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ORDER-THEORETIC FIXED POINT RESULTS FOR $(\psi, \phi, \eta)_g$-GENERALIZED WEAK CONTRACTIVE MAPPINGS

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Abstract. In this paper, we prove common fixed point results for a pair of self-mappings satisfying a $(\psi, \phi, \eta)_g$-generalized contractive condition. Our results generalize several core results of the existing literature especially involving weak and rational contractive conditions.

1. Introduction

Fixed point theory is a very wide domain of mathematical research. It has extensive applications in various fields within and beyond mathematics which also include varied real word problems. Indeed, the fundamental result of metric fixed point theory is the classical Banach contraction principle which has been extended and generalized in many directions. In 1986, Turinici [1] initiated the theory of order-theoretic fixed point results. Later on, Ran and Reurings [2] formulated yet another relatively more natural formulation of Banach contraction principle and also used his result to discuss the existence and uniqueness of solution of a system of linear equations. This paper was well followed by Nieto and Rodriguez-Lopez [3,4]. Aydi et al. [5] and Imdad et al. [6] almost simultaneously gave some fixed point results involving Boyd-Wong-type contractions in partially ordered metric spaces. Nowadays, a vigorous research activity is in progress on this theme and for this kind of activity one can consult [7–14].

In 2009, Harjani and Sadarangani [15,16] proved an order-theoretic analogue of a well known result due to Rhoades [17] involving a weak contractive condition. Luong et al. [18] proved a fixed point theorem for a generalized rational-type contractions which extends main results proved in [15,16]. In [19], Arab extended the main results of Luong et al. [18] and proved common fixed point theorems for mappings satisfying a rational contractive condition. In [20], Chandok and Postolache employed a Chatterjea-type cyclic weak contraction and proved some existence fixed point results for such mappings. Simultaneously, Choudhury et al. [21] proved some coincidence point results for generalized weak contractions via discontinuous control functions. Very recently, Gubran and Imdad [22] generalized the main results

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contained in [15,16] and utilized the same to prove a result for mappings employing an integral type $(\psi, \varphi)_g$-generalized weak contractive condition.

2. Mathematical Preliminaries

In this section, we collect notions, definitions and auxiliary results which are needed in our subsequent discussions.

Definition 2.1. *[23] A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called an altering distance function if it is continuous, increasing and satisfies $\psi(t) = 0$ if and only if $t = 0$. We denote the set of all altering distance functions by Ψ.*

Let Φ denotes the set of all lower semi-continuous function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(t) = 0$ if and only if $t = 0$.

Definition 2.2. *[24] Let (X, d) be a metric space. A self-mapping f on X is said to be (ψ, ϕ)-weakly contractive (with $\psi, \phi \in \Psi$) if for all $x, y \in X$,*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)). \tag{2.1}$$

Definition 2.3. *[25] A triple $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set. Moreover, two elements $x, y \in X$ are said to be comparable if either $x \preceq y$ or $x \succeq y$. For brevity, we write it $x \prec\succ y$.*

Remark. *In the setting of ordered metric spaces, the inequality* (2.1) *is requited to hold merely for comparable elements rather as opposed to every pair of elements in X.*

Definition 2.4. *[26] Let (f, g) be a pair of self-mappings on an ordered set $(X, \preceq)$. Then the mapping:*

(1) f is said to be g-increasing if $gx \preceq gy \Rightarrow fx \preceq fy$, for all $x, y \in X$,
(2) f is said to be g-decreasing if $gx \preceq gy \Rightarrow fx \succeq fy$, for all $x, y \in X$,
(3) f is said to be g-monotone if f is either g-increasing or g-decreasing.

Definition 2.5. *[27] Let (f, g) be a pair of self-mappings on an ordered metric space $(X, d, \preceq)$ and $x \in X$. Then f is called (g,$\overline{\text{O}}$)- continuous (resp. (g,$\underline{\text{O}}$)-continuous, (g,O)-continuous) at $x \in X$ if $fx_n \xrightarrow{d} fx$, for every sequence $\{x_n\} \subset X$ with $gx_n \uparrow gx$ (resp. $gx_n \downarrow gx$, $gx_n \uparrow\downarrow gx$). Moreover, f is called (g,$\overline{\text{O}}$)-continuous (resp. (g,$\underline{\text{O}}$)-continuous, (g,O)-continuous) if it is so at every point of X.*

Remark. *In an ordered metric space, continuity $\Rightarrow$ (g,O)-continuity $\Rightarrow$ (g,$\overline{\text{O}}$)-continuity as well as (g,$\underline{\text{O}}$)-continuity.*

Definition 2.6. *[27] An ordered metric space $(X, d, \preceq)$ is called $\overline{\text{O}}$-complete (resp. $\underline{\text{O}}$-complete, O-complete) if every monotonically increasing (resp. monotonically deceasing, monotone) Cauchy sequence in X converges to a point of X.*

Remark. *In an ordered metric space, completeness $\Rightarrow$ O-completeness $\Rightarrow$ $\overline{\text{O}}$-completeness (as well as $\underline{\text{O}}$-completeness).*

Definition 2.7. *Let (f, g) be a pair of a self-mappings on an ordered metric space $(X, d, \preceq)$. Then the pair (f, g) is said to be*

(i) [28] compatible if $\lim\limits_{n\to\infty} d(gfx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} fx_n$,

(ii) *[27]* $\overline{\mathrm{O}}$*-compatible (resp.* $\underline{\mathrm{O}}$*-compatible,* O*-compatible) if*

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$$

whenever $\{x_n\}$ *is a sequence in* X *with* $\{gx_n\}$ *and* $\{fx_n\}$ *are increasing (resp. decreasing, monotone) sequence such that* $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$,

(iii) *[29] weak compatible if* $gfx = fgx$, *for every coincidence point (i.e.,* $fx = gx$*)* x *in* X.

Remark. *In an ordered metric space, compatibility* $\Rightarrow$ O*-compatibility* $\Rightarrow$ $\overline{\mathrm{O}}$*- compatibility (as well as* $\underline{\mathrm{O}}$*-compatibility)* $\Rightarrow$ *weak compatibility.*

Definition 2.8. *[27] Let* (f, g) *be a pair of a self-mappings on an ordered metric space* $(X, d, \preceq)$. *We say that*

(i) $(X, d, \preceq)$ *is said to have g-increasing-convergence-comparable (in short g-ICC) property if every g-increasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that g-image of every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$, *i.e.*

$$x_n \uparrow x \;\Rightarrow\; \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } gx_{n_k} \prec\succ gx \;\forall\, k \in \mathbb{N}_0,$$

(ii) $(X, d, \preceq)$ *is said to have g-decreasing-convergence-comparable (in short g-DCC) property if every g-decreasing convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that g-image of every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$, *i.e.*

$$x_n \downarrow x \;\Rightarrow\; \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } gx_{n_k} \prec\succ gx \;\forall\, k \in \mathbb{N}_0,$$

(iii) $(X, d, \preceq)$ *is said to have g-monotone-convergence-comparable (in short g-MCC) property if every g-monotone convergent sequence* $\{x_n\}$ *in* X *has a subsequence* $\{x_{n_k}\}$ *such that g-image of every term of* $\{x_{n_k}\}$ *is comparable with the limit of* $\{x_n\}$, *i.e.*

$$x_n \uparrow\downarrow x \;\Rightarrow\; \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } gx_{n_k} \prec\succ gx \;\forall\, k \in \mathbb{N}_0.$$

Remark. *On setting* $g = I_X$, *the g-ICC property in Definition 2.8 reduces to ICC property. Observe that, Definition 2.8 is relatively weaker than the notion utilized in condition* (ii) *of [19, Theorem 2.1].*

Definition 2.9. *[30] Let* Y *be a subset of an ordered set* $(X, \preceq)$ *and* g *a self-mapping on* X. *We say that* Y *is g-directed if for every pair of elements* $x, y \in Y$, *there exists* $z \in X$ *such that* $x \prec\succ gz$ *and* $y \prec\succ gz$.

Lemma 2.10. *[31] Let* (f, g) *be a pair of weak compatible self-mappings defined on a non-empty set* X. *Then every point of coincidence of the pair* (f, g) *also remains a coincidence point.*

Lemma 2.11. *[16] Let* $(X, d, \preceq)$ *be an ordered metric space and* $\{x_n\}$ *a sequence in* X *such that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If* $\{x_n\}$ *is not a Cauchy sequence, then there exist* $\varepsilon > 0$ *and two subsequences* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$, *such that* $n_k > m_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$, $d(x_{m_k}, x_{n_k-1}) < \varepsilon$ *and the sequences* $d(x_{m_k}, x_{n_k})$, $d(x_{m_k+1}, x_{n_k})$, $d(x_{m_k}, x_{n_k+1})$, $d(x_{m_k+1}, x_{n_k+1})$ *tend to* ε *when* $k \to \infty$.

The aim of this article is to prove common fixed point results for a pair of self-mappings satisfying a generalized weak contractive condition. Our results generalize some core results of the existing literature (e.g., [18, 19, 32]). Particularly, we improve Theorem 2.1 of Arab [19] in the following four-respects:

(a) the involved contractive condition is replaced by a relatively weaker one,
(b) relatively weaker notions of completeness and continuity are utilized,
(c) the completeness is merely required on any subspace of X containing $f(X)$,
(d) the property utilized in condition (ii) is replaced by a relatively weaker one.

3. Main results

The following definition is utilized in our results:

Definition 3.1. *Let (f,g) be a pair of self-mappings on an ordered metric space $(X,d,\preceq)$. Then f is said to be a $(\psi,\phi,\eta)_g$ -generalized weak contractive mapping if for all $x,y\in X$ such that $gx\preceq gy$, we have*

$$\psi(d(fx,fy))\le\psi(M(x,y))-\phi(M(x,y)) \tag{3.1}$$

where,

$$M(x,y)=\max\Big\{d(gx,gy),\eta\big(d(gx,fx),d(gx,gy),d(gy,fy)\big),\frac{d(gx,fy)+d(gy,fx)}{2}\Big\},$$

$\psi\in\Psi$, $\phi\in\Phi$ and $\eta:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function such that $\eta(a,b,b)=a$ and $\eta(0,b,c)=0$ for all $a,b,c\in\mathbb{R}_+$.

Our main result runs as follows:

Theorem 3.2. *Let $(X,d,\preceq)$ be an ordered metric space, Y an $\overline{\mathrm{O}}$-complete subspace of X and $f,g:X\to X$ such that f is g-increasing. Moreover, suppose that the following conditions hold:*

(i) there exists an $x_0\in X$ such that $gx_0\preceq fx_0$,
(ii) f is a $(\psi,\phi,\eta)_g$-generalized weak contractive mapping,
(iii) $f(X)\subseteq Y\subseteq g(X)$,
(iv) either
   (a) f is $(g,\overline{\mathrm{O}})$-continuous or
   (b) $(Y,d,\preceq)$ enjoys the g-ICC-property.

Then the pair (f,g) has a coincidence point.

Proof. Take $x_0\in X$ such that $gx_0\preceq fx_0$. As the mapping f is g-increasing and $f(X)\subseteq g(X)$, we can define increasing sequences $\{gx_n\}$ and $\{fx_n\}$ in Y such that for all $n\in\mathbb{N}_0$

$$gx_{n+1}=fx_n. \tag{3.2}$$

Observe that, if $d(gx_n,gx_{n+1})=0$ for some $n\in\mathbb{N}_0$, then x_n is a coincidence point and we are done. Henceforth, we assume that $d(gx_n,gx_{n+1})>0$ for all $n\in\mathbb{N}_0$. We assert that $\lim\limits_{n\to\infty}d(gx_{n-1},gx_n)=0$. On setting $x=x_n$ and $y=x_{n-1}$ in (3.1), we get

$$\begin{aligned}\psi(d(gx_{n+1},gx_n)) &= \psi(d(fx_n,fx_{n-1}))\\ &\le \psi(M(x_n,x_{n-1}))-\phi(M(x_n,x_{n-1})),\end{aligned} \tag{3.3}$$

for all $n\in\mathbb{N}_0$, where

$$M(x_n,x_{n-1}) = \max\Big\{d(gx_n,gx_{n-1}),d(gx_n,gx_{n+1}),\frac{d(gx_{n-1},gx_{n+1})}{2}\Big\}.$$

Obviously, $\max\{d(gx_n, gx_{n-1}), d(gx_n, gx_{n+1})\} \geq \frac{1}{2}d(gx_{n-1}, gx_{n+1})$. If possible, assume $d(gx_n, gx_{n-1}) \leq d(gx_n, gx_{n+1})$ so that (3.3) reduces to

$$\begin{aligned}\psi(d(gx_{n+1}, gx_n)) &\leq \psi(d(gx_{n+1}, gx_n)) - \phi(d(gx_{n+1}, gx_n))\\ &< \psi(d(gx_{n+1}, gx_n)),\end{aligned}$$

a contradiction. Thus, $d(gx_n, gx_{n+1}) < d(gx_n, gx_{n-1})$ and (3.3) deduces that $\psi(d(gx_n, gx_{n+1})) < \psi(d(gx_{n-1}, gx_n))$ which (due to the fact that ψ is an increasing function) implies that $\{d(gx_{n-1}, gx_n)\}$ is a decreasing sequence of non-negative real numbers so that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = \alpha \geq 0.$$

By taking limit superior as $n \to \infty$ in inequality (3.3), we get $\psi(\alpha) \leq \psi(\alpha) - \phi(\alpha)$, a contraction unless $\alpha = 0$.

Now, we assert that $\{gx_n\}$ is a Cauchy sequence. For if it is not so, owing to Lemma 2.11, there exist $\epsilon > 0$ and two subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $n_k > m_k \geq k$, $d(gx_{m_k}, gx_{n_k}) \geq \epsilon$, $d(gx_{m_k}, gx_{n_k-1}) < \epsilon$ and each of the sequences $d(gx_{m_k}, gx_{n_k})$, $d(gx_{m_k+1}, gx_{n_k})$, $d(gx_{m_k}, gx_{n_k+1})$ and $d(gx_{m_k+1}, gx_{n_k+1})$ tends to ϵ as k tends to ∞. Since $n_k > m_k$, on putting $x = x_{n_k}$ and $y = y_{m_k}$ in (3.1), we have (for all $k \in \mathbb{N}$)

$$\begin{aligned}\psi(d(gx_{n_k+1}, gx_{m_k+1})) &= \psi(d(fx_{n_k}, fx_{m_k}))\\ &\leq \psi(M(x_{n_k}, x_{m_k})) - \phi(M(x_{n_k}, x_{m_k})), \qquad (3.4)\end{aligned}$$

where

$$\begin{aligned}M(x_{n_k}, x_{m_k}) = \max\Big\{&d(gx_{n_k}, gx_{m_k}), \eta\big(d(gx_{n_k}, gx_{n_k+1}), d(gx_{n_k}, gx_{m_k}),\\ &d(gx_{m_k}, gx_{m_k+1})\big), \frac{d(gx_{n_k}, gx_{m_k+1}) + d(gx_{m_k}, gx_{n_k+1})}{2}\Big\}.\end{aligned}$$

Taking limit superior as $k \to \infty$ in (3.4), one gets that

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

a contradiction as $\phi(\epsilon) > 0$. Thus, $\{gx_n\}$ is a Cauchy sequence in Y. Therefore, there exists $x \in Y$ such that $gx_n \uparrow x$. Owing to condition (iii), there exists $z \in X$ such that $x = gz$ so that

$$gx_n \uparrow gz. \qquad (3.5)$$

Now, using the condition (iv), we show that z is a coincidence point of the pair (f, g). Assume that f is $(g, \overline{\text{O}})$-continuous. In view of (3.5), we have $fx_n \to fz$ which (in view of (3.2)) gives rise (due to the uniqueness of the limit) $gz = fz$.
Alternately, assume that $(Y, d, \preceq)$ enjoys g-ICC-property. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $gx_{n_k} \preceq gz$, $\forall\ k \in \mathbb{N}$. On setting $x = x_{n_k}$, $y = z$ in (3.1), we have (for all $k \in \mathbb{N}_0$)

$$\begin{aligned}\psi(d(gx_{n_k+1}, fz)) &= \psi(d(fx_{n_k}, fz))\\ &= \psi(M(x_{n_k}, z)) - \phi(M(x_{n_k}, z)), \qquad (3.6)\end{aligned}$$

where

$$\begin{aligned}M(x_{n_k}, z) = \max\Big\{&d(gx_{n_k}, gz), \eta\Big(d(gx_{n_k}, gx_{n_k+1}), d(gx_{n_k}, gz), d(gz, fz)\Big),\\ &\frac{d(gx_{n_k}, fz) + d(gz, gx_{n_k+1})}{2}\Big\}.\end{aligned}$$

In view of (3.2) an (3.5), taking limit superior of 3.6 as $k \to \infty$, we have

$$\psi(d(gz, fz)) \leq \psi(d(gz, fz)) - \phi(d(gz, fz)),$$

which is a contradiction unless $gz = fz$. This conclude the proof. □

On setting $Y = g(X)$ and $\eta(a, b, c) = a\varphi(b, c)$ in Theorem 3.2, we get:

Corollary 3.3. *Let $(X, d, \preceq)$ be an ordered metric space and $f, g : X \to X$ such that f is a g-increasing. Moreover, suppose that the following conditions hold:*

(i) there exists an $x_0 \in X$ such that $gx_0 \preceq fx_0$,

(ii) for all $x, y \in X$ such that $gx \preceq gy$, we have

$$\psi(d(fx, fy)) \leq \psi(n(x, y)) - \phi(n(x, y)),$$

where, $n(x, y) = \max\{d(gx, gy), d(gx, fx)\varphi(d(gx, gy), d(gy, fy)), \frac{1}{2}[d(gx, fy) + d(gy, fx)]\}$, $\psi \in \Psi$, $\phi \in \Phi$ and $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in \mathbb{R}_+$,

(iii) $f(X) \subseteq g(X)$,

(iv) either

(a) f is $(g, \overline{O})$-continuous or

(b) $g(X)$ is complete and enjoys the ICC-property.

Then the pair (f, g) has a coincidence point.

Remark. *Corollary 3.3 is a sharpened version of Theorem 2.1 of [19] and hence that of Theorem 2.2 in [32]. In fact, the continuity of g and the compatibility of the pair (f, g) are not necessary there. Notice that, the completeness is required only on $g(X)$.*

Choosing η suitably in Theorem 3.2, we can deduce several results under rational contraction:

Corollary 3.4. *Let $(X, d, \preceq)$ be an ordered metric space and $f, g : X \to X$ such that f is a g-increasing and $g(X)$ is complete subspace of X. Moreover, suppose that the following conditions hold:*

(i) there exists an $x_0 \in X$ such that $gx_0 \preceq fx_0$,

(ii) for every two comparable gx and gy, we have

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where $\psi \in \Psi$, $\phi \in \Phi$ and

$$M(x, y) = \max\left\{d(gx, gy), \frac{d(gx, fx)d(gy, fy)}{d(gx, gy)}, \frac{d(gx, fy) + d(gy, fx)}{2}\right\}.$$

(iii) $f(X) \subseteq g(X)$,

(iv) either

(a) f is $(g, \overline{O})$-continuous or

(b) $(g(X), d, \preceq)$ enjoys the ICC-property.

Then the pair (f, g) has a coincidence point.

Proof. This result follows from Theorem 3.2 by taking $Y = g(X)$ and $\eta(a, b, c) = \frac{ab}{c}$, (for all $a, b \in \mathbb{R}_+$ and $c \neq 0$). □

Remark. *On setting $g = I_X$ in Corollary 3.4, we obtain a generalized version of Theorem 2.1 contained in [18].*

Remark. *On setting* $\psi = I_{\mathbb{R}_+}$ *and* $\phi(t) = (1-\lambda)t$, *(for all* $t \geq 0$,*) in Corollary 3.4, one can deduce a sharpened version of Corollary 2.2 of Thuan et al. [18].*

Corollary 3.5. *Corollary 3.4 remains true if condition (ii) is replaced by the following: for every two comparable* gx *and* gy*, we have*

$$\psi(d(fx, fy)) \leq \psi(M(x,y)) - \phi(M(x,y)),$$

where $\psi \in \Psi$, $\phi \in \Phi$ *with*

$$M(x,y) = \max\left\{d(gx,gy), \frac{d(gx,fx)(1+d(gy,fy))}{1+d(gx,gy)}, \frac{d(gx,fy)+d(gy,fx)}{2}\right\}.$$

Proof. Setting $\eta(a,b,c) = \frac{a(1+b)}{1+c}$, (for all $a, b \in \mathbb{R}_+$) in Corollary 3.4, one gets the result. □

Example 3.6. *Consider* $X = (-1/2, 0]$ *with usual metric. Obviously* (X,d) *is an* $\overline{\text{O}}$*-complete metric space. Define an order relation on* X *as under:*

$$x \preceq y \Leftrightarrow \text{ either } x = y \text{ or } x < y \text{ wherein } x, y \in \{0\} \cup \big\{-1/n : n = 3, 4, ...\big\}.$$

Here, $\leq$ *is the usual order on* $\mathbb{R}$*. Consider* $g = I_X$ *and define* $f : X \to X$ *as follows:*

$$fx = \begin{cases} 0, & \text{if } x = 0 \\ -1/(n+1), & \text{if } x = -1/n, n = 3, 4, ... \\ -0.2, & \text{otherwise.} \end{cases}$$

Observe that f *is increasing and* X *has the ICC-property. Let* $\psi = I_{\mathbb{R}_+}$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $\phi(t) = t^3$ *and* $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is given by* $\eta(a,b,c) = ac/b$ *for all* $a, b, c \in \mathbb{R}_+$*. If* $x = y$*, then condition (3.1) holds trivially. Otherwise, we distinguish the following two cases:*

Case 1. *Taking* $x = -1/n$, *(wherein* $n = 3, 4, ..$*) and* $y = 0$*. Then, from (3.1) we have*

$$\psi(d(fx,fy)) = \frac{1}{(n+1)} = \frac{1}{n} - \frac{1}{n(n+1)} \leq \frac{1}{n} - \frac{1}{n^3} = \psi(M(x,y)) - \phi(M(x,y)),$$

wherein

$$M(x,y) = \max\left\{\frac{1}{n}, \eta\Big(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n}, 0\Big), \frac{1}{2}\Big[\frac{1}{n} + \frac{1}{n+1}\Big]\right\} = \frac{1}{n}.$$

Case 2. *Taking* $x = -1/n, y = -1/m$ *for* $m > n \geq 3$*. Then, we have*

$$\begin{aligned} \psi(d(fx,fy)) = d(fx,fy) &= \frac{1}{n+1} - \frac{1}{m+1} \\ &\leq \Big(\frac{1}{n} - \frac{1}{n^3}\Big) - \Big(\frac{1}{m} - \frac{1}{m^3}\Big) \\ &\leq \frac{1}{n} - \frac{1}{m} - \Big(\frac{1}{n} - \frac{1}{m}\Big)^3 \\ &= \psi(M(x,y)) - \phi(M(x,y)), \end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\Big\{\frac{1}{n}-\frac{1}{m}, \frac{\big(\frac{1}{n}-\frac{1}{n+1}\big)\big(\frac{1}{m}-\frac{1}{m+1}\big)}{\frac{1}{n}-\frac{1}{m}}, \\ &\quad \frac{1}{2}\Big[\Big(\frac{1}{n}-\frac{1}{m+1}\Big)+\Big(\frac{1}{n+1}-\frac{1}{m}\Big)\Big]\Big\} \\ &= \frac{1}{n}-\frac{1}{m}. \end{aligned}$$

Thus, all the conditions of Theorems 3.2 are satisfied and hence the pair (f,g) have a coincidence point (namely $x=0$). Observe that, Theorem 3.2 is genuinely different from [19, Theorem 2.1] because the present example can not be covered by it due to the absence of completeness of X.

4. Uniqueness results

Theorem 4.1. *In addition to the hypothesis of Theorem 3.2, if $f(X)$ is g-directed, then the pair (f,g) has a unique point of coincidence.*

Proof. Let $x,y\in X$ be such that $fx=gx$ and $fy=gy$. We show that $gx=gy$. Owing to the g-directedness of $f(X)$, there exists $z\in X$ such that gz is comparable to both fx and fy. Without loss of generality, we may assume that $fx\preceq gz$ and $fy\preceq gz$. Put $z=z_0$. Since $f(X)\subseteq g(X)$, one can define a sequence $z_n\subset X$ such that

$$gz_{n+1}=fz_n \text{ and } gx\preceq gz_n \text{ for all } n\in\mathbb{N}.$$

Using (3.1), we have (for all $n\in\mathbb{N}$)

$$\begin{aligned} \psi(d(gx,gz_{n+1})) &= \psi(d(fx,fz_n)) \\ &\le \psi(M(x,z_n))-\phi(M(x,z_n)), \end{aligned} \tag{4.1}$$

where

$$M(x,z_n) = \max\Big\{d(gx,gz_n), \frac{d(gx,gz_{n+1})+d(gz_n,gx)}{2}\Big\}.$$

Assume that $d(gx,gz_{n+1})>d(gx,gz_n)$. Then (4.1) implies that

$$\psi\big(d(gx,gz_{n+1})\big)\le\psi\Big(\frac{d(gx,gz_{n+1})+d(gz_n,gx)}{2}\Big).$$

Since ψ is increasing, $d(gx,gz_{n+1})\le d(gx,gz_n)$ which is a contradiction. Hence, (4.1) can be written as

$$\psi(d(gx,gz_{n+1}))\le\psi(d(gx,gz_n)) \text{ for all } n\in\mathbb{N}.$$

Now, $\{d(gx,gz_n)\}$ is a decreasing sequence of non-negative real numbers possessing a limit $r\ge 0$. If $r>0$, then, from (4.1), as earlier, we deduce that $\psi(r)\le\psi(r)-\phi(r)$ which is a contradiction. Therefore,

$$\lim_{n\to\infty} d(gx,gz_n)=0. \tag{4.2}$$

Similarly, one can also show that

$$\lim_{n\to\infty} d(gy,gz_n)=0. \tag{4.3}$$

On using (4.2) and (4.3), we have

$$d(gx,gy)\le d(gx,gz_n)+d(gz_n,gy)\ \to 0 \text{ as } n\to\infty,$$

which shows that the pair (f, g) has a unique point of coincidence. □

Theorem 4.2. *In addition to the hypothesis of Theorem 4.1, if the pair (f, g) is weak compatible, then the pair has a unique common fixed point.*

Proof. Let $x \in X$ be an arbitrary coincidence point of the pair (f, g). Appealing to Theorem 4.1, there exists a unique point of coincidence $x^* \in X$ (say) such that $fx = gx = x^*$. In view of Lemma 2.10, x^* is a coincidence point, i.e., $fx^* = gx^*$. Again, Theorem 4.1 ensures that $fx^* = gx^* = x^*$, i.e., x^* is a unique common fixed point of the pair (f, g). □

Theorem 4.3. *Theorems 3.2, 4.1 and 4.2 remain true if certain involved terms namely: $\overline{\mathrm{O}}$-complete, $(g, \overline{\mathrm{O}})$-continuous and g-ICC property are respectively replaced by $\underline{\mathrm{O}}$-complete (resp. O-complete), $(g, \underline{\mathrm{O}})$-continuous (resp. (g, O)-continuous) and g-DCC (resp. g-MCC) property and assumption $gx_0 \preceq fx_0$ by $gx_0 \succeq fx_0$ (or $gx_0 \prec\succ fx_0$).*

Conclusions. In the framework of ordered metric spaces, we prove common fixed point results for a pair of self-mappings (f, g) satisfying a generalized contractive condition. The advantage of our utilized contraction condition heavily banks on the continuous function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying certain conditions which can deduce weak as well as rational contractive conditions under suitable chooses. Consequently, the main results of Luong and Thuan [18], Harjani and Sadarangani [32] and Arab [19] are generalized and improved. In doing so, we utilize the usual technique of constructing a nondecreasing Cauchy sequence whose limit turns out to be the point of coincidence of the pair. Finally, we furnish an examples to support the usability of our results.

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