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SOME COMMON FIXED POINT THEOREMS WITHOUT ORBITAL CONTINUITY VIA C-CLASS FUNCTIONS AND AN APPLICATION

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Abstract. By using C-class functions, some existence and uniqueness results for common fixed points of three self-mappings in orbitally complete metric spaces are established. To demonstrate the usability of the hypotheses of our results, an application is provided to show the existence of solutions for certain system of integral equations.

1. Introduction

In 1976, Jungck [9] initiated a study of common fixed points of commuting mappings. On the other hand in 1982, Sessa [17] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Then Jungck generalized this idea, first to compatible mappings [10] and then to weakly compatible mappings. For common fixed points results, see [2, 4, 5, 6, 7, 8, 14, 18, 19, 20, 22].

In the following, we give some useful notations and concepts needed to prove our results.

Definition 1.1. [13] *Let (X, d) be a metric space. Let f, S self-mappings of X. A point x in X is called a coincidence point of f and g iff $fx = Sx$. We shall call $w = fx = Sx$ a point of coincidence of f and g.*

Let $C(f, S)$ and $PC(f, S)$ denote the set of coincidence points and points of coincidence of the pair (f, S), respectively.

Definition 1.2. [10] *Let (X, d) be a metric space. Two self-mappings f and S of X are said to be compatible if and only if $\lim_{n\to\infty} d(fSx_n, Sfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.*

Definition 1.3. [13] *Let (X, d) be a metric space. Two maps f and S of X are said to be weakly compatible if they commute at their coincidence points.*

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Definition 1.4. [13] *Let (X,d) be a metric space. Two self-mappings f and g of X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.*

Let (X,d) be a metric space. Given $x_0 \in X$ and self-mappings f, S and T on X, if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that

$$Sx_{2n} = fx_{2n+1} \quad \text{and} \quad Tx_{2n+1} = fx_{2n+2}, \tag{1.1}$$

then $O(S,T,f,x_0) = \{fx_n : n = 0, 1, 2, \cdots\}$ is called a (S,T)-orbit at x_0 with respect to f.

Definition 1.5. [15] *The space X is called orbitally complete at x_0 if and only if every Cauchy sequence in $O(S,T,f,x_0)$ converges in X.*

Definition 1.6. [15] *The pair (S,T) is asymptotically regular at x_0 with respect to f if there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $Sx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = fx_{2n+2}$ where $d(fx_n, fx_{n+1}) \to 0$ as $n \to \infty$.*

Definition 1.7. *A self-mapping f on X is orbitally continuous at x_0 if and only if it is continuous on $O(S,T,f,x_0)$. Obviously, every continuous self-mapping on X is orbitally continuous at each $x_0 \in X$. However the converse is not true (see* [15]*).*

Definition 1.8. [1] *The self-mappings f and S are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.*

Definition 1.9. [3] *A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies following axioms:*
(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

We denote the set of C-class functions as $\mathcal{C}$.

Example 1.10. [3] *The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:*
(1) $F(s,t) = s - t$;
(2) $F(s,t) = ms$ where $0<m<1$;
(3) $F(s,t) = s\beta(s)$ where $\beta : [0,\infty) \to [0,1)$ is is continuous;
(4) $F(s,t) = s - \varphi(s)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
(5) $F(s,t) = \phi(s)$ where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$.

Definition 1.11. [11] *A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the set of altering distance functions.

Definition 1.12. *A tripled (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ is said to be monotone if for any $x, y \in [0,\infty)$*

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

Example 1.13. *Let* $F(s,t) = s - t, \varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2 & \text{if } x > 1 \end{cases},$$

then (ψ, φ, F) *is monotone.*

In this paper, we prove some existence and uniqueness results for common fixed points of three self-mappings in orbitally complete metric spaces via C-class functions and without using the orbital continuity hypothesis. These results improve some known ones in literature, as [12] and [16], where the orbital continuity was used to prove the existence of common fixed points. We furnish a suitable example to demonstrate the usability of the hypotheses of our results. Finally, we apply these results to prove the existence of solutions for a system of integral equations.

2. Main results

In this section, we will prove some common fixed point results without the orbital continuity.

Theorem 2.1. *Let* f, S *and* T *be self-mappings on a metric space* (X, d) *satisfying*

$$\psi(d(Sx, Ty) \le F\Big(\psi(M(x,y)), \varphi(w(M(x,y)))\Big), \tag{2.1}$$

for all $x, y \in X$*, where* $\psi, \varphi \in \Psi$ *and* $F \in \mathcal{C}$*, with*

$$M(x,y) = \max\{d(fx, fy), d(fx, Sx), d(fy, Ty), d(fx, Ty), d(fy, Sx)\}, \tag{2.2}$$

and $w : [0,\infty) \to [0,\infty)$ *is a continuous map such that* $w(t) < t$ *for* $t > 0$*. Suppose that*

(a) (f, S) *or* (f, T) *satisfies the property (E.A);*
(b) $f(X)$ *is an orbitally complete subspace of* X*;*
(c) (f, S) *or* (f, T) *is weakly compatible.*

Then f, S *and* T *have a unique common fixed point.*

Proof. First, assume that the pair (f, S) satisfies the (E.A) property. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z \text{ for some } z \in X. \tag{2.3}$$

Put $\lim_{n\to\infty} Tx_n = p$. We shall prove that $p = z$. By using (2.1) for $x = x_n$ and $y = x_n$, we have

$$\begin{aligned} \psi(d(Sx_n, Tx_n) \le F\Big(&\psi(\max\{d(fx_n, fx_n), d(fx_n, Sx_n), \\ &d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_n, Sx_n)\}), \\ &\varphi(w(\max\{d(fx_n, fx_n), d(fx_n, Sx_n), \\ &d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_n, Sx_n)\}))\Big). \end{aligned}$$

Applying the limit as $n \to \infty$ and using (2.3),

$$\begin{aligned} \psi(d(z,p) &\le F\Big(\psi(\max\{0, 0, d(z,p), d(z,p), 0\}), \varphi(w(\max\{0, 0, d(z,p), d(z,p), 0)\})\Big) \\ &= F\Big(\psi(d(z,p)), \varphi(w(d(z,p))\Big). \end{aligned}$$

Suppose that $z \neq p$, so $w(d(z,p) < d(z,p)$. By Definition 1.12, we get

$$\psi(d(z,p)) \leq F\Big(\psi(d(z,p)), \varphi(w(d(z,p))\Big) \leq F\Big(\psi(d(z,p)), \varphi(d(z,p)\Big) \leq \psi(d(z,p)),$$

which implies that $\psi(d(z,p)) = 0$ or $\varphi(d(z,p)) = 0$. Since ψ and φ are in Ψ, we have $z = p$, which is a contradiction. Thus

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z. \tag{2.4}$$

Note that (2.4) can also be obtained in similar lines whenever (f,T) satisfies property (E.A). From the orbital completeness of $f(X)$, we see that $z \in f(X)$, so that $z = fu$ for some $u \in X$. Now, taking $x = u$ and $y = x_n$ in (2.1),

$$\begin{aligned}\psi(d(Su, Tx_n) \leq F\Big(&\psi(\max\{d(fu, fx_n), d(fu, Su),\\ &d(fx_n, Tx_n), d(fu, Tx_n), d(fx_n, Su)\}),\\ &\varphi(w(\max\{d(fu, fx_n), d(fu, Su),\\ &d(fx_n, Tx_n), d(fu, Tx_n), d(fx_n, Su)\}))\Big).\end{aligned}$$

Applying again the limit as $n \to \infty$, then by (2.4) and the fact that $fu = z$,

$$\begin{aligned}\psi(d(Su, fu) &\leq F\Big(\psi(\max\{0, d(fu, Su), 0, 0, d(fu, Su)\}),\\ &\qquad \varphi(w(\max\{0, d(fu, Su), 0, 0, d(fu, Su)\})\Big)\\ &= F\Big(\psi(d(fu, Su)), \varphi(w(d(fu, Su))\Big).\end{aligned}$$

Assume that $fu \neq Su$. Then $w(d(fu, Su)) < d(fu, Su)$. We have

$$\psi(d(Su, fu) \leq F\Big(\psi(d(fu, Su)), \varphi(w(d(fu, Su))\Big) \leq F\Big(\psi(d(fu, Su)), \varphi(d(fu, Su)\Big) \leq \psi(d(fu, Su)).$$

Similarly, $\psi(d(fu, Su)) = 0$ or $\varphi(d(fu, Su)) = 0$. We deduce $fu = Su$, which is a contradiction. So $fu = Su = z$. By the weak compatibility of (f,S), we see that $fSu = Sfu$, i.e., $fz = Sz$. Taking again $x = y = z$ in the inequality (2.1) and using $fz = Sz$, it follows that

$$\begin{aligned}\psi(d(Sz, Tz) &\leq F\Big(\psi(d(Sz, Tz)), \varphi(w(d(Sz, Tz))\Big)\\ &\leq \psi(d(Sz, Tz)).\end{aligned}$$

Proceeding similarly, we obtain

$$fz = Sz = Tz. \tag{2.5}$$

Again, for $x = x_n$ and $y = z$ in (2.1), we have

$$\begin{aligned}\psi(d(Sx_n, Tz) \leq F\Big(&\psi(\max\{d(fx_n, fz), d(fx_n, Sx_n),\\ &d(fz, Tz), d(fx_n, Tz), d(fz, Sx_n)\}),\\ &\varphi(w(\max\{d(fx_n, fz), d(fx_n, Sx_n),\\ &d(fz, Tz), d(fx_n, Tz), d(fz, Sx_n)\}))\Big).\end{aligned}$$

Letting $n \to \infty$ and using (2.4) and (2.5),

$$\begin{aligned}\psi(d(z,Tz) &\leq F\Big(\psi(d(z,Tz)), \varphi(w(d(z,Tz))\Big)\\ &\leq \psi(d(z,Tz)).\end{aligned}$$

Similarly, $z = Tz$. Thus z is a common fixed point of self-mappings f, S and T.

On the other hand, with minor changes in the above proof, we can prove that $fu = Tu = z$. Suppose that the pair (f,T) is weakly compatible. Then $fTu = Tfu$ or $fz = Tz$. Proceeding as in before, we get that $fz = Tz = Sz = z$.
Uniqueness: Let z and z' be two common fixed points of f, S and T. Taking $x = y = z$ (2.1), we get

$$\begin{aligned}\psi(d(z,z') = \psi(d(Sz,Tz') \leq F\Big(&\psi(\max\{d(fz,fz'), d(fz,Sz),\\ &d(fz',Tz'), d(fz,Tz'), d(fz',Sz)\}),\\ &\varphi(w(\max\{d(fz,fz'), d(fz,Sz),\\ &d(fz',Tz'), d(fz,Tz'), d(fz',Sz)\}))\Big).\end{aligned}$$

We deduce

$$\begin{aligned}\psi(d(z,z') &\leq F\Big(\psi(d(z,z')), \varphi(w(d(z,z'))\Big)\\ &\leq \psi(d(z,z')).\end{aligned}$$

The same strategy implies that $z = z'$. Hence the common fixed point is unique. $\square$

Proceeding similarly as the proof of Theorem 2.1, we have the following corollary.

Corollary 2.2. *If in Theorem 2.1, we replace the condition (2.1) by*

$$\psi(d(Sx,Ty)) \leq F\Big(\psi(M(x,y) - w(M(x,y))), \varphi(M(x,y) - w(M(x,y)))\Big), \tag{2.6}$$

then f, S and T have a unique common fixed point.

Remark 1. *Taking $F(s,t) = s - t$ and $\psi(t) = \varphi(t) = t$, in Theorem 2.1, we obtain Theorem B of Swatmaram et al. in* [21].

Theorem 2.3. *Let f, S and T be self-mappings on a metric space (X,d) satisfying (2.1). Suppose that*

(a) *the pair (S,T) is asymptotically regular with respect to f at some $x_0 \in X$;*
(b) *the space X is orbitally complete at such x_0;*
(c) *(f,S) or (f,T) is a commuting pair.*

Then f, S and T have a unique common fixed point.

Proof. Since (S,T) is asymptotically regular with respect to f at x_0, there exists a sequence $\{x_n\}$ in X defined by

$$Sx_{2n} = fx_{2n+1} \quad \text{and} \quad Tx_{2n+1} = fx_{2n+2} \quad \text{for } n = 0, 1, 2, \cdots,$$

such that

$$d_n = d(fx_n, fx_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.7}$$

In order to show that $\{fx_n\}$ is a Cauchy sequence, by (2.7), it is sufficient to show that $\{fx_{2n}\}$ is a Cauchy sequence. Suppose that the result is not true. Then there exists $\varepsilon > 0$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > 2k$ and

$$d(fx_{2m(k)}, fx_{2n(k)}) > \varepsilon. \tag{2.8}$$

For each integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (2.8). So

$$d(fx_{2m(k)-2}, fx_{2n(k)}) \leq \varepsilon. \tag{2.9}$$

Then

$$d(fx_{2m(k)}, fx_{2n(k)}) \leq d(fx_{2m(k)-2}, fx_{2n(k)}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

Referring to (2.7)-(2.9), we get

$$d(fx_{2m(k)}, fx_{2n(k)}) \to \varepsilon \ \text{ as } \ k \to \infty. \tag{2.10}$$

By triangular inequality, we obtain

$$\begin{aligned}
&|d(fx_{2m(k)+1}, fx_{2n(k)}) - d(fx_{2m(k)}, fx_{2n(k)})| \leq d_{2m(k)}, \\
&|d(fx_{2m(k)+1}, fx_{2n(k)+1}) - d(fx_{2m(k)+1}, fx_{2n(k)})| \leq d_{2n(k)}, \\
&|d(fx_{2m(k)+2}, fx_{2n(k)+1}) - d(fx_{2m(k)+1}, fx_{2n(k)+1})| \leq d_{2m(k)+1},
\end{aligned}$$

and

$$|d(fx_{2m(k)+2}, fx_{2n(k)}) - d(fx_{2m(k)+1}, fx_{2n(k)})| \leq d_{2m(k)+1}.$$

From (2.7), (2.10) and the above inequalities, we have

$$\begin{aligned}
\varepsilon &= \lim_{k\to\infty} d(fx_{2m(k)+1}, fx_{2n(k)}) = \lim_{k\to\infty} d(fx_{2m(k)+1}, fx_{2n(k)+1}) \\
&= \lim_{k\to\infty} d(fx_{2m(k)+2}, fx_{2n(k)+1}) = \lim_{k\to\infty} d(fx_{2m(k)+2}, fx_{2n(k)}).
\end{aligned}$$

By (2.1), it follows that

$$\begin{aligned}
&\psi(d(fx_{2n(k)+1}, fx_{2m(k)+2})) = \psi(d(Sx_{2n(k)}, Tx_{2m(k)+1})) \\
&\leq F\Big(\psi(\max\{d(fx_{2n(k)}, fx_{2m(k)+1}), d_{2n(k)}, d_{2m(k)+1}, \\
&d(fx_{2n(k)}, fx_{2m(k)+2}), d(fx_{2m(k)+1}, fx_{2n(k)+1})\}), \\
&\varphi(w(\max\{d(fx_{2n(k)}, fx_{2m(k)+1}), d_{2n(k)}, d_{2m(k)+1}, \\
&d(fx_{2n(k)}, fx_{2m(k)+2}), d(fx_{2m(k)+1}, fx_{2n(k)+1})\}))\Big).
\end{aligned}$$

As $k \to \infty$, we get

$$\begin{aligned}
\psi(\varepsilon) &\leq F\Big(\psi(\max\{\varepsilon, 0, 0, \varepsilon, \varepsilon\}), \varphi(w(\max\{\varepsilon, 0, 0, \varepsilon, \varepsilon\}))\Big) \\
&= F\Big(\psi(\varepsilon), \varphi(w(\varepsilon))\Big) \\
&\leq F\Big(\psi(\varepsilon), \varphi(\varepsilon)\Big) \\
&\leq \psi(\varepsilon).
\end{aligned}$$

Thus $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. So $\varepsilon = 0$, which is a contradiction. Hence $\{fx_n\}$ is a Cauchy sequence. Thus by the orbital completeness of X at x_0, we can find some $z \in X$ such that $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} fx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = z$, which immediately implies that the pairs (f, T) and (S, T) satisfy the property (E.A).

Having in mind that every commuting pair is weakly compatible, so the uniqueness of the common fixed point follows easily from Theorem 2.1. □

Corollary 2.4. *If in Theorem 2.3, we replace the condition (2.1) by (2.6), then f, S and T have a unique common fixed point.*

Remark 2. *(a) Taking $F(s,t) = s - t$ and $\psi(t) = \varphi(t) = t$, in Theorem 2.3, we conclude that Theorem 2.1 of Z. Liu et al. in* [12] *could be proved without the need of the orbital continuity condition.*
(b) Taking $F(s,t) = s - t$, $\psi(t) = \varphi(t) = t$, and $w(t) = (1 - r)r$, $r \in (0,1)$, in Theorem 2.3, we conclude that Theorem 1 of Sastry et al. in [16] *could be proved without the need of the orbital continuity condition.*

We provide the following example.

Example 2.5. *Let $X = [0,1) \cup \{2\}$ with the usual metric $|.|$. Define $f, S, T : X \to X$ by $Sx = Tx = \frac{1}{3}x$ and $fx = x$ for $x \in X$. Consider $F(s,t) = s - t, \psi(t) = 2t$ and $\varphi(t) = t$. Take $x_0 = 2$ and $w(t) = \frac{1}{2}t$ for $t \geq 0$. Then $O(x_0, S, T, f) = \{\frac{2}{3^n} : n = 0, 1, 2, \cdots\}$, $f(X) = X$ is orbitally complete at x_0. Also, (f, S) (also (f, T)) satisfies the property (E.A). Moreover, (f, S) (also (f, T)) is weakly compatible. For all $x, y \in X$, we have*

$$\begin{aligned}\psi(d(Sx, Ty)) = \frac{2}{3}|x - y| &\leq \frac{3}{2}\max\{|x - y|, \frac{2}{3}x, \frac{2}{3}y, |x - \frac{1}{3}y|, |y - \frac{1}{3}x|\} \\ &= F\Big(\psi(M(x,y)), \varphi(w(M(x,y)))\Big).\end{aligned}$$

All the conditions of Theorem 2.1 are satisfied and $x = 0$ is the unique common fixed point of f, S and T.

3. Application to systems of integral equations

Consider the following system of integral equations:

$$\begin{cases} u(a) = \int_0^A k_1(a, b, u(b))db + q(a), \\ u(a) = \int_0^A k_2(a, b, u(b))db + q(a), \end{cases} \tag{3.1}$$

where $a \in J = [0, A]$, with $A > 0$. The purpose of this section is to give an existence theorem for a solution of the system (3.1) using Theorem 2.1.
Let $\mathcal{X} := C(J, R^n)$ with the usual supremum norm, i.e., $\|x\|_{\mathcal{X}} = \max_{a \in J} \|x(a)\|$, for $x \in C(J, R^n)$. Also $(\mathcal{X}, \|.\|_{\mathcal{X}})$ is a complete metric space.
Define $f, S, T : \mathcal{X} \to \mathcal{X}$ by

$$fx(a) = x(a), \quad Sx(a) = \int_0^A k_1(a, b, x(b))db + q(a), \quad a \in [0, A],$$

and

$$Tx(a) = \int_0^A k_2(a, b, x(b))db + q(a), \quad a \in [0, A].$$

Theorem 3.1. *Consider the integral equations (3.1). Assume that*

(i) *$k_1, k_2 : [0, A] \times [0, A] \times R^n \to R^n$ and $q : [0, A] \to R^n$ are continuous,*

(ii) *There exists* $x \in \mathcal{X}$ *such that*

$$x(a) = \int_0^A k_1(a, b, x(b))db + q(a), \quad a, b \in [0, A],$$

or

$$x(a) = \int_0^A k_2(a, b, x(b))db + q(a), \quad a, b \in [0, A];$$

(iii) *There exists a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *such that*

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_1(a, b, x_n(b))db + q(a) = z, \quad a, b \in [0, A], \ z \in \mathcal{X},$$

or

$$\lim_{n\to\infty} x_n(a) = \lim_{n\to\infty} \int_0^A k_2(a, b, x_n(b))db + q(a) = z, \quad a, b \in [0, A], \ z \in \mathcal{X};$$

(iv) *For each* $a, b \in J$ *and* $u, v \in \mathcal{X}$,

$$\begin{aligned}&\int_0^A \|k_1(a, b, u(b)) - k_2(a, b, v(b))\|db \\ &\leq \frac{3}{4} \max\{\|u(a) - v(a)\|, \|u(a) - Su(a)\|, \|v(a) - Tv(a)\|, \|u(a) - Tv(a)\|, \\ &\|v(a) - Su(a)\|\}.\end{aligned}$$

Then the system of integral equations (3.1) has a unique solution u^* *in* $C(J, R^n)$.

Proof. By condition (i), f, S and T are self-mappings on $\mathcal{X}$. By assumption (ii) and knowing that f is the identity mapping on $\mathcal{X}$, so (f, S) (also (f, T)) is weakly compatible. By assumption (iii), (f, S) (also (f, T)) satisfies the property (E.A). Also, for each $u, v \in \mathcal{X}$, by (iv) we have

$$\begin{aligned}&\|Su(a) - Tv(a)\| \leq \int_0^A \|k_1(a, b, u(b)) - k_2(a, b, v(b))\|db \\ &\leq \frac{3}{4} \max\{\|u(a) - v(a)\|, \|u(a) - Su(a)\|, \|v(a) - Tv(a)\|, \|u(a) - Tv(a)\|, \\ &\|v(a) - Su(a)\|\} \\ &= \frac{3}{4} \max\{d(u, v), d(u, Su), d(v, Tv), d(u, Tv), d(v, Su)\},\end{aligned}$$

which implies that

$$\psi(d(Su, Tv)) \leq F\Big(\psi(M(u, v)), \varphi(w(M(u, v)))\Big),$$

where $\psi(t) = 2t, \varphi(t) = t, F(s, t) = s - t$ and $w(t) = \frac{1}{2}t$. Since $(\mathcal{X}, \|.\|_{\mathcal{X}})$ is a complete metric space, every Cauchy sequence in $O(S, T, f, x_0) = \{x_n : n = 0, 1, 2, \cdots\}$ (for some $x_0 \in \mathcal{X}$) converges in $\mathcal{X}$. Hence $f(\mathcal{X}) = \mathcal{X}$ is orbitally complete at x_0. Then Theorem 2.1 is applicable, where f is the identity mapping. So S and T have a common fixed point. Thus there exists $u^* \in C(J, R^n)$ a common fixed point of S and T, that is, u^* is the unique solution to (3.1). □

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