Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 5 (2017), Pages 145-160.

EMBEDDINGS OF NEAR VECTOR SPACES AND APPLICATIONS IN PRE-HILBERT SPACES, FILTRATIONS, MARTINGALES, AND METRIC SPACES

A. ABKAR, M. RASTGOO, M.B. KAZEMI, A. AZIZI

Abstract. In this paper we introduce a family of embeddings j_n, $n \geq 2$, of a near vector space into a vector space. We give some examples for such embeddings and show that j_n's invariant metric on a near vector space S defines an isometry on the vector space $R_n(S)$, and if S is a near vector lattice then j_n's are join preserving on the vector lattices $R_n(S)$. Finally, we will find applications of this embeddings in Hilbert spaces, filtrations, martingales, and metric spaces.

1. Introduction

Initially Rådström in [6] proved that any near vector space S can be embedded into a vector space $R(S)$ via an embedding j, moreover, if there exists an invariant metric on S, then $R(S)$ admits a norm such that j is distance preserving. Later on, S. Bochner in [1] proved that any martingale $(f_i, \mathcal{E}_i)_{i\in\mathbb{N}}$ on a near vector space S induces a martingale $\big(j(f_i), \mathcal{E}_i\big)_{i\in\mathbb{N}}$ on the vector space $R(S)$.

Recently, C.C.A. Labuschagne, A.L. Pinchuck and C.J. van Alten in [5] entered topics related to lattices in this discussion and showed that this embedding is join preserving and embeds a near vector lattice into a vector lattice. They also proved that if we have a Riesz metric on S, we can define a Riesz norm on $R(S)$.

In this paper, we show that a near vector space S can be embedded into an innumerable vector spaces $R_n(S)$, $(n = 2, 3, 4, ...)$ via j_n's (see Definition 2.5). By investigating properties of $R_n(S)$ and j_n, we prove that these embeddings preserve the inner product and the basis. Finally, we shall provide some applications in the theory of filtrations, and martingales.

This paper is organized as follows: In Section 2, after introducing the embedding j_n and vector space $R_n(S)$, we study the impact of j_n on metric spaces, Banach spaces, Hilbert spaces and on invariant metrics. In Section 3, we study j_n on a

2000 *Mathematics Subject Classification.* 46B85; 46B99.
Key words and phrases. Near vector space, near vector lattice, ordered near vector space, invariant metric, inner product space, filtration, martingale.

Submitted April 23, 2017. Published September 3, 2017.
Communicated by Romi Shamoyan.

partially ordered near vector space, and finally, in Section 4, we study MS-filtration and martingales.

2. The embeddings

In this section, for a given near vector space S, we shall construct countably many embeddings $j_n,\ n = 2, 3, \cdots$, each of which will embed S into a vector space $R_n(S)$. We begin this section by recalling the definition of a near vector space.

Definition 2.1. *A nonempty set S is said to be a near vector space provided that addition and scalar multiplication by positive numbers satisfy the following conditions; more precisely, addition $+ : S \times S \longrightarrow S$ is defined in such a way that $(S, +)$ is a cancellative commutative semigroup; i.e., for all $x, y, z \in S$:*

$$x + z = y + z \Longrightarrow x = y,$$
$$x + y = y + x,$$
$$(x + y) + z = x + (y + z);$$

moreover, multiplication $. : \mathbb{R}_+ \times S \longrightarrow S$ by positive scalars is defined in such a way that for all $x, y \in S$ and $\lambda, \delta \in \mathbb{R}_+$:

$$\lambda x + \lambda y = \lambda(x + y),$$
$$(\lambda + \delta)x = \lambda x + \delta x,$$
$$(\lambda\delta)x = \lambda(\delta x),$$
$$1x = x.$$

Definition 2.2. *Let S be a near vector space, for $n = 2, 3, 4, ...$ we define $\sim_n$ on $\underbrace{S \times S \times \cdots \times S}_{n-times}$ by*

$$(x_1, x_2, ..., x_n) \sim_n (y_1, y_2, ..., y_n) \iff x_1+y_2+y_3+\cdots+y_n = y_1+x_2+x_3+\cdots+x_n.$$

Clearly $\sim_n$ is an equivalence relation on $\underbrace{S \times S \times \cdots \times S}_{n-times}$. Let

$$[x_1, x_2, ..., x_n] := \{(y_1, y_2, ..., y_n) \in S\times S\times\cdots\times S : (x_1, x_2, ..., x_n) \sim_n (y_1, y_2, ..., y_n)\},$$

Now define the quotient

$$R_n(S) := (S \times S \times \cdots \times S)/\sim_n = \{[x_1, x_2, ..., x_n] : (x_1, x_2, ..., x_n) \in S \times S \times \cdots \times S\}.$$

Also, on the quotient $R_n(S)$, define addition by
$[x_1, x_2, ..., x_n]+[y_1, y_2, ..., y_n] = [x_1+y_1, x_2+y_2, ..., x_n+y_n]$ *and, scalar multiplication* $\cdot : \mathbb{R} \times R_n(S) \longrightarrow R_n(S)$ *by*

$$\lambda\cdot[x_1, x_2, ..., x_n] := \begin{cases} [\lambda x_1, \lambda x_2, ..., \lambda x_n], & \lambda \in \mathbb{R}_+, \\ [(n-1)x_1, x_1, x_1, ..., x_1], & \lambda = 0, \\ -\lambda[(n-2)x_1 + x_2 + x_3 + ... + x_n, x_1, x_1, ..., x_1], & -\lambda \in \mathbb{R}_+. \end{cases}$$

Lemma 2.3. *Let S be a near vector space. Then for any $x_1 \in S$, $[(n-1)x_1, x_1, ..., x_1]$ is the additive identity in $R_n(S)$.*

Proof. For any $(x_1, x_2, ..., x_n) \in S \times S \times \cdots \times S$ we have

$$[x_1, x_2, ..., x_n] + [(n-1)x_1, x_1, x_1, ..., x_1] \\ = [nx_1, x_1 + x_2, x_1 + x_3, ..., x_1 + x_n] = [x_1, x_2, ..., x_n],$$

thus, we have a right neutral member, and by using commutative property, in the same way, we can verify that $[(n-1)x_1, x_1, x_1, ..., x_1]$ is a left neutral member. It can be easily proved that the neutral element is unique. Therefore $R_n(S)$ has a unique neutral member. □

Lemma 2.4. *Let S be a near vector space, then*

$$-[x_1, x_2, ..., x_n] := [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]$$

is the additive inverse of $[x_1, x_2, ..., x_n]$ in $R_n(S)$.

Proof. Let $[x_1, x_2, ..., x_n] \in R_n(S)$, then

$$[x_1, x_2, ..., x_n] + [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1] \\ = [(n-1)x_1 + x_2 + x_3 + \cdots + x_n \quad , x_1 + x_2, x_1 + x_3, ..., x_1 + x_n] \\ = [(n-1)x_1, x_1, x_1, ..., x_1],$$

thus, it is a right inverse for $[x_1, x_2, ..., x_n]$, and by using the commutative property, we can verify that $[(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]$ is a left inverse as well. Now suppose $[w_1, w_2, ..., w_n] \in R_n(S)$ is another inverse for $[x_1, x_2, ..., x_n]$, so we have

$$[w_1, w_2, ..., w_n] + [x_1, x_2, ..., x_n] = [(n-1)x_1, x_1, x_1, ..., x_1],$$

or

$$[w_1 + x_1, w_2 + x_2, ..., w_n + x_n] = [(n-1)x_1, x_1, x_1, ..., x_1].$$

Thus

$$w_1 + x_1 + (n-1)x_1 = (n-1)x_1 + w_2 + x_2 + w_3 + x_3 + \cdots + w_n + x_n,$$

hence

$$w_1 + (n-1)x_1 = (n-2)x_1 + x_2 + x_3 + \cdots + x_n + w_2 + w_3 + \cdots + w_n,$$

and finally

$$[w_1, w_2, ..., w_n] = [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1].$$

Therefore, each element has a unique inverse. □

Definition 2.5. *Let S be a near vector space. We define*

$$j_n : S \longrightarrow R_n(S)$$
$$j_n(x) = [x + (n-1)z, z, z, ..., z] \quad \forall x, z \in S. \tag{2.1}$$

Definition 2.6. *If S is a near vector space and $d : S \times S \longrightarrow \mathbb{R}_+$ is a metric on S, then d is said to be an invariant metric on S, provided that*
(1) addition and scalar multiplication by positive scalars are continuous operations in the topology defined by d,
(2) $d(\lambda x, \lambda y) = \lambda d(x, y)$ for all $\lambda \in \mathbb{R}_+$ and $x, y \in S$,
(3) $d(x + z,\ y + z) = d(x,\ y)$ for all $x, y, z \in S$.

Definition 2.7. *If S is a near vector space and d is an invariant metric on S, then we define*

$$\|.\|_{dn}\colon R_n(S) \longrightarrow \mathbb{R}_+$$
$$\|[x_1, x_2, ..., x_n]\|_{dn} := d(x_1,\ x_2 + x_3 + \cdots + x_n) \tag{2.2}$$

for all $[x_1, x_2, ..., x_n] \in R_n(S)$.

Lemma 2.8. *If S is a near vector space and d is an invariant metric on S, then $\|.\|_{dn}$ defined by (2.2) is a norm on $R_n(S)$.*

Proof. Suppose that for $[x_1, x_2, ..., x_n]$ and $[y_1, y_2, ..., y_n] \in R_n(S)$ we have

$$[x_1, x_2, ..., x_n] = [y_1, y_2, ..., y_n].$$

Then $x_1 + y_2 + y_3 + \cdots + y_n = y_1 + x_2 + x_3 + \cdots + x_n$, and hence

$$\begin{aligned}\|[x_1, x_2, ..., x_n]\|_{dn} &= d(x_1,\ x_2 + x_3 + \cdots + x_n)\\ &= d(x_1 + y_1,\ y_1 + x_2 + x_3 + \cdots + x_n)\\ &= d(x_1 + y_1,\ x_1 + y_2 + y_3 + \cdots + y_n)\\ &= d(y_1,\ y_2 + y_3 + \cdots + y_n)\\ &= \|[y_1, y_2, ..., y_n]\|_{dn}.\end{aligned}$$

Thus the norm is well-defined. We also have

$$\begin{aligned}&\|[x_1, x_2, ..., x_n] + [y_1, y_2, ..., y_n]\|_{dn}\\ &= \|[x_1 + y_1, x_2 + y_2, ..., x_n + y_n]\|_{dn}\\ &= d(x_1 + y_1,\quad x_2 + y_2 + x_3 + y_3 + \cdots + x_n + y_n)\\ &\le d(x_1 + y_1,\quad x_1 + y_2 + y_3 + \cdots + y_n)\\ &+ d(x_1 + y_2 + y_3 + \cdots + y_n,\quad x_2 + y_2 + x_3 + y_3 + \cdots + x_n + y_n)\\ &= d(y_1,\quad y_2 + y_3 + \cdots + y_n) + d(x_1,\quad x_2 + x_3 + \cdots + x_n)\\ &= \|[x_1, x_2, ..., x_n]\|_{dn} + \|[y_1, y_2, ..., y_n]\|_{dn}.\end{aligned}$$

Other properties of a norm are clearly satisfied. □

Now we state and prove the main result of this section.

Theorem 2.9. *Let S be a near vector space, then the following statements hold:*
(a) There exists a vector space $R_n(S)$ and a map $j_n : S \longrightarrow R_n(S)$ for $n = 2, 3, 4, \cdots$ such that
(1) j_n is injective,
(2) $j_n(\alpha x + \beta y) = \alpha j_n(x) + \beta j_n(y)$ for all $\alpha, \beta \in \mathbb{R}_+$ and $x, y \in S$,
(3)

$$\begin{aligned}R_n(S) &= j_n(S) - \big(j_n(S) + j_n(S) + \cdots + j_n(S)\big)\\ &:= \{j_n(x_1) - \big(j_n(x_2) + j_n(x_3) + \cdots + j_n(x_n)\big) : x_1, x_2, ..., x_n \in S\}.\end{aligned}$$

(b) If $d : S \times S \longrightarrow \mathbb{R}_+$ is an invariant metric, then there exists a norm $\|.\|_{dn}$ on $R_n(S)$ such that $d(x,\ y) = \|j_n(x) - j_n(y)\|_{dn}$ for all $x, y \in S$.

Proof. By Definition 2.2 and Lemmas 2.3 and 2.4 it is clear that $R_n(S)$ is a vector space with additive identity $[(n-1)x_1, x_1, x_1, ..., x_1]$ and additive inverse $-[x_1, x_2, ..., x_n] := [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]$.

Note also that the map $j_n : S \longrightarrow R_n(S)$, defined by (2.1), has the desired properties. For (3) suppose that $x_1, x_2, ..., x_n \in S$ and $z \in S$,

$$\begin{aligned} j_n(x_1) - \big(j_n(x_2) + j_n(x_3) + \cdots + j_n(x_n)\big) &= [x_1 + (n-1)z, z, z, ..., z] \\ &- [x_2 + x_3 + \cdots + x_n + (n-1)z, z, z, ..., z] \\ &= [x_1 + (n-1)z, z, z, ..., z] \\ +[(n-2)(x_2+x_3+\cdots+x_n+(n-1)z)&+(n-1)z, x_2+x_3+\cdots+x_n+(n-1)z, x_2+x_3 \\ &+ \cdots + x_n + (n-1)z, ..., x_2 + x_3 + \cdots + x_n + (n-1)z] \\ &= [x_1 + (n-1)z, x_2 + x_3 + \cdots + x_n + z, z, z, ..., z] \\ &= [x_1 + (n-1)z, x_2 + z, x_3 + z, ..., x_n + z] = [x_1, x_2, ..., x_n]. \end{aligned}$$

Therefore

$$R_n(S) = j_n(S) - \big(j_n(S) + j_n(S) + \cdots + j_n(S)\big).$$

To prove part (b), let d be an invariant metric on S, then $\|.\|_{dn}$, defined by (2.2) is a norm on $R_n(S)$. Moreover, by using Lemma 2.8 we prove that $\|.\|_{dn}$ has the desired property:

$$\begin{aligned} \|j_n(x) - j_n(y)\|_{dn} &= \|[x + (n-1)z, z, z, ..., z] - [y + (n-1)z, z, z, ..., z]\|_{dn} \\ &= \|[x + (n-1)z, z, z, ..., z] + [(n-2)\big(y + (n-1)z\big) \\ &+ z + \cdots + z, y + (n-1)z, y + (n-1)z, ..., y + (n-1)z]\|_{dn} \\ &= \|[x + (n-2)y + n(n-1)z, y + nz, y + nz, ..., y + nz]\|_{dn} \\ &= d\big(x + (n-2)y + n(n-1)z,\ (n-1)y + (n-1)nz\big) \\ &= d\big(x + (n-2)y,\ (n-1)y\big) \\ &= d\big(x + (n-2)y,\ (n-2)y + y\big) \\ &= d(x,\ y). \end{aligned}$$

□

Example 2.10. *Consider $\mathbb{R}_+$ with usual addition and scalar multiplication, we embed $(\mathbb{R}_+, +, .)$ into $R_n(\mathbb{R}_+)$. Define $\sim_n$ on $\underbrace{\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n-times}$ by*

$$(x_1, x_2, ..., x_n) \sim_n (y_1, y_2, ..., y_n) \iff x_1+y_2+y_3+\cdots+y_n = y_1+x_2+x_3+\cdots+x_n.$$

We consider $R_n(\mathbb{R}_+)$ as the equivalence classes of this equivalence relation and define addition and scalar multiplication on $R_n(\mathbb{R}_+)$ by

$$[x_1, x_2, ..., x_n] + [y_1, y_2, ..., y_n] = [x_1 + y_1, x_2 + y_2, ..., x_n + y_n],$$

$$\lambda \cdot [x_1, x_2, ..., x_n] := \begin{cases} [\lambda x_1, \lambda x_2, ..., \lambda x_n], & \lambda \in \mathbb{R}_+, \\ [(n-1)x_1, x_1, x_1, ..., x_1], & \lambda = 0, \\ -\lambda\big(-[x_1, x_2, ..., x_n]\big), & -\lambda \in \mathbb{R}_+. \end{cases}$$

In this case $(R_n(\mathbb{R}_+), +, .)$ is a vector space with additive identity $[(n-1)x_1, x_1, x_1, ..., x_1]$ and additive inverse $-[x_1, x_2, ..., x_n] := [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]$, *for any* $(x_1, x_2, ..., x_n) \in \underbrace{\mathbb{R}_+ \times \mathbb{R}_+ \ \times \cdots \times \mathbb{R}_+}_{n-times}$. *Now, we can embed* $\mathbb{R}_+$, *for* $n = 2, 3, 4, ...$, *into* $(R_n(\mathbb{R}_+), +, .)$ *with the following embedding:*

$$j_n(x) = [x + (n-1)z, z, z, ..., z] \ ,$$

for all $x, z \in \mathbb{R}_+$.
Note that $d(x, y) = |x - y|$ *is an invariant metric on* $\mathbb{R}_+$, *and* $\|.\|_{dn}$ *which is defined as bellow, is a norm on* $R_n(\mathbb{R}_+)$

$$\|[x_1, x_2, ..., x_n]\|_{dn} = |x_1 - (x_2 + x_3 + \cdots + x_n)|.$$

This embedding, preserves distance, namely:

$$\begin{aligned} d(x,\ y) &= |x - y| \\ &= d_n\big(j_n(x),\ j_n(y)\big) \\ &= \|j_n(x) - j_n(y)\|_{dn}, \end{aligned}$$

where d_n *is the induced metric from* $\|j_n(x) - j_n(y)\|_{dn}$.

Theorem 2.11. *Let* V *be a vector space and* $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ *be a basis for* V. *Then*

$$\{j_n(\alpha_1), j_n(\alpha_2), ..., j_n(\alpha_k)\}$$

is a basis for $R_n(V)$.

Proof. The proof follows easily from the following statements:
(1) If V is a vector space, then $j_n(V)$ is onto, because, for any $[x_1, x_2, ..., x_n] \in R_n(V)$

$$\begin{aligned} j_n(x_1 - x_2 - x_3 - \cdots - x_n) &= [x_1 - x_2 - x_3 - \cdots - x_n, 0, 0, ..., 0] \\ &= [x_1, x_2, ..., x_n]. \end{aligned}$$

(2) The set $\{j_n(\alpha_1), j_n(\alpha_2), ..., j_n(\alpha_k)\}$ is linearly independent. □

In the following theorem, we prove that each j_n preserves inner products and completeness.

Theorem 2.12. *Suppose that* V *is an inner product space over* F. *Then the following assertions hold:*
(a) There is an inner product on $R_n(V)$ *such that* j_n *preserves the inner product.*
(b) V *is a Hilbert space if and only if* $R_n(V)$ *is a Hilbert space.*
(c) If V *or* $R_n(V)$ *is finite dimensional, then* j_n *is continuous on* V.
(d) If V *is a Hilbert space, then* $(x_k)_{k\in\mathbb{N}}$ *is an orthonormal basis for* V *if and only if* $\big(j_n(x_k)\big)_{k\in\mathbb{N}}$ *is an orthonormal basis for* $R_n(V)$.

Proof. (a) Suppose that $\langle\ ,\ \rangle$ is an inner product on V; define $\langle\ ,\ \rangle_n$ for

$$[x_1, x_2, ..., x_n],\ [y_1, y_2, ..., y_n] \in R_n(V)$$

by:

$$\langle [x_1, x_2, ..., x_n],\ [y_1, y_2, ..., y_n] \rangle_n = \langle x_1 + y_2 + y_3 + \cdots + y_n,\ y_1 + x_2 + x_3 + \cdots, x_n \rangle.$$

It is easy to verify that

$$\langle [x_1, x_2, ..., x_n],\ [y_1, y_2, ..., y_n] \rangle_n = \langle x_1 - x_2 - x_3 - \cdots - x_n,\ y_1 - y_2 - y_3 - \cdots - y_n \rangle.$$

So that $\langle j_n(x),\ j_n(y)\rangle = \langle [x,0,0,...,0],\ [y,0,0,...,0]\rangle_n = \langle x,\ y\rangle$.
The proof of (b) is easy and (c) follows from [7] Theorem 7B.
(d) Suppose $[y_1, y_2, ..., y_n] \in R_n(V)$ is a vector that satisfies the condition

$$\langle [y_1, y_2, ..., y_n],\ j_n(x_k)\rangle = 0$$

for all $k \in \mathbb{N}$. Then

$$\langle y_1 - y_2 - y_3 - \cdots - y_n,\ x_k\rangle = 0$$

so

$$y_1 - y_2 - y_3 - \cdots - y_n = 0$$

or

$$y_1 = y_2 + y_3 + \cdots + y_n.$$

Therefore

$$[y_1, y_2, ..., y_n] = 0_{R_n(V)}.$$

Similarly, if $x \in V$ is a vector that satisfies the condition $\langle x,\ x_k\rangle = 0$ for all $k \in \mathbb{N}$, then $\langle j_n(x),\ j_n(x_k)\rangle = 0$ and so $j_n(x) = 0_{R_n(V)}$, therefore $x = 0$. □

3. Near vector lattices

This section is devoted to the study of partially ordered near vector spaces, as well as vector lattices.

Definition 3.1. *A partially ordered set $(P, \leq)$ is called a join-semilattice if the least upper bound (join) of x and y, denoted $x \vee y$, exists for all $x, y \in P$. Moreover, if it has the greatest lower bound (meet) of x and y; denoted $x \wedge y$, then P is called a lattice.*

Definition 3.2. *Let S be a near vector space. If $(S, \leq)$ is a partially ordered set such that $\leq$ is compatible with addition and multiplication by positive scalars; i.e., for all x, y,*

$$x \leq y \Longrightarrow \begin{cases} x + z \leq y + z, \\ \alpha x \leq \alpha y. \end{cases}$$

Then S is called an ordered near vector space. If S is an ordered near vector space and $(S, \leq)$ is a join-semilattice for which

$$(x \vee y) + z = (x + z) \vee (y + z), \quad x, y, z \in S,$$

then S is called a near vector lattice.

A Riesz space, a lattice-ordered vector space or a vector lattice is defined similarly.

Definition 3.3. *Let $d : S \times S \longrightarrow \mathbb{R}_+$ be an invariant metric on a near vector lattice S. Then d is said to be a Riesz metric on S provided that*
(i) $x \leq y \leq z \Longrightarrow d(x,\ y) \leq d(x,\ z)$, and
(ii) $d(x,\ y) = d\big(2(x \vee y),\ x + y\big)$ for all $x, y, z \in S$.
Moreover if $\|.\|: E \longrightarrow \mathbb{R}_+$ is a (semi) norm, then $\|\cdot\|: E \longrightarrow \mathbb{R}_+$ is called a Riesz (semi) norm, provided that $x, y \in E$ and $0 \leq y \leq x$, then $\|y\| \leq \|x\|$, and $\||x|\| = \|x\|$ for all $x \in E$.

Let $(S, \leq)$ be an ordered near vector space, we define

$$[x_1, x_2, ..., x_n] \leq_n [y_1, y_2, ..., y_n] \Longleftrightarrow x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n,$$

for all $[x_1, x_2, ..., x_n], [y_1, y_2, ..., y_n] \in R_n(S)$.

Lemma 3.4. *Let $(S, \leq)$ be an ordered near vector space, then $\big(R_n(S), \leq_n\big)$ is an ordered vector space.*

Proof. For all $[x_1, x_2, ..., x_n], [y_1, y_2, ..., y_n], [z_1, z_2, ..., z_n] \in R_n(S)$ if $[x_1, x_2, ..., x_n] \leq_n [y_1, y_2, ..., y_n] \leq_n [z_1, z_2, ..., z_n]$ then we have

$$x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n$$

and

$$y_1 + z_2 + z_3 + \cdots + z_n \leq z_1 + y_2 + y_3 + \cdots + y_n$$

which imply

$$x_1 + y_2 + y_3 + \cdots + y_n + z_2 + z_3 + \cdots + z_n \leq y_1 + x_2 + x_3 + \cdots + x_n + z_2 + z_3 + \cdots + z_n$$

and

$$y_1 + z_2 + z_3 + \cdots + z_n + x_2 + x_3 + \cdots + x_n \leq z_1 + y_2 + y_3 + \cdots + y_n + x_2 + x_3 + \cdots + x_n.$$

From these inequalities we get

$$x_1 + y_2 + y_3 + \cdots + y_n + z_2 + z_3 + \cdots + z_n \leq z_1 + y_2 + y_3 + \cdots + y_n + x_2 + x_3 + \cdots + x_n.$$

Thus

$$[x_1 + y_2 + y_3 + \cdots + y_n, x_2 + y_2, x_3 + y_3, ..., x_n + y_n] \leq_n [z_1, z_2, ..., z_n],$$

and, $[x_1, x_2, ..., x_n] \leq_n [z_1, z_2, ..., z_n]$. Other properties of an ordered vector space easily follow. □

If E is a vector lattice, then $E_+ := \{x \in E : x \geq 0\}$ denotes the positive cone of E. Furthermore, $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x \vee (-x)$ are the positive part, negative part and absolute value of $x \in E$, respectively.

Theorem 3.5. *If S is a near vector lattice, then we have:*
(a) $R_n(S)$ is a vector lattice, with positive cone

$$R(S)_+ := \{[x_1, x_2, ..., x_n] : x_2 + x_3 + \cdots + x_n \leq x_1\},$$

in which the following equalities hold:

(1) $[x_1, x_2, ..., x_n]^+ = [x_1 \vee (x_2 + x_3 + \cdots + x_n), x_2, x_3, ..., x_n]$,
(2) $[x_1, x_2, ..., x_n]^- = [\big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big) + (n-2)x_1, x_1, x_1, ..., x_1]$,
(3) $|[x_1, x_2, ..., x_n]| = [2(x_1 \vee (x_2 + x_3 + \cdots + x_n)), x_1 + x_2, x_3, x_4, ..., x_n]$,
(4) $[x_1, x_2, ..., x_n] \vee [y_1, y_2, ..., y_n] = [(x_1 + y_2 + y_3 + \cdots + y_n) \vee (y_1 + x_2 + x_3 + \cdots + x_n), x_2 + y_2, x_3 + y_3, ..., x_n + y_n]$,
(5) $[x_1, x_2, ..., x_n] \wedge [y_1, y_2, ..., y_n] = [x_1 + x_3 + x_4 + \cdots + x_n + y_1, (x_1 + y_2 + y_3 + \cdots + y_n) \vee (y_1 + x_2 + x_3 + \cdots + x_n), x_3, x_4, ..., x_n]$,
(6) $[x_1, x_2, ..., x_n] \wedge -[x_1, x_2, ..., x_n] = -|[x_1, x_2, ..., x_n]|$,
(7) $-[x_1, x_2, ..., x_n] \wedge [(n-1)x_1, x_1, x_1, ..., x_1] = -[x_1, x_2, ..., x_n] - [x_1, x_2, ..., x_n]^-$.

(b) The embedding $j_n : S \longrightarrow R_n(S)$ is join preserving.
(c) If $d : S \times S \longrightarrow \mathbb{R}_+$ is an invariant metric, then d is a Riesz metric on S if and only if $\|.\|_{dn}$ is a Riesz norm on the vector lattice $R_n(S)$.

Proof. To prove (1), we note that $[x_1 \vee (x_2+x_3+\cdots+x_n), x_2, x_3, ..., x_n]$ is an upper bound for $\{[x_1, x_2, ..., x_n], [(n-1)x_1, x_1, x_1, ..., x_1]\}$; since

i) $x_2 + x_3 + \cdots + x_n \leq x_1 \vee (x_2 + x_3 + \cdots + x_n)$ we obtain
$(n-1)x_1 + x_2 + x_3, ... + x_n \leq \big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big) + (n-1)x_1$, so
$[(n-1)x_1, x_1, x_1, ..., x_1] \leq_n [x_1 \vee (x_2 + x_3 + \cdots + x_n), x_2, x_3, ..., x_n]$.

ii) $x_1 \leq x_1 \vee (x_2 + x_3 + \cdots + x_n)$, so
$x_1 + x_2 + \cdots + x_n \leq \big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big) + x_2 + x_3 + \cdots + x_n$
therefore $[x_1, x_2, ..., x_n] \leq_n [x_1 \vee (x_2 + x_3 + \cdots + x_n), x_2, x_3, ..., x_n]$.

iii) If $[y_1, y_2, ..., y_n]$ is also an upper bound for

$$\{[x_1, x_2, ..., x_n], [(n-1)x_1, x_1, x_1, ..., x_1]\},$$

then $[x_1, x_2, ..., x_n] \leq_n [y_1, y_2, ..., y_n]$, so

$$x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n.$$

Hence $[x_2 + x_3 + \cdots + x_n, x_2, x_3, ..., x_n] \leq_n [y_1, y_2, ..., y_n]$, which implies that

$$x_2 + x_3 + \cdots + x_n + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n.$$

Therefore, from these inequalities we conclude that
$(x_1+y_2+y_3+\cdots+y_n)\vee(x_2+x_3+\cdots+x_n+y_2+y_3+\cdots+y_n) \leq y_1+x_2+x_3+\cdots+x_n$, so
$\big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big) + (y_2 + y_3 + \cdots + y_n) \leq y_1 + x_2 + x_3 + \cdots + x_n$, hence
$[x_1 \vee (x_2 + x_3 + \cdots + x_n), x_2, x_3, ..., x_n] \leq_n [y_1, y_2, ..., y_n]$, therefore
$[x_1, x_2, ..., x_n]^+ = [x_1 \vee x_2 + x_3 + \cdots + x_n, x_2, x_3, ..., x_n]$.

To prove (2), note that

$$\begin{aligned}
[x_1, x_2, ..., x_n]^- &= [(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]^+ \\
&= [\big((n-2)x_1 + x_2 + x_3 + \cdots + x_n\big) \vee (n-1)x_1, x_1, x_1, ..., x_1] \\
&= [\big((n-2)x_1 + x_2 + \cdots + x_n\big) \vee \big((n-2) + x_1\big) + x_1, x_1, ..., x_1] \\
&= [\big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big) + (n-2)x_1, x_1, x_1, ..., x_1].
\end{aligned}$$

This also proves that $R_n(S)$ is a vector lattice. It is clear that $R_n(S)$ is a vector lattice with positive cone $R(S)_+ := \{[x_1, x_2, ..., x_n] : x_2 + x_3 + \cdots + x_n \leq x_1\}$.
For (3), if E is a vector lattice, we make use of the following well-known equality (see [4], p. 17) in the sequel:
$2(x \vee y) - (x+y) = |x-y|$, so $2(x \vee 0) - (x+0) = |x-0|$ *for all* $x, y \in E$. Therefore $|x| = 2x^+ - x$ *for all* $x \in E$.
Since, $(x-y)^+ = x \vee y - y$ for all $x, y \in E$ (see [5]) we conclude that $x \vee y = (x-y)^+ + y$. So (4) is proved.
Parts (5) and (6) can be proved by using the fact that, $x \vee y + x \wedge y = x + y$ for all $x, y \in E$ (see [5]) because $x \wedge y = x + y - (x \vee y)$ so

$$[x_1, x_2, ..., x_n] \wedge -[x_1, x_2, ..., x_n] = -|[x_1, x_2, ..., x_n]|.$$

Now (4) and (5) imply that $R_n(S)$ is a vector lattice. It is clear that $R_n(S)$ is a vector lattice with positive cone $R(S)_+ := \{[x_1, x_2, ..., x_n] : x_2 + x_3 + \cdots + x_n \leq x_1\}$.

To prove (7) we use the fact that $x \vee y + x \wedge y = x + y$ for all $x, y \in E$ (see [5]), so if $x := -x$ and $y := 0$ then we have

$$-[x_1, x_2, ..., x_n] \wedge [(n-1)x_1, x_1, x_1, ..., x_1] = -[x_1, x_2, ..., x_n] - [x_1, x_2, ..., x_n]^-.$$

As for part (b), let $x, y \in S$. Then for any $z \in S$:

$$\begin{aligned} j_n(x) \vee j_n(y) &= [x + (n-1)z, z, z, ..., z] \vee [y + (n-1)z, z, z, ..., z] \\ &= [\big(x + \big(2(n-1)\big)z\big) \vee \big(y + 2(n-1)z\big), 2z, 2z, ..., 2z] \\ &= [(x \vee y) + \big(2(n-1)\big)z, 2z, 2z, ..., 2z] \\ &= [(x \vee y) + (n-1)z, z, z, ..., z] \\ &= j_n(x \vee y). \end{aligned}$$

To prove (c), suppose that d is a Riesz metric on S and $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S$ and

$$[y_2 + y_3 + \cdots + y_n, y_2, y_3, ..., y_n] \leq_n [x_1, x_2, ..., x_n] \leq_n [y_1, y_2, ..., y_n],$$

then $x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n$ and,

$$x_2 + x_3 + \cdots + x_n + y_2 + y_3 + \cdots + y_n \leq x_1 + y_2 + y_3 + \cdots + y_n.$$

It now follows that
$x_2 + x_3 + \cdots + x_n + y_2 + y_3 + \cdots + y_n \leq x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n$,
so
$d(x_2 + x_3 + \cdots + x_n + y_2 + y_3 + \cdots + y_n,\ x_1 + y_2 + y_3 + \cdots + y_n)$
$\leq d(x_2 + x_3 + \cdots + x_n + y_2 + y_3 + \cdots + y_n,\ y_1 + x_2 + x_3 + \cdots + x_n)$, hence
$d(x_2 + x_3 + \cdots + x_n,\ x_1) \leq d(y_2 + y_3 + \cdots + y_n,\ y_1)$, thus
$d(x_1,\ x_2 + x_3 + \cdots + x_n) \leq d(y_1,\ y_2 + y_3 + \cdots + y_n)$, and finally
$\|[x_1, x_2, ..., x_n]\|_{dn} \leq_n \|[y_1, y_2, ..., y_n]\|_{dn}$.

And from the fact that

$$|[x_1, x_2, ..., x_n]| = [2(x_1 \vee (x_2 + x_3 + \cdots + x_n)), x_1 + x_2, x_3, x_4, ..., x_n],$$

we have

$$\begin{aligned} \|[x_1, x_2, ..., x_n]\|_{dn} &= d(x_1, x_2 + x_3 + \cdots + x_n) \\ &= d(2(x_1 \vee (x_2 + x_3 + \cdots + x_n)), x_1 + x_2 + \cdots + x_n) \\ &= \|[2(x_1 \vee (x_2 + \cdots + x_n)), x_1 + x_2, x_3, ..., x_n]\|_{dn} \\ &= \||[x_1, x_2, ..., x_n]|\|_{dn}. \end{aligned}$$

Therefore $\|.\|_{dn}$ is a Riesz norm on $R_n(S)$.
Conversely, if $\|.\|_{dn}$ is a Riesz norm on $R_n(S)$, $x, y, z \in S$ and $x \leq y \leq z$, then for all $x_1, x_2, ..., x_{2n-6} \in S$ if $y \leq z$, then

$$x + 2y + z + x_1 + x_2 + \cdots + x_{2n-6} \leq x + y + 2z + x_1 + x_2 + \cdots + x_{2n-6}.$$

Hence,

$$\begin{gathered} [y + x_{n-3} + x_{n-2} + \cdots + x_{2n-6}, x, x_{n-3}, x_{n-2}, ..., x_{2n-6}] \\ \leq_n [y + 2z + x_1 + x_2 + \cdots + x_{n-4}, x, y, z, x_1, x_2, ..., x_{n-4}], \end{gathered}$$

so

$$\begin{gathered} \|[y + x_{n-3} + x_{n-2} + \cdots + x_{2n-6}, x, x_{n-3}, x_{n-2}, ..., x_{2n-6}]\|_{dn} \\ \leq_n \|[y + 2z + x_1 + x_2 + \cdots + x_{n-4}, x, y, z, x_1, x_2, ..., x_{n-4}]\|_{dn}, \end{gathered}$$

and

$$d(y + x_{n-3} + x_{n-2} + \cdots + x_{2n-6},\ x + x_{n-3} + x_{n-2} + \cdots + x_{2n-6}) \leq$$

$$d(y + 2z + x_1 + x_2 + \cdots + x_{n-4},\ x + y + z + x_1 + x_2 + \cdots + x_{n-4}).$$

Therefore $d(x,\ y) \leq d(x,\ z)$. Moreover,

$$\begin{aligned} d(x_1, x_2 + x_3 + \cdots + x_n) &= \|[x_1, x_2, ..., x_n]\|_{dn} \\ &= \||[x_1, x_2, ..., x_n]|\|_{dn} \\ &= \|[2(x_1 \vee (x_2 + \cdots + x_n)), x_1 + x_2, x_3, ..., x_n]\|_{dn} \\ &= d(2(x_1 \vee (x_2 + \cdots + x_n)),\ x_1 + \cdots + x_n) \end{aligned}$$

which shows that d is a Riesz metric on S. □

By the same argument as in the previous theorem we obtain the following theorem.

Theorem 3.6. *If S is a vector lattice, then the followings hold:*

(1) $[x_1, x_2, ..., x_n]^+ = [x_1 \vee (x_2 + x_3 + \cdots + x_n), x_2 + x_3 + \cdots + x_n, 0, 0, ..., 0]$,

(2) $[x_1, x_2, ..., x_n]^- = [\big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big), x_1, 0, 0, ..., 0]$,

(3) $|[x_1, x_2, ..., x_n]| = [2\big(x_1 \vee (x_2 + x_3 + \cdots + x_n)\big), x_1 + x_2 + \cdots + x_n, 0, 0, ..., 0]$,

(4) $[x_1, x_2, ..., x_n] \vee [y_1, y_2, ..., y_n] = [(x_1 + y_2 + y_3 + \cdots + y_n) \vee (y_1 + x_2 + x_3 + \cdots + x_n), x_2 + x_3 + \cdots + x_n, y_2 + y_3 + \cdots + y_n, 0, 0, ..., 0]$,

(5) $[x_1, x_2, ..., x_n] \wedge [y_1, y_2, ..., y_n] = [x_1 + y_1, (x_1 + y_2 + y_3 + \cdots + y_n) \vee (y_1 + x_2 + x_3 + \cdots + x_n), 0, 0, ..., 0]$.

Example 3.7. *As in the previous example, consider $\mathbb{R}_+$ with the usual ordering and the join $\vee$ given by $x \vee y = \max\{x, y\}$. Now we have a vector lattice. Moreover, $R_n(\mathbb{R}_+)$ for each $n = 2, 3, 4, ...$ is a vector lattice and embedding j_n which is join preserving:*

$$[x_1, x_2, ..., x_n] \leq_n [y_1, y_2, ..., y_n] \Longleftrightarrow x_1 + y_2 + y_3 + \cdots + y_n \leq y_1 + x_2 + x_3 + \cdots + x_n,$$

$[x_1, x_2, ..., x_n] \vee [y_1, y_2, ..., y_n] = [max\{(x_1 + y_2 + y_3 + \cdots + y_n), (y_1 + x_2 + x_3 + \cdots + x_n)\}, x_2 + y_2, x_3 + y_3, ..., x_n + y_n]$,
$[x_1, x_2, ..., x_n] \wedge [y_1, y_2, ..., y_n] = [x_1 + x_3 + x_4 + \cdots + x_n + y_1, max\{(x_1 + y_2 + y_3 + \cdots + y_n) \vee (y_1 + x_2 + x_3 + \cdots + x_n)\}, x_3, x_4, ..., x_n]$.

Note that the metric defined on $\mathbb{R}_+$ is a Riesz metric and the norm on $R_n(\mathbb{R}_+)$ for each $n = 2, 3, 4, ...,$ is a Riesz norm.

Theorem 3.8. *Let S_1 and S_2 be near vector spaces and $T : S_1 \longrightarrow S_2$ be addition preserving. Define $\hat{T} : R_n(S_1) \longrightarrow R_n(S_2)$ by*

$$\hat{T}([x_1, x_2, ..., x_n]) = [T(x_1), T(x_2), ..., T(x_n)] \qquad \text{for all } x_1, x_2, ..., x_n \in S_1.$$

(a) If T is $\mathbb{R}_+$-linear, then $\hat{T}$ is linear.
(b) If d_1 is an invariant metric on S_1, d_2 is an invariant metric on S_2 and T is non-expansive, then $\|\hat{T}\| \leq 1$.

Proof. For (a) we only need to prove if $\alpha \in \mathbb{R}_+$, then

$$\begin{aligned}\hat{T}\big(\alpha[x_1, x_2, ..., x_n]\big) &= \hat{T}\big(-\alpha[(n-2)x_1 + x_2 + x_3 + \cdots + x_n, x_1, x_1, ..., x_1]\big)\\ &= \hat{T}\big([\big(-\alpha(n-2)\big)x_1 - \alpha x_2 - \alpha x_3 - \cdots - \alpha x_n, -\alpha x_1, -\alpha x_1, ..., -\alpha x_1]\big)\\ &= [\big(-\alpha(n-2)\big)\big(T(x_1)\big) - \alpha\big(T(x_2) + T(x_3) + \cdots + T(x_n)\big), -\alpha T(x_1),\\ &\quad -\alpha T(x_2), ..., -\alpha T(x_n)] = \alpha[T(x_1), T(x_2), ..., T(x_n)]\\ &= \alpha\hat{T}([x_1, x_2, ..., x_n]).\end{aligned}$$

For (b) first we define:

$$\|\hat{T}\| := \sup\{\|\hat{T}[x_1, x_2, ..., x_n]\|_{d_2} \colon \|[x_1, x_2, ..., x_n]\|_{d_1} \leq 1\}$$

where

$$\begin{aligned}\|\hat{T}[x_1, x_2, ..., x_n]\|_{d_2} &= d_2\big(T(x_1), T(x_2) + T(x_3) + \cdots + T(x_n)\big)\\ &\leq d_1(x_1, x_2 + x_3 + \cdots + x_n)\\ &= \|[x_1, x_2, ..., x_n]\|_{d_1}\\ &\leq 1.\end{aligned}$$

Since T is non-expansive. So

$$\|\hat{T}\| \leq 1.$$

□

Let S_2 be a near vector lattice and S_1 a nonempty subset of S_2. Then S_1 is said to be a sub-near vector lattice of S_2 provided that S_1 is closed under the operations addition, multiplication by positive scalars and join. The notion of sub-near vector space is defined similarly.

Corollary 3.9. *If S_1 is a sub-near vector space (lattice) of a near vector space (lattice) S_2, then $R_n(S_1)$ is a vector subspace (sublattice) of $R_n(S_2)$.*

Proof. Since S_1 is closed under addition and multiplication operations, it is clear that $R_n(S_1)$ is closed under this operations, so that $R_n(S_1)$ is a vector sublattice of $R_n(S_2)$. □

4. Filtration, martingales and metric spaces

In this section we study the embedding theorem on filtration and Martingales.

Definition 4.1. *Let (P, d) be a metric space and f be a function on P. Then f is called a non-expansive idempotent if for each x and y in P, $d\big(f(x), f(y)\big) \leq d(x, y)$ and $f(f(x)) = f(x)$.*

Definition 4.2. *Let (X, d) be a metric space and f be a linear function on X. Then f is called a contractive linear projection if for each x and y in X, $d\big(f(x), f(y)\big) < d(x, y)$ and $f(f(x)) = f(x)$.*

Definition 4.3. *Let X be a Banach space. If $T_i : X \longrightarrow X$ is a contractive linear projection and $T_i = T_iT_k = T_kT_i$ for each $i \leq k$ where $i, k \in \mathbb{N}$, then the sequence of projections (T_i) is called a BS-filtration on X. If (T_i) is a BS-filtration on X, the pair (f_i, T_i) is called a martingale in X if $T_if_k = f_i$ for each $i \leq k$, and $(f_i) \subseteq X$.*

This motivates the following definition.

Definition 4.4. *Let (P, d) be a complete metric space. A Sequence $(\mathcal{E}_i)$ of non-expansive idempotents on P is called an MS-filtration on P if we have*

$$\mathcal{E}_i\mathcal{E}_k = \mathcal{E}_k\mathcal{E}_i = \mathcal{E}_i \quad \forall i \leq k.$$

Moreover if there exists $(f_i) \subseteq P$ such that $f_i = \mathcal{E}_i f_k$ for all $i \leq k$, then $(f_i, \mathcal{E}_i)$ is called a martingale in P.

It is obvious that $\mathcal{R}(\mathcal{E}_i) \subseteq \mathcal{R}(\mathcal{E}_{i+1})$ where $(\mathcal{E}_i)$ is an MS-filtration on P and $\mathcal{R}(\mathcal{E}_i)$ denotes the range of $\mathcal{E}_i$.

Definition 4.5. *In Definition 4.4 if we replace (P, d) by (S, d) where S is a complete near vector space with respect to the invariant metric d, each $(\mathcal{E}_i)$ is $\mathbb{R}_+$-linear, and each $\mathcal{R}(\mathcal{E}_i)$ is a (closed) near vector subspace of S, then this set is denoted by $\big(S, d, \mathcal{E}_i\big)$ and is called a complete MS-filtration space.*

Lemma 4.6. *Let $\big(S, d, \mathcal{E}_i\big)$ be a complete MS-filtration space, then $(\tilde{\mathcal{E}}_{in})$ is a BS-filtration on $\tilde{R}_n(S)$, where $n = 2, 3, 4, ...$ and each $(\tilde{\mathcal{E}}_{in})$ is the continuous extension of $\hat{\mathcal{E}}_{in}$ defined by*

$$\hat{\mathcal{E}}_{in}([x_1, x_2, ..., x_n]) = [\mathcal{E}_i x_1, \mathcal{E}_i x_2, ..., \mathcal{E}_i x_n].$$

Moreover, $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}\big(\tilde{\mathcal{E}}_{in}|_{j_n(S)}\big)} = j_n\big(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathcal{E}_i)}\big)$; the former closure is the $\|.\|_{dn}$ -closure in $\tilde{R}_n(S)$ and the latter is the d-closure in S.

Proof. Since $\mathcal{E}_i$ is $\mathbb{R}_+$-linear and non-expansive, it follows from Theorem 3.8 that $\hat{\mathcal{E}}_{in}$ is linear and $\|\hat{\mathcal{E}_{in}}\| \leq 1$. As $\mathcal{E}_i\mathcal{E}_k = \mathcal{E}_k\mathcal{E}_i = \mathcal{E}_i$ for all $i \leq k$, then $\hat{\mathcal{E}}_{in}\hat{\mathcal{E}}_{kn} = \hat{\mathcal{E}}_{kn}\hat{\mathcal{E}}_{in} = \hat{\mathcal{E}}_{in}$ for all $i \leq k$.
As $(\tilde{\mathcal{E}}_{in})$ is the continuous extension to $\tilde{R}_n(S)$ of $\hat{\mathcal{E}}_{in}$, it follows that $(\tilde{\mathcal{E}}_{in})$ is a linear contractive projection with $\|\tilde{\mathcal{E}}_i\| \leq 1$ and $\tilde{\mathcal{E}}_i\tilde{\mathcal{E}}_k = \tilde{\mathcal{E}}_k\tilde{\mathcal{E}}_i = \tilde{\mathcal{E}}_i$ for all $i \leq k$. Consequently, $(\tilde{\mathcal{E}}_{in})$ is a BS-filtration on $\tilde{R}_n(S)$.
It remains to show that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}\big(\tilde{\mathcal{E}}_{in}|_{j_n(S)}\big)} = j_n\big(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathcal{E}_i)}\big)$. We first note that $\mathcal{R}\big(\tilde{\mathcal{E}}_{in}|_{j_n(S)}\big) = j_n\big(\mathcal{R}(\mathcal{E}_i)\big)$ for all $i \in \mathbb{N}$, because
Let $[k + (n-1)w, w, w, ..., w] \in \mathcal{R}\big(\tilde{\mathcal{E}}_{in}|_{j_n(S)}\big)$. Then, there are $x, z \in S$ such that $\tilde{\mathcal{E}}_{in}(z) = \hat{\mathcal{E}}_{in}(z) = w$ and $\tilde{\mathcal{E}}_{in}(x) = \hat{\mathcal{E}}_{in}(x) = k$. Thus

$$\begin{aligned}
[k + (n-1)w, w, w, ..., w] &= \tilde{\mathcal{E}}_{in}\big([x + (n-1)z, z, z, ..., z]\big) \\
&= \hat{\mathcal{E}}_{in}\big([x + (n-1)z, z, z, ..., z]\big) \\
&= [\mathcal{E}_i(x) + (n-1)\mathcal{E}_i(z), \mathcal{E}_i(z), \mathcal{E}_i(z), ..., \mathcal{E}_i(z)] \\
&= j_n\big(\mathcal{E}_i(x)\big) \in j_n\big(\mathcal{R}(\mathcal{E}_i)\big).
\end{aligned}$$

Let $[k + (n-1)w, w, w, ..., w] \in j_n\big(\mathcal{R}(\mathcal{E}_i)\big)$. Then, there are $x, z \in S$ such that $\tilde{\mathcal{E}}_{in}(z) = \hat{\mathcal{E}}_{in}(z) = w$ and $\tilde{\mathcal{E}}_{in}(x) = \hat{\mathcal{E}}_{in}(x) = k$. Thus

$$\begin{aligned}
[k + (n-1)w, w, w, ..., w] &= [\mathcal{E}_i(x) + (n-1)\mathcal{E}_i(z), \mathcal{E}_i(z), \mathcal{E}_i(z), ..., \mathcal{E}_i(z)] \\
&= [\mathcal{E}_i(x + (n-1)z), \mathcal{E}_i(z), \mathcal{E}_i(z), ..., \mathcal{E}_i(z)] \\
&= \hat{\mathcal{E}}_{in}\big([x + (n-1)z, z, z, ..., z]\big) \\
&= \tilde{\mathcal{E}}_{in}\big([x + (n-1)z, z, z, ..., z]\big) \in \mathcal{R}\big(\tilde{\mathcal{E}}_{in}|\ j_n(S)\big).
\end{aligned}$$

Consequently, $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}\big(\tilde{\mathcal{E}_{in}}|_{j_n(S)}\big)} = \overline{\bigcup_{i=1}^{\infty} j_n\big(\mathcal{R}(\mathcal{E}_i)\big)}$. By the completeness of S and the continuity of j_n, it is readily verified that $\overline{\bigcup_{i=1}^{\infty} j_n\big(\mathcal{R}(\mathcal{E}_i)\big)} = j_n\big(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathcal{E}_i)}\big)$. Thus $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}\big(\tilde{\mathcal{E}_{in}}|j_n(S)\big)} = j_n\big(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathcal{E}_i)}\big)$. $\square$

Lemma 4.7. *Let $\big(S, d, \mathcal{E}_i\big)$ be a complete MS-filtration space, then $(f_i, \mathcal{E}_i)$ is a martingale in S (and (f_i) is d-convergent) if and only if $\big(j_n(f_i), \tilde{\mathcal{E}}_{in}\big)$ is a martingale in $\tilde{R}_n(S)$ (and $j_n\big((f_i)\big)$ is $\|.\|_d$-convergent).*

Proof. Recall that $j_n(x) = [x + (n-1)z, z, z, ..., z]$ for all $x \in S$ and for any $z \in S$. If $(f_i, \mathcal{E}_i)$ is a martingale in S, then $\big(j_n(f_i), \tilde{\mathcal{E}}_{in}\big)$ is a martingale in $\tilde{R}_n(S)$, because, for $i \leq k$,

$$\begin{aligned}\tilde{\mathcal{E}}_{in} j_n(f_k) &= \hat{\mathcal{E}}_{in}[f_k + (n-1)z, z, z, ..., z] \\ &= [\mathcal{E}_i f_k + (n-1)\mathcal{E}_i z, \mathcal{E}_i z, \mathcal{E}_i z, ..., \mathcal{E}_i z] \\ &= [f_i + (n-1)z, z, z, ..., z] \\ &= j_n(f_i).\end{aligned}$$

Conversely, suppose $\big(j_n(f_i), \tilde{\mathcal{E}}_{in}\big)$ is a martingale in $\tilde{R}_n(S)$. Then, for $i \leq k$ we have $\tilde{\mathcal{E}}_k\big(j_n(f_i)\big) = j_n(f_i)$, so $[\mathcal{E}_k f_i + (n-1)\mathcal{E}_k z, \mathcal{E}_k z, \mathcal{E}_k z, ..., \mathcal{E}_k z] = [f_i + (n-1)z, z, z, ..., z]$, and hence $\mathcal{E}_k f_i + (n-1)\mathcal{E}_k z + (n-1)z = f_i + (n-1)z + (n-1)\mathcal{E}_k z$. Therefore $\mathcal{E}_k f_i = f_i$, from which we conclude that $\mathcal{E}_i f_k = f_i$; this in turn means that $\big(j_n(f_i), \tilde{\mathcal{E}}_{in}\big)$ is a martingale in S. It now follows that the martingale $(f_i, \mathcal{E}_i)$ is d-convergent if and only if the martingale $\big(j_n(f_i), \tilde{\mathcal{E}}_{in}\big)$ is $\|.\|_{dn}$-convergent in $R_n(S)$, because

$$\begin{aligned}\lim_{i\to\infty} \|j_n(f_i) - j_n(f)\|_{dn} &= \lim_{i\to\infty} \|[f_i + (n-1)z, z, ..., z] - [f + (n-1)z, z, ..., z]\|_{dn} \\ &= \lim_{i\to\infty} \|[f_i + (n-1)z, z, z, ..., z] \\ &\quad + [(n-2)f + (n-1)(n-1)z, f + (n-1)z, ..., \\ &\quad f + (n-1)z]\|_{dn} \\ &= \lim_{i\to\infty} \|[f_i + (n-2)f, f, ..., f]\|_{dn} \\ &= \lim_{i\to\infty} d\big(f_i + (n-2)f,\ (n-1)f\big) \\ &= \lim_{i\to\infty} d(f_i,\ f).\end{aligned}$$

This completes the proof. □

Definition 4.8. *Let $\big(S, d, \mathcal{E}_i\big)$ be a complete MS-filtration space. Denote by $\mathcal{M}_d(S, \mathcal{E}_i)$ the set of all martingales $(f_i, \mathcal{E}_i)$ in S for which (f_i) is d-convergent. Define $d_{\mathcal{M}}$ by*

$$d_{\mathcal{M}}\big((f_i, \mathcal{E}_i), (g_i, \mathcal{E}_i)\big) = \sup_{i\in\mathbb{N}} d(f_i, g_i)$$

for all $(f_i, \mathcal{E}_i), (g_i, \mathcal{E}_i) \in \mathcal{M}_d(S, \mathcal{E}_i)$. Define addition and positive scalar multiplication on $\mathcal{M}_d(S, \mathcal{E}_i)$, by

$$(f_i, \mathcal{E}_i) + (g_i, \mathcal{E}_i) = (f_i + g_i, \mathcal{E}_i), \quad \lambda(f_i, \mathcal{E}_i) = (\lambda f_i, \mathcal{E}_i)$$

for all $(f_i, \mathcal{E}_i), (g_i, \mathcal{E}_i) \in \mathcal{M}_d(S, \mathcal{E}_i)$ and $\lambda \in \mathbb{R}_+$.

It is readily verified that $\big(\mathcal{M}_d(S, \mathcal{E}_i), d_{\mathcal{M}}\big)$ is a metric space and $\mathcal{M}_d(S, \mathcal{E}_i)$ is a near vector space. We use Rådström's embedding result on the complete MS-filtration spaces $\big(S, d, \mathcal{E}_i\big)$ and $\mathcal{M}_d(S, \mathcal{E}_i)$. The first problem to deal with is the fact that $R_n(S)$ need not be norm complete. So, instead of $R_n(S)$ we consider its norm completion $\tilde{R}_n(S)$.

Definition 4.9. *Let X be a Banach space and $(T_i)_{i\in\mathbb{N}}$ be a BS-filtration on X. Denote by $\mathcal{M}_{nc}(X, T_i)$ the set of martingales (f_i, T_i) in X for which (f_i) is norm convergent. The addition and scalar multiplication are defined by*

$$(f_i, T_i) + (g_i, T_i) = (f_i + g_i, T_i), \qquad \lambda(f_i, T_i) = (\lambda f_i, T_i)$$

for each $(f_i,T_i),(g_i,T_i)\in\mathcal{M}_{nc}(X,T_i)$ *and* $\lambda\in\mathbb{R}$. *Moreover, the norm in the vector space* $\mathcal{M}_{nc}(X,T_i)$ *is defined by*

$$\|(f_i,T_i)\|_{\mathcal{M}}=\sup_{i\in\mathbb{N}}\|f_i\|.$$

Note that $\|.\|_{\mathcal{M}}$ is a norm on $\mathcal{M}_{nc}(X,T_i)$ which makes $\mathcal{M}_{nc}(X,T_i)$ into a Banach space (see [3]). The following result shows how $\mathcal{M}_d(S,\mathcal{E}_i)$ and $\mathcal{M}_{nc}(X,T_i)$ are related via the Rådström completion of S.

Theorem 4.10. *Let* $\big(S,d,\mathcal{E}_i\big)$ *be a complete MS-filtration space and* $(f_i,e_i)\in M_d(S,E_i)$. *Then the map* $K:\mathcal{M}_d(S,\mathcal{E}_i)\longrightarrow\mathcal{M}_{nc}\big(\tilde{R}_n(S),\tilde{\mathcal{E}}_{in}\big)$ *defined by*

$$K\big((f_i,\mathcal{E}_i)\big)=\big(j_n(f_i),\tilde{\mathcal{E}}_{in}\big)$$

is an $\mathbb{R}_+$*-linear isometry (into) and* $\mathcal{M}_d(S,\mathcal{E}_i)$ *is complete.*

Proof. It is clear that K is injective and $\mathbb{R}_+$-linear. We verify that K is an isometry. Let $(f_i,\mathcal{E}_i),(g_i,\mathcal{E}_i)\in\mathcal{M}_d(S,\mathcal{E}_i)$. Then

$$\begin{aligned}
d_{\mathcal{M}}\big((f_i,\mathcal{E}_i),(g_i,\mathcal{E}_i)\big)&=\sup_{i\in N}d(f_i,g_i)\\
&=\sup_{i\in N}d\big(f_i+(n-2)g_i+n(n-1)z,\ (n-1)g_i+\big((n-1)n\big)z\big)\\
&=\sup_{i\in N}\|[f_i+(n-2)g_i+n(n-1)z,g_i+nz,...,g_i+nz]\|_{dn}\\
&=\sup_{i\in N}\|[f_i+(n-1)z,z,...,z]+[(n-2)g_i\\
&\quad+(n-1)(n-1)z,g_i+(n-1)z,...,g_i+(n-1)z]\|_{dn}\\
&=\sup_{i\in N}\|[f_i+(n-1)z,z,...,z]-[g_i+(n-1)z,z,...,z]\|_{dn}\\
&=\|\big(([f_i+(n-1)z,z,...,z]-[g_i+(n-1)z,z,...,z]),\tilde{\mathcal{E}}_{in}\big)\|_{\mathcal{M}}\\
&=\|\big(j_n(f_i),\tilde{\mathcal{E}}_{in}\big)-\big(j_n(g_i),\tilde{\mathcal{E}}_{in}\big)\|_{\mathcal{M}}.
\end{aligned}$$

Since $\mathcal{M}_{nc}(\tilde{R}_n(S),\tilde{\mathcal{E}}_{in})$ is complete and $j_n(S)$ is closed in $\tilde{R}_n(S)$, it follows that $\mathcal{M}_d(S,\mathcal{E}_i)$ is complete. □

References

1. S. Bochner, Partial ordering in the theory of martingales, Ann. of Math. **62**(1955), 162-169.
2. W.W.L Chen, Linear functional analysis, Macquarie University, Balaclava, Australia, (2008).
3. S.F. Cullender, C.C.A. Labuschagne, A description of norm-convergent martingales on vector valued L^p-spaces, J. Math. Anal. Appl. **23**(2006), 119–130.
4. C.C.A. Labuschagne, A Banach lattice approach to convergent integrably bounded set-valued martingales and their positive parts, J. Math. Anal. Appl. **342**(2008), 780–797.
5. C.C.A. Labuschagne, A.L. Pinchuck, C.J. van Alten, A vector lattice version of Rådström's embedding theorem, Quaest. Math. **30**(2007), 285–308.
6. H. Rådström, An embedding theorem for spaces of convex sets, Proc. Amer. Math. Soc. **3**(1952), 165–169.
7. A.C. Zaanen, Introduction to Operator Theory in Riesz Space, Springer-Verlag, Berlin, (1997).

A. Abkar
Department of Mathemathics, Imam Khomeini International University, Qazvin 34149, Iran
E-mail address: abkar@sci.ikiu.ac.ir

M. Rastgoo
Department of Mathemathics, Imam Khomeini International University, Qazvin 34149, Iran
E-mail address: m.rastgoo89@gmail.com

M.B. Kazemi
Department of Mathemathics, University of Zanjan, P. O. Box 45195–313, Zanjan, Iran
E-mail address: mbkazemi@znu.ac.ir

A. Azizi
Department of Mathemathics, Imam Khomeini International University, Qazvin 34149, Iran
E-mail address: azizi@sci.ikiu.ac.ir