Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 4 (2017), Pages 90-96.

A NOTE ON CONVEX CONTRACTION MAPPINGS AND DISCONTINUITY AT FIXED POINT

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Abstract. In this paper we prove some fixed point theorems by dropping the continuity requirement of convex contraction mappings introduced by Vasile I. Istrătescu [Fixed point theory: An introduction, Mathematics and Its Applications, Vol. 7, D. Reidel Publishing Company, Dordrecht, Holland, 1981]. Further, we show that convex contraction mappings are strong enough to generate a fixed point but do not force the mapping to be continuous at the fixed point. Our work generalizes and unifies some well-known fixed point theorems due to Banach-Picard-Caccioppoli, Kannan and Reich.

1. Introduction

The well-known Banach-Picard-Caccioppoli contraction principle states that:

Theorem 1.1. *Let T be a self-mapping of a complete metric space (X, d) satisfying the condition; $d(Tx, Ty) \leq ad(x, y), 0 \leq a < 1$, for each $x, y \in X$. Then T has a unique fixed point. The Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n, (n = 0, 1, 2, ...)$ converges to a point $x_* \in X$ for any initial value $x_0 \in X$.*

The mapping T of the Banach-Picard-Caccioppoli contraction is continuous in the entire domain of X.

In 1968, Kannan [16] proved the following theorem:

Theorem 1.2. *Let T be a self-mapping of a complete metric space (X, d) satisfying the condition; $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], 0 \leq b < 1/2$, for each $x, y \in X$. Then T has a unique fixed point.*

It is important to note that the Kannan contraction is independent of the Banach-Picard-Caccioppoli contraction and does not force the mapping to be continuous in the entire domain X.

Another noteworthy generalizations of both the Banach-Picard-Caccioppoli contraction principle and the Kannan fixed point theorem was obtained by Reich [21]. He proved the following theorem:

2000 *Mathematics Subject Classification.* 47H09; 47H10.
Key words and phrases. Fixed point; convex contraction mappings; orbital continuity.

Submitted April 18, 2017. Published August 13, 2017.
Communicated by Wasfi Shatanawi.

Theorem 1.3. *Let T be a self-mapping of a complete metric space (X, d) satisfying the condition; $d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)], 0 \leq a + 2b < 1$, for each $x, y \in X$. Then T has a unique fixed point.*

Similar to the Kannan contraction, the Reich contraction does not require the mapping to be continuous in the entire domain X for the existence of the fixed point. However, a mapping T satisfying the Reich contraction turns out to be continuous at the fixed point. To see this, suppose that $z = Tz$ is a fixed point of T and $x_n \to z$. Then

$$\begin{aligned} d(Tx_n, z) = d(Tx_n, Tz) &\leq ad(x_n, z) + b[d(x_n, Tx_n) + d(z, Tz)] \\ &\leq ad(x_n, z) + b[d(x_n, z) + d(z, Tx_n)], \end{aligned} \tag{1.1}$$

that is, $(1 - b)d(Tx_n, z) \leq (a + b)d(x_n, z)$. This implies that $Tx_n \to z = Tz$ and T is continuous at the fixed point z.

Kannan's and Reich's papers [16, 21] generated a widespread interest in the study of fixed points of generalized contractive mappings and soon these were followed by a flood of papers involving contractive definitions many of which do not require continuity of the mapping in the entire domain ([1, 3, 4, 8, 10, 11, 12]). However, in most of the cases the mapping is continuous at the fixed point. Few answers to the open question [22] whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the mapping to be continuous at the fixed point have been given in [5, 6, 7, 20].

In 1981, Istrătescu [14] extended the well-known Banach-Picard-Caccioppoli contraction principle by introducing a convexity condition, namely convex contraction mapping of type 2 and proved the following theorem:

Theorem 1.4. *Let T be a continuous self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq c_0d(x, y) + c_1d(Tx, Ty) + a_1d(x, Tx) + a_2d(Tx, T^2y) + b_1d(y, Ty) + b_2d(Ty, T^2y),$$

where $0 \leq c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$, for distinct $x, y \in X$. Then T has a unique fixed point.

In this paper, we show that the assumption of continuity condition of Theorem 1.4 can be replaced by a relatively weaker condition of orbital continuity. Recall that the set $O(x; T) = \{T^nx : n = 0, 1, 2, ...\}$ is called the orbit of the self-mapping T at the point $x \in X$.

Definition 1.5. *A self-mapping T of a metric space (X, d) is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ (for some $x \in X$) $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.*

It is easy to check that every continuous self-mapping of a metric space is orbitally continuous, but the converse need not be true (see Example 2.4 below).

2. Main results

Our first main result is the following. The constructive proof of the theorem up to Cauchy sequence is similar to that in ([13], see also [2]). We include it for the sake of completeness.

Theorem 2.1. *Let T be a self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq c_0 d(x, y) + c_1 d(Tx, Ty) + a_1 d(x, Tx) + a_2 d(Tx, T^2y) + b_1 d(y, Ty) + b_2 d(Ty, T^2y), \tag{2.1}$$

where $0 \leq c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1$, for distinct $x, y \in X$.

Suppose T is orbitally continuous. Then T has a unique fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Set

$$M = \max\{d(x_0, Tx_0), d(Tx_0, T^2x_0)\}. \tag{2.2}$$

Then we have

$$d(T^2x_0, T^3x_0) \leq (\frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2})M. \tag{2.3}$$

Similarly,

$$d(T^3x_0, T^4x_0) \leq (\frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2})M, \tag{2.4}$$

as well as

$$d(T^4x_0, T^5x_0) \leq (\frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2})^2 M. \tag{2.5}$$

An induction argument shows that

$$d(T^m x_0, T^{m+1}x_0) \leq (\frac{c_0 + c_1 + a_1 + a_2 + b_1}{1 - b_2})^{m-2} M, \tag{2.6}$$

holds for all $m \geq 4$.

From these estimates we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Orbital continuity of T implies that $lim_{n\to\infty} Tx_n = Tz$. This yields $Tz = z$, that is, z is a fixed point of T. Uniqueness of the fixed point follows easily. □

In the next theorem, we completely drop the condition of continuity or orbital continuity of the mapping T assumed in Theorem 2.1.

Theorem 2.2. *Let T be a self-mapping of a complete metric space (X, d) satisfying equation (2.1) of Theorem 2.1 and*

$$d(Tx, Ty) \leq \phi(max\{d(x, y), d(x, Tx), d(y, Ty)\}, \tag{2.7}$$

where $\phi : \mathbb{R}_+(: [0, \infty)) \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$.

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$.

Proof. In view of (2.1) we can easily establish (see Theorem 2.1 above) that the sequence $\{x_n\}$ in X is Cauchy. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. We claim that $Tz = z$. For if $Tz \neq z$, we get

$$d(Tz, Tx_n) \leq \phi(max\{d(z, x_n), d(z, Tz), d(x_n, Tx_n)\}). \tag{2.8}$$

On letting $n \to \infty$ this yields, $d(Tz, z) \leq \phi(d(Tz, z)) < d(Tz, z)$, a contradiction. Thus z is a fixed point of T. Uniqueness of the fixed point follows easily. This concludes the theorem. $\square$

Corollary 2.3. *Let T be a self-mapping of a complete metric space (X, d) satisfying equation (2.1) of Theorem 2.1 and*

$$d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

whenever the right hand side is positive. Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$.

We now give an example [20] to show that convex contraction mapping of order 2 is strong enough to generate a fixed point but does not force the mapping to be continuous at the fixed point.

Example 2.4. *Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by*

$$T(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{if } 1 < x \leq 2. \end{cases}$$

Then T satisfies all the conditions of Theorems 2.1 and 2.2 and has a unique fixed point $x = 1$ at which T is discontinuous. The mapping T satisfies equation (2.7) of Theorem 2.2 with

$$\phi(t) = \begin{cases} 1, & \text{if } t > 1; \\ 0, & \text{if } t \leq 1. \end{cases}$$

Putting $c_0 = a_1 = a_2 = b_1 = b_2 = 0$ in Theorem 2.1, we get the following corollary which is an extension of the Banach-Picard-Caccioppoli contraction principle.

Corollary 2.5. *Let T be an orbitally continuous self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq c_1 d(Tx, Ty), \tag{2.9}$$

where $0 \leq c_1 < 1$, for distinct $x, y \in X$. Then T has a unique fixed point.

Setting $c_1 = a_1 = a_2 = b_1 = b_2 = 0$ in Theorem 2.1 we get a power contraction version of the Banach-Picard-Caccioppoli contraction principle.

Corollary 2.6. *Let T be an orbitally continuous self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq c_0 d(x, y), \tag{2.10}$$

where $0 \leq c_0 < 1$, for distinct $x, y \in X$. Then T has a unique fixed point.

Putting $c_0 = c_1 = a_1 = b_1 = 0$ and $a_2 = b_2 = b$ in Theorem 2.1 we get a power contraction version of the Kannan fixed point theorem [16].

Corollary 2.7. *Let T be an orbitally continuous self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq b[d(Tx, T^2x) + d(Ty, T^2y)], \tag{2.11}$$

where $0 \leq b < 1/2$, for distinct $x, y \in X$. Then T has a unique fixed point.

Setting $c_0 = a_1 = b_1 = 0, c_1 = a$, and $a_2 = b_2 = b$ in Theorem 2.1 we get a power contraction version of the Reich fixed point theorem [21].

Corollary 2.8. *Let T be an orbitally continuous self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq ad(Tx, Ty) + b[d(Tx, T^2x) + d(Ty, T^2y)], \tag{2.12}$$

where $0 \leq a + 2b < 1$, for distinct $x, y \in X$. Then T has a unique fixed point.

In view of the result of Bryant (Put $k = 2$ in [7]), Corollary 2.6 also holds if we drop the condition of orbital continuity of the mapping.

Theorem 2.9. *Let T be a self-mapping of a complete metric space (X, d) satisfying the condition;*

$$d(T^2x, T^2y) \leq c_0 d(x, y), \tag{2.13}$$

where $0 \leq c_0 < 1$, for distinct $x, y \in X$. Then T has a unique fixed point.

Proof. In view of the Banach-Picard-Caccioppoli contraction principle and equation (2.13) we see that T^2 has a unique fixed point $z \in X$, i.e., $T^2(z) = z$. Since the fixed point of T^2 is unique, hence $Tz = z$. □

Remark. *Corollaries 2.5, 2.6, 2.7 and 2.8 show that power contraction need not be continuous at the fixed point.*

The concept of $\alpha - (\psi_k)_1^n$- admissible mapping was introduced by Hussain and Salimi [9].

Definition 2.10. *Let $T : X \to X$ and $\alpha, \psi_1, \psi_2, ..., \psi_n : X^2 \to [0, \infty)$. T is said to be $\alpha - (\psi_k)_1^n$- admissible mapping if for each $1 \leq k \leq n$ and $x, y \in X$;*

$\alpha(x, y) \geq 1, \psi_{k(x,y)} \leq r_k$ *implies* $\alpha(Tx, Ty) \geq 1, \psi_k(Tx, Ty) \leq r_k$,
where $(r_k)_1^n \subset \mathbb{R}_+$.

We now prove two fixed point theorems for $\alpha - (\psi_k)_1^n$- admissible mapping without assuming continuity condition of the given mapping. The results of Hussain and Salimi [9] are true if we replace continuity of the mapping by a weaker notion of orbital continuity. Hence our results improve the results of Hussain and Salimi.

Theorem 2.11. *Let T be a self-mapping of a complete metric space (X, d) and T be an $\alpha - (\psi_k)_1^4$- admissible mapping such that for each $x, y \in X$;*

$$\alpha(x, y)d(T^2x, T^2y) \leq \psi_1(x, y)d(x, Tx) + \psi_2(x, y)d(Tx, T^2x) + \psi_3(x, y)d(y, Ty) + \psi_4(x, y)d(Ty, T^2y),$$

where $0 \leq r_1 + r_2 + r_3 + r_4 < 1$. Suppose T is orbitally continuous and if there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \psi_k(x_0, Tx_0) \leq r_k$, then T has a fixed point. Moreover, if $\alpha(x, y) \geq 1$ and $\psi_k(x, y) \leq r_k$ for each $x, y \in X$, then T has a unique fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Following the same steps as in [9] we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Orbital continuity of T implies that $lim_{n\to\infty} Tx_n = Tz$.

This yields $Tz = z$, that is, z is a fixed point of T. Uniqueness of the fixed point follows easily.

□

Theorem 2.12. *Let T be a self-mapping of a complete metric space (X, d) and T be an $\alpha - (\psi_k)_1^2$- admissible mapping such that for each $x, y \in X$;*

$$\alpha(x, y)d(T^2x, T^2y) \leq \psi_1(x, y)d(Tx, Ty) + \psi_2(x, y)d(x, y),$$

where $0 \leq r_1 + r_2 < 1$. Suppose T is orbitally continuous and if there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \psi_k(x_0, Tx_0) \leq r_k$, then T has a fixed point. Moreover, if $\alpha(x, y) \geq 1$ and$\psi_k(x, y) \leq r_k$ for each $x, y \in X$, then T has a unique fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Following the same steps as in [9] we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Orbital continuity of T implies that $lim_{n\to\infty}Tx_n = Tz$. This yields $Tz = z$, that is, z is a fixed point of T. Uniqueness of the fixed point follows easily.

□

Remark. *Results given in* [2, 17, 18, 19] *are also valid if we drop continuity requirementt by orbital continuity of the mapping.*

Acknowledgments. The authors are thankful to the learned referees for suggesting some improvements in the presentation of the paper.

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