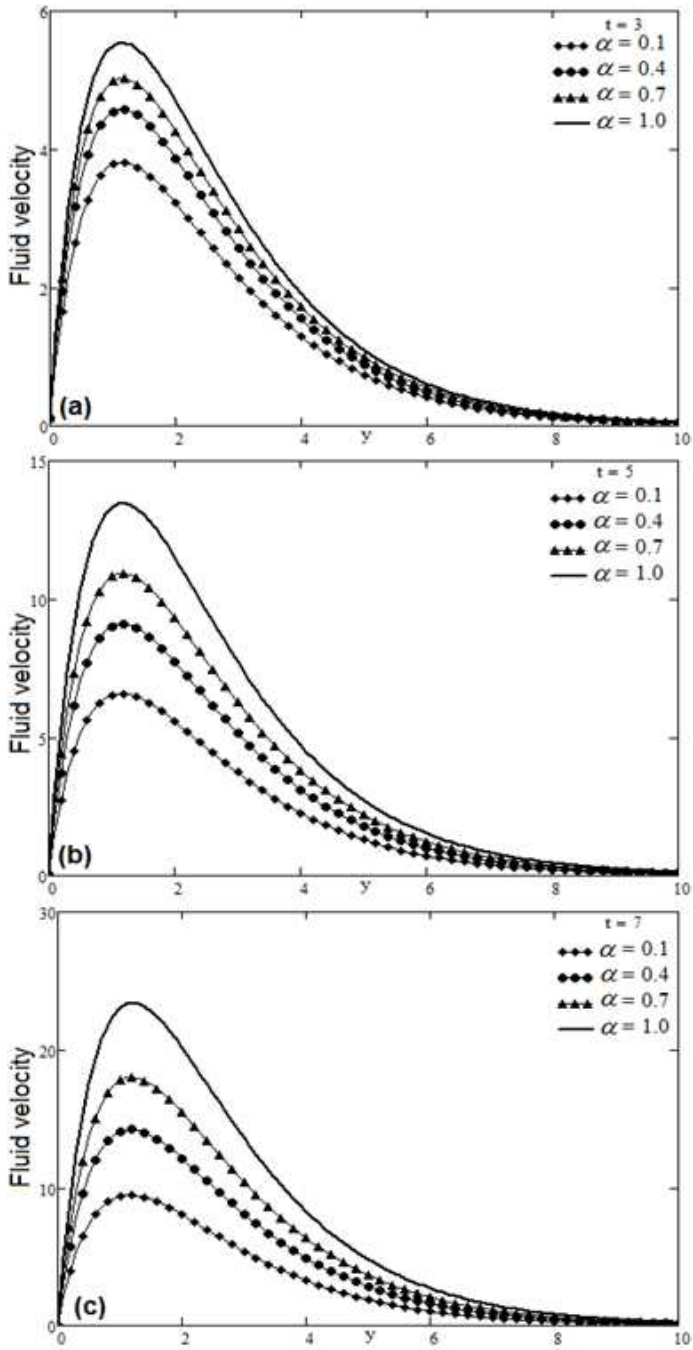

FIGURE 2. Profiles of fluid velocity for α variation versus y for $Pr = 5$. $Gr = 8$, $Gm = 10$, $Sc = 0.1$, $\alpha_2 = 2$, $S = 0.5$, $\lambda = 0.7$, $M = 0.4$, $a = -0.75$, for different large values of time

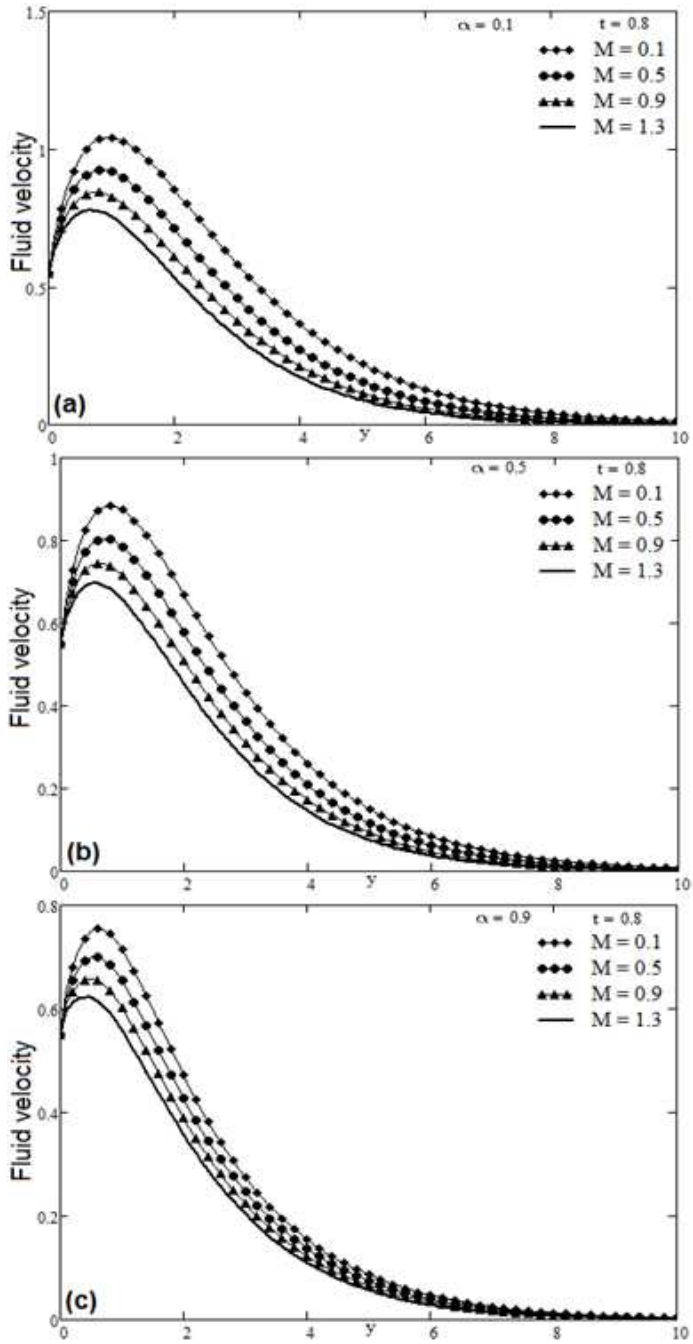

FIGURE 3. Profiles of fluid velocity for M variation versus y for $Pr = 5$. $Gr = 8$, $Gm = 10$, $Sc = 0.1$, $\alpha_2 = 2$, $S = 0.5$, $\lambda = 0.7$, $a = -0.75$, for different α and small values of time

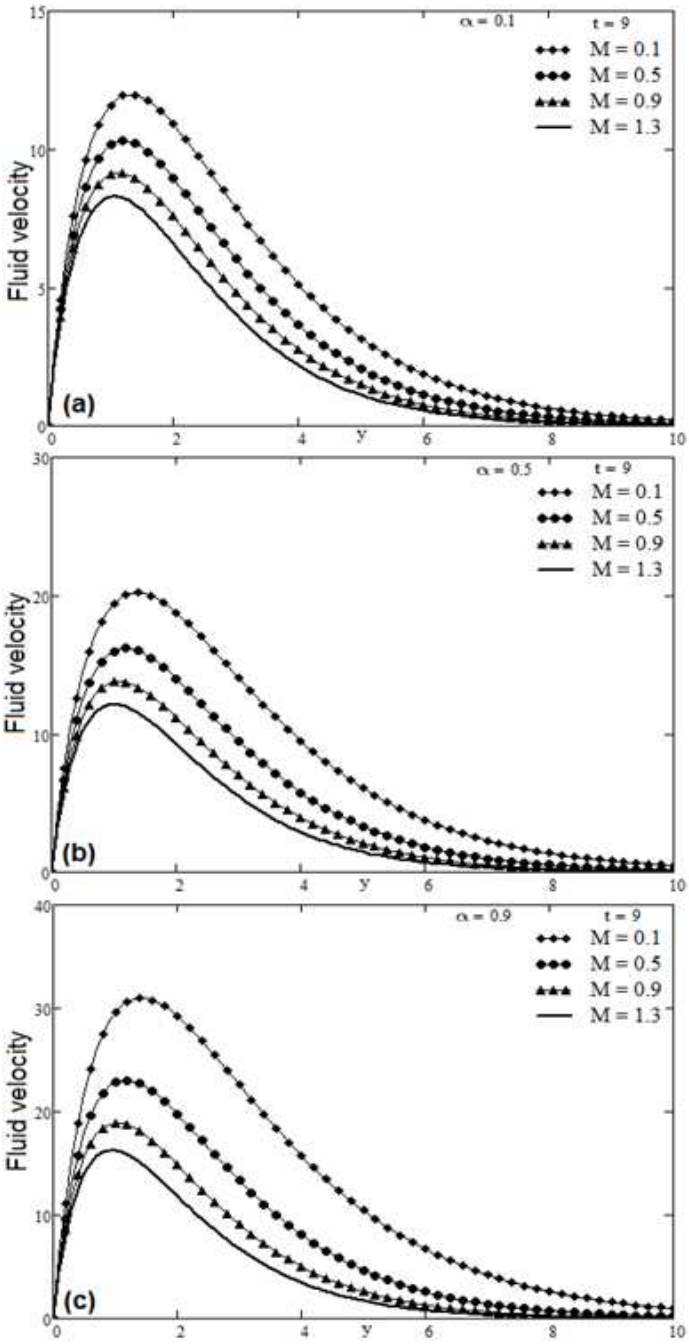

FIGURE 4. Profiles of fluid velocity for M variation versus y for $Pr = 5$. $Gr = 8$, $Gm = 10$, $Sc = 0.1$, $\alpha_2 = 2$, $S = 0.5$, $\lambda = 0.7$, $a = -0.75$, for different α and large values of time

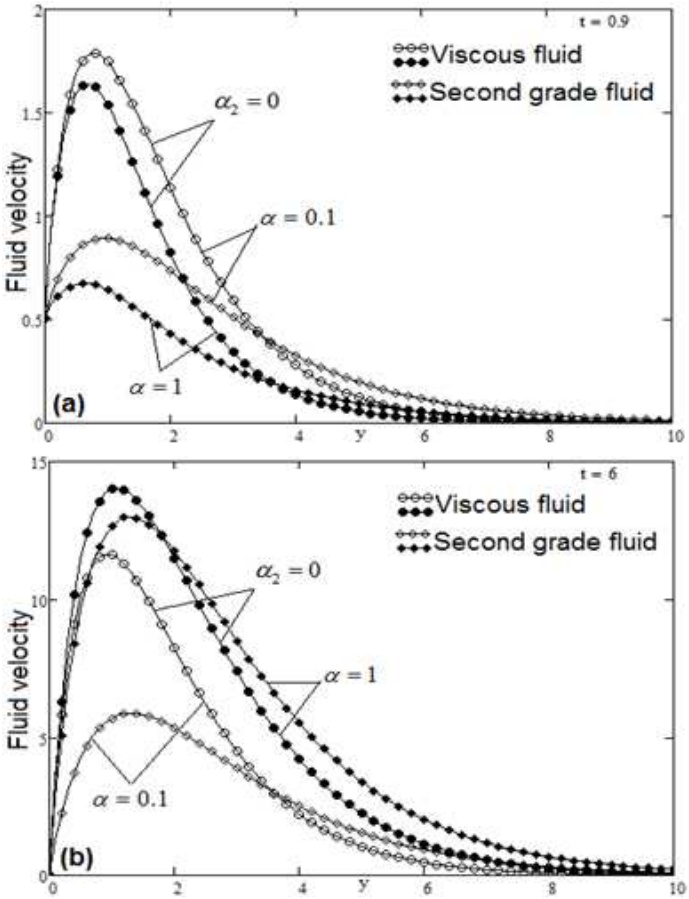

FIGURE 5. Comparison between fractional viscous and fractional second grade fluid $Pr = 5$. $Gr = 8$, $Gm = 10$, $Sc = 0.1$, $\alpha_2 = 2$, $S = 0.5$, $\lambda = 0.7$, $a = -0.75$, $M - 0.4$ for small and large times

5. CONCLUSION

Some concluding remarks of the present study
1: For small value of time and different values of non-integer fluid velocity increases by decreasing the value of fractional parameter but for large value of time opposite behavior can be observed.
2: For small and large time and different value of magnetic parameter it is fund that the fluid velocity is deceases function of magnetic parameter.
3: Viscous fluid (fractional and ordinary) is faster near the plate than second grade fluid (fractional and ordinary) but this vicinity changes away from the plate.

6. Appendix

$$\phi(\beta,-\sigma;z)=\sum_{n=0}^{\infty}\frac{z^n}{n\Gamma(\beta-\sigma n)} \tag{6.1}$$

$$L^{-1}\left\{\frac{1}{q^2}exp\left(-a\sqrt{q^\alpha+b}\right)\right\}=\int_0^\infty\int_0^t erf\left(\frac{a}{2\sqrt{x}}\right)exp(-bx)\frac{1}{\tau}\phi(0,\alpha,-x\tau^{-\alpha})\left[\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}+b(t-\tau)\right] \tag{6.2}$$

$$L^{-1}\left\{\frac{exp\left(-a\sqrt{q+b}\right)}{q^2}\right\}=$$
$$\frac{1}{2}\left[\left(t-\frac{a}{2\sqrt{b}}\right)exp(-a\sqrt{b})erfc\left(\frac{a}{2\sqrt{t}-\sqrt{bt}}\right)+\left(t+\frac{a}{2\sqrt{b}}\right)exp(a\sqrt{b})erfc\left(\frac{a}{2\sqrt{t}+\sqrt{bt}}\right)\right] \tag{6.3}$$

$$\Phi(y,t,a)=\frac{1}{2}\left[e^{y\sqrt{a}}erfc\left(\frac{y}{a\sqrt{t}}+\sqrt{at}\right)+e^{-y\sqrt{a}}erfc\left(\frac{y}{a\sqrt{t}}-\sqrt{at}\right)\right] \tag{6.4}$$

$$L^{-1}\left\{\frac{e^{-y\sqrt{q+a}}}{q-b}\right\}=e^{bt}\Phi(y,t,a+b) \tag{6.5}$$

$$L^{-1}\{F(q+a)\}=exp(-at)f(t)\text{where}L^{-1}\{F(q)\}=f(t) \tag{6.6}$$

$$L^{-1}\left\{\frac{q^{a-b}}{q^a+c}\right\}=t^{b-1}E_{a,b}(-ct^a);\quad \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)};\quad a>0,b>0 \tag{6.7}$$

$$L^{-1}\left\{\frac{e^{-a\sqrt{s^\alpha+b}}}{s^\alpha+b}\right\}=\int_0^\infty erfc\left(\frac{a}{2\sqrt{u}}\right)exp(-bu)\frac{1}{t}\phi(0,-\alpha,-ut^{-\alpha})du \tag{6.8}$$

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Mudassar Nazar
Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan
E-mail address: mudssarnazar666@yahoo.com

Mushtaq Ahmad
Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University Multan, Pakistan
E-mail address: math7690@yahoo.com

Muhammad Imran Asjad
Department of Mathematics, University of Management and Technology Lahore, Pakistan
E-mail address: imran.asjad@umt.edu.pk

N. A. Shah
Abdus Salam School of Mathematical Science, GC, University Lahore, Pakistan
E-mail address: nehadali199@yahoo.com