Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 4 (2017), Pages 136-146.

COMMON FIXED POINTS OF (ϕ, ψ)-CONTRACTION ON G-METRIC SPACE USING E.A PROPERTY

ABDULLAH, MUHAMMAD SARWAR*, ZEAD MUSTAFA, AND M.M.M. JARADAT

Abstract. The aim of this manuscript is to present a unique common fixed point theorem for six mappings satisfying (ϕ, ψ)- contraction and E.A property in the framework of $G-$metric space. An illustrative example is also given to justify the established result.

1. Introduction

Banach contraction mapping principle [3] is one of the pivotal results of analysis. According to the contraction mapping principle, any mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq kd(x, y) \;\; \forall \;\; x, y \in X \;\;\; and \;\;\; 0 \leq k < 1$$

must have a unique fixed point. Generalization of the above contraction mapping principle has been a very active field of research during recent years. Many authors generalized this principle with different approaches in various spaces.

Yan et al. [9] gave the idea of (ϕ, ψ)-contraction who extended the results of Harjani and Sadarangani [4] and some other authors and proved a fixed point theorem of a contraction mapping in a complete metric space endowed with a partial order by using altering distance functions. Different authors used (ϕ, ψ)-contraction to obtain common fixed point results in different spaces. Some of the work on (ϕ, ψ)-contraction is given in [2, 6, 8, 10].

In 1996, Jungck [17] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. However, the study of common fixed point of non-compatible mappings has recently been initiated by Pant ([19]) and ([20]).

In 2002, Amari and El Moutawakil [18] defined a new property called an (E.A) property which generalized the concept of non-compatible mappings and they proved some common fixed point theorem.

In 2005, Mustafa and Sims [13] introduced a new generalization of metric space by assigning to every triplet $(x, y, z) \in X \times X \times X$ a real number and named it a G-metric space. In 2008, Mustafa et al. [14] obtained some fixed point results in

2010 *Mathematics Subject Classification.* Primary 47H10; secondary 54H25.
Key words and phrases. Complete G-metric space, Cauchy sequence, (ϕ, ψ) contraction, E.A property, common fixed point, altering distance function, weakly compatible.

Submitted June 26, 2017. Published August 3, 2017.
Communicated by Wasfi Shatanawi.
* Corresponding author .

G-metric space for mappings satisfying different contractive conditions. Shatanawi [16] proved a fixed point theorem for contractive mappings satisfying Φ-maps in G-metric spaces. After that several fixed point results were proved in these spaces. Some of these works are noted in $\big[$[7], [8], [10], [11], [13], [16], [[21]-[27]] etc$\big]$.

In the current work we will obtain a unique common fixed point result in G-metric spaces using (ϕ, ψ)-contraction and E.A property.

2. Preliminaries

The following definitions and results will be needed in the sequel.

Definition 2.1. [13] *A G-metric space is a pair (X, G) where X is a nonempty set and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$, we have*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$;
(G2) $0 < G(x, x, y)$; *for all* $x, y \in X$, *with* $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ *(symmetry in all three variables);*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality).*

The function G is called a G-metric on X. Note that every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Example 2.1. [13] *Let (X, d) be a metric space and define $G_s, G_m : X \times X \times X \longrightarrow [\mathbf{0}, \infty)$ by*

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z),$$

and

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then (X, G_s) and (X, G_m) are G-metric spaces.

Definition 2.2. [13] *A sequence $\{x_n\}$ in a G-metric space (X, G) is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one say that the sequence $\{x_n\}$ is G-convergent to x. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.*

Throughout this paper we mean by $\mathbf{N}$ the set of all natural numbers.

Proposition 2.1. [13] *Let (X, G) be G-metric space. Then the following statements are equivalent.*

(1) $\{x_n\}$ *is G-convergent to x.*
(3) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$.
(4) $G(x_n, x, x) \to 0$, *as* $n \to \infty$.
(5) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 2.3. [13] *In a G-metric space (X, G), a sequence $\{x_n\}$ is said to be G-Cauchy if given $\epsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$. That is $G(x_n, x_m, x_l) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.*

Proposition 2.2. [13] *In a G-metric space (X, G), the following statements are equivalent.*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 2.4. [13] *A G-metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$, and called nonsymmetric if its not symmetric.*

Example 2.2. [15] *Let $X = \mathbf{N}$ be the set of all natural numbers, and define $G : X \times X \times X \to \mathbf{R}$ such that for all $x, y, z \in X$:*

$$\begin{array}{ll} G(x,y,z) = 0, & \text{if; } x = y = z \\ G(x,y,y) = x + y, & \text{if; } x < y \\ G(x,y,y) = x + y + \frac{1}{2}, & \text{if; } x > y \\ G(x,y,z) = x + y + z, & \text{if } x \neq y \neq z \end{array}$$

and symmetry in all three variables.
Then, (X, G) is G-metric space and non-symmetric. In fact, if $x < y$, then $G(x, y, y) = x + y \neq x + y + \frac{1}{2} = G(y, x, x)$.

Proposition 2.3. [13] *Let X be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.5. [13] *A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.*

Definition 2.6. [5] *A mapping $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:*
1. ψ is continuous and non-decreasing.
2. $\psi(t) = t$ if and only if $t = 0$.
The set consist of all altering distance functions is denoted by Ψ.

Definition 2.7. [12] *Let f and g be self maps of a set X. If $w = fx = gx$ for some $x \in X$. Then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

The following definition was given by Jungck [17].

Definition 2.8. [17] *Two maps f and g are said to be weakly compatible if they commute at their coincidence points, that is if x is a coincidence point, then $f(g(x)) = g(f(x))$.*

Definition 2.9. [18] *Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the (E.A) property if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

3. Main Results

We start this section by presenting the main result.

Theorem 3.1. *Let (X, G) be a complete G-metric space and $f, g, h, R, S, T : X \to X$ be self mappings such that*
(1) (f, R) and (g, S) satisfy (E.A) property.
(2) $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ and $h(X) \subseteq S(X)$.
(3) $R(X)$ is a closed subspace of X.
(4) $(f, R), (g, S)$ and (h, T) are weakly compatible pairs of mappings.

(5)

$$\psi\left(G(fx,gy,hz)\right)\leq\psi\left(M(x,y,z)\right)-\phi\left(M(x,y,z)\right)\qquad\forall x,y,z\in X,\tag{3.1}$$

where $\psi,\phi\in\Psi$ *and*

$$\begin{aligned}M(x,y,z)=\max\{&G(Rx,Sy,Tz),G(fx,Sy,Tz),G(Rx,gy,Tz),\\&G(Rx,Sy,hz),G(fx,gy,Tz),G(fx,Sy,hz),G(Rx,gy,hz)\}\end{aligned}\tag{3.2}$$

Then f,g,h,R,S,T *have a unique common fixed point in* X.

Proof. Since the pair (f,R) satisfy (E.A) property, there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty}fx_n=\lim_{n\to\infty}Rx_n=r_1.\tag{3.3}$$

But $f(X)\subseteq T(X)$, so there exists a sequence $\{z_n\}\subseteq X$ such that

$$fx_n=Tz_n\text{ and }\lim_{n\to\infty}fx_n=\lim_{n\to\infty}Tz_n=\lim_{n\to\infty}Rx_n=r_1.\tag{3.4}$$

Since the pair (g,S) satisfy E.A property, there exists a sequence $\{y_n\}$ such that

$$\lim_{n\to\infty}gy_n=\lim_{n\to\infty}Sy_n=r_2.\tag{3.5}$$

But $g(X)\subseteq R(X)$, so as in the above there exists a sequence $\{\beta_n\}\subseteq X$ such that

$$gy_n=R\beta_n\text{ and }\lim_{n\to\infty}gy_n=\lim_{n\to\infty}R\beta_n=\lim_{n\to\infty}Sy_n=r_2.\tag{3.6}$$

Now, we shall show that $\lim_{n\to\infty}hz_n=r_1$. Replacing x,y and z by x_n,y_n and z_n, respectively, in equation (3.2), we get that

$$\begin{aligned}M(x_n,y_n,z_n)=\max\{&G(Rx_n,Sy_n,Tz_n),G(fx_n,Sy_n,Tz_n),G(Rx_n,gy_n,Tz_n),\\&G(Rx_n,Sy_n,hz_n),G(fx_n,gy_n,Tz_n),G(fx_n,Sy_n,hz_n),\\&G(Rx_n,gy_n,hz_n)\}.\end{aligned}\tag{3.7}$$

Taking the limit as $n\to\infty$ in (3.7), and using (3.4), (3.6) and the continuity of G, we obtain that

$$\begin{aligned}\lim_{n\to\infty}M(x_n,y_n,z_n)&=\lim_{n\to\infty}\max\{G(Rx_n,Sy_n,Tz_n),G(fx_n,Sy_n,Tz_n),\\&\quad G(Rx_n,gy_n,Tz_n),G(Rx_n,Sy_n,hz_n),G(fx_n,gy_n,Tz_n),\\&\quad G(fx_n,Sy_n,hz_n),G(Rx_n,gy_n,hz_n),\}\\&=\max\{G(r_1,r_2,r_1),G(r_1,r_2,r_1),G(r_1,r_2,r_1),\\&\quad G(r_1,r_2,\lim_{n\to\infty}hz_n),G(r_1,r_2,r_1),\\&\quad G(r_1,r_2,\lim_{n\to\infty}hz_n),G(r_1,r_2,\lim_{n\to\infty}hz_n)\Big\}\\&=\max\Big\{G(r_1,r_2,\lim_{n\to\infty}hz_n),G(r_1,r_2,r_1)\Big\}\\&=G(r_1,r_2,\lim_{n\to\infty}hz_n).\end{aligned}\tag{3.8}$$

Now, in equation (3.1) replacing x,y and z by x_n,y_n and z_n, respectively, we have

$$\psi\big(G(fx_n,gy_n,hz_n)\big)\leq\psi\big(M(x_n,y_n,z_n)\big)-\phi\big(M(x_n,y_n,z_n)\big).\tag{3.9}$$

Letting $n\to\infty$ in (3.9), we get

$$\lim_{n\to\infty}\psi\big(G(fx_n,gy_n,hz_n)\big)\leq\lim_{n\to\infty}\psi\big(M(x_n,y_n,z_n)\big)-\lim_{n\to\infty}\phi\big(M(x_n,y_n,z_n)\big),$$

which implies by (3.4), (3.6) and the continuity of G, ψ and ϕ that

$$\psi\big(G(r_1, r_2, \lim_{n\to\infty} hz_n)\big) \le \psi\big(\lim_{n\to\infty} M(x_n, y_n, z_n)\big) - \phi\big(\lim_{n\to\infty} M(x_n, y_n, z_n)\big). \quad (3.10)$$

Substituting (3.8) into (3.10), we get

$$\psi\big(G(r_1, r_2, \lim_{n\to\infty} hz_n)\big) \le \psi\big(G(r_1, r_2, \lim_{n\to\infty} hz_n)\big) - \phi\big(G(r_1, r_2, \lim_{n\to\infty} hz_n)\big),$$

which implies $\phi\big(G(r_1, r_2, \lim_{n\to\infty} hz_n)\big) = 0$, which further gives from the properties of ϕ that $G(r_1, r_2, \lim_{n\to\infty} hz_n) = 0$, hence $\lim_{n\to\infty} hz_n = r_1 = r_2$. Therefore,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tz_n = \lim_{n\to\infty} R\beta_n = r \quad (3.11)$$

for some $r \in X$. As $R(X)$ is a closed subspace of X, so $Ru = r$ for some $u \in X$. Now we shall prove that $fu = r$. Specifying $x = u, y = y_n$ and $z = z_n$ in (3.8) and (3.2), then taking the limit as $n \to \infty$ and using (3.11), we obtain

$$\begin{aligned}\lim_{n\to\infty} M(u, y_n, z_n) = \lim_{n\to\infty} \max\{&G(Ru, Sy_n, Tz_n), G(fu, Sy_n, Tz_n), G(Ru, gy_n, Tz_n),\\ &G(Ru, Sy_n, hz_n), G(fu, gy_n, Tz_n), G(fu, Sy_n, hz_n), G(Ru, gy_n, hz_n)\}\\ = \max\Big\{&G(r, r, r), G(fu, r, r), G(r, r, r), G(r, r, r), G(fu, r, r),\\ &G(fu, r, r), G(r, r, r)\Big\}\\ = G&(fu, r, r), \end{aligned} \quad (3.12)$$

and

$$\begin{aligned}\psi(G(fu, r, r)) &= \lim_{n\to\infty} \psi\left(G(fu, gy_n, hz_n)\right)\\ &\le \lim_{n\to\infty} \psi\left(M(u, y_n, z_n)\right) - \lim_{n\to\infty} \phi\left(M(u, y_n, z_n)\right)\\ &= \psi\big(\lim_{n\to\infty} M(u, y_n, z_n)\big) - \phi\big(\lim_{n\to\infty} M(u, y_n, z_n)\big). \end{aligned} \quad (3.13)$$

By substituting (3.12) into (3.13), we find that

$$\psi(G(fu, r, r)) \le \psi(G(fu, r, r)) - \phi(G(fu, r, r)).$$

Thus, $\phi(G(fu, r, r)) = 0$ which implies from the properties of ϕ that $G(fu, r, r) = 0$, and so

$$fu = r = Ru. \quad (3.14)$$

Now, we shall prove that $hw = Tw = r$. As $f(X) \subseteq T(X)$, there exists $w \in X$ such that $fu = Tw = r$. By using a similar argument as above, taking limit as $n \to \infty$ in (3.8) and (3.2) after specifying $x = u, y = y_n$ and $z = w$ and using (3.11), we obtain that

$$\begin{aligned}\lim_{n\to\infty} M(u, y_n, w) = \lim_{n\to\infty} \max\{&G(Ru, Sy_n, Tw), G(fu, Sy_n, Tw), G(Ru, gy_n, Tw),\\ &G(Ru, Sy_n, hw), G(fu, gy_n, Tw), G(fu, Sy_n, hw), G(Ru, gy_n, hw)\}\\ = \max\{&G(r, r, r), G(r, r, r), G(r, r, r), G(r, r, hw), G(r, r, r),\\ &G(r, r, hw), G(r, r, hw)\}\\ = G&(r, r, hw). \end{aligned} \quad (3.15)$$

and

$$\begin{aligned}\psi(G(r,r,hw)) &= \lim_{n\to\infty}\psi\big(G(fu,gy_n,hw)\big)\\ &\leq \lim_{n\to\infty}\psi\big(M(u,y_n,w)\big)-\lim_{n\to\infty}\phi\big(M(u,y_n,w)\big)\\ &= \psi\big(\lim_{n\to\infty}M(u,y_n,w)\big)-\phi\big(\lim_{n\to\infty}M(u,y_n,w)\big). \qquad (3.16)\end{aligned}$$

Substituting (3.15) into (3.16), we find that

$$\psi(G(r,r,hw))\leq\psi(G(r,r,hw))-\phi(G(r,r,hw)).$$

Hence, $\phi(G(r,r,hw))=0$, which implies that $G(r,r,hw)=0$, and so

$$hw=r=Tw. \qquad (3.17)$$

Now, we shall prove that $gp=Sp=r$. Similarly, as $h(X)\subseteq S(X)$, there exists $p\in X$ such that $hw=Sp=r$. Again, by taking $x=u,y=p$ and $z=w$ in (3.1) and using (3.14) and (3.17), we find that

$$\begin{aligned}\psi\left(G(r,gp,r)\right) &= \psi\left(G(fu,gp,hw)\right)\\ &\leq \psi\left(M(u,p,w)\right)-\phi\left(M(u,p,w)\right), \qquad (3.18)\end{aligned}$$

where

$$\begin{aligned}M(u,p,w) &= \max\{G(Ru,Sp,Tw),G(fu,Sp,Tw),G(Ru,gp,Tw),\\ &\qquad G(Ru,Sp,hw),G(fu,gp,Tw),G(fu,Sp,hw),G(Ru,gp,hw)\}\\ &= \max\{G(r,r,r),G(r,r,r),G(r,gp,r),G(r,r,r),G(r,gp,r),\\ &\qquad G(r,r,r),G(r,gp,r)\}\\ &= G(r,gp,r). \qquad (3.19)\end{aligned}$$

Hence, by substituting (3.19) into (3.18), we get that

$$\psi(G(r,gp,r))\leq\psi(G(r,gp,r))-\phi(G(r,gp,r)).$$

Thus, $\phi(G(r,gp,r))=0$, which implies that $G(r,gp,r)=0$ and so

$$gp=r=Sp. \qquad (3.20)$$

Therefore, from (3.14), (3.17) and (3.20), we conclude that r is a point of coincidences of each pair of $(f,R),(g,S)$ and (h,T), that is

$$fu=Ru=gp=Sp=hw=Tw=r. \qquad (3.21)$$

To this end, we shall show that r is a common fixed point of f,g,h,R,S and T. From (3.21) and since the functions of the pair (f,R) are weakly compatible mappings, then

$$f(fu)=f(r)=f(Ru)=R(fu)=R(r)\Rightarrow f(r)=R(r), \qquad (3.22)$$

which shows that r is a coincidence point of f and R. As above, applying (3.1) for $x=r$, $y=p$ and $z=w$ and using (3.21) and (3.22), we get that

$$\begin{aligned}\psi\left(G(fr,p,w)\right) &= \psi\left(G(fr,gp,hw)\right)\\ &\leq \psi\left(M(r,p,w)\right)-\phi\left(M(r,p,w)\right) \qquad (3.23)\end{aligned}$$

where

$$\begin{aligned} M(r,p,w) &= \max\{G(Rr,Sp,Tw), G(fr,Sp,Tw), G(Rr,gp,Tw), \\ &\qquad G(Rr,Sp,hw), G(fr,gp,Tw), G(fr,Sp,hw), G(Rr,gp,hw)\} \\ &= \max\{G(Rr,r,r), G(fr,r,r), G(Rr,r,r), \\ &\qquad G(Rr,r,r), G(fr,r,r), G(fr,r,r), G(Rr,r,r)\} \\ &= G(fr,r,r). \end{aligned}$$

Hence equation (3.23) becomes

$$\psi\left(G(fr,p,w)\right) \le \psi\left(G(fr,r,r)\right) - \phi\left(G(fr,r,r)\right),$$

which yields that $\phi(G(fr,r,r)) = 0$, which further implies by properties of ϕ that $G(fr,r,r) = 0$. Thus $fr = r = Rr$. That is r is a common fixed point of f and R. Similarly, using the same steps and procedure as in the case of f and R, we can prove that $gr = Sr = r$ and $hr = Tr = r$. Therefore, r is a common fixed point of f, g, h, R, S and T.

Now, we shall prove that the obtained common fixed point is unique. Suppose that v is another common fixed point of the mappings f, g, h, R, S and T. Then applying equation (3.1) for $x = r, y = v$ and $z = v$, we find that

$$\psi\left(G(r,v,v)\right) = \psi\big(G(fr,gv,hv)\big) \le \psi\big(M(r,v,v)\big) - \phi\big(M(r,v,v)\big), \tag{3.24}$$

where

$$\begin{aligned} M(r,v,v) &= \max\{G(Rr,Sv,Tv), G(fr,Sv,Tv), G(Rr,gv,Tv), G(Rr,Sv,hv), \\ &\qquad G(fr,gv,Tv), G(fr,Sv,hv), G(Rr,gv,hv)\} \\ &= \max\{G(r,v,v), G(r,v,v), G(r,v,v), G(r,v,v), G(r,v,v), \\ &\qquad G(r,v,v), G(r,v,v)\} \\ &= G(r,v,v). \end{aligned} \tag{3.25}$$

So (3.24) becomes

$$\psi\big(G(r,v,v)\big) \le \psi\big(G(r,v,v)\big) - \phi\big(G(r,v,v)\big),$$

which yields that $\phi\big(G(r,v,v)\big) = 0$. Hence by properties of ϕ we have $G(r,v,v) = 0$ and so $r = v$. Therefore, r is a unique common fixed point of f, g, h, R, S and T. □

The following result is an immediate consequence of Theorem (3.1) by taking $\phi(t) = t$.

Corollary 3.1. *Let (X,G) be a complete G-metric space and $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(f,R) and (g,S) satisfy (E.A) property.*
(2) *$f(X) \subseteq T(X), g(X) \subseteq R(X)$ and $h(X) \subseteq S(X)$.*
(3) *$R(X)$ is a closed subspace of X.*
(4) *$(f,R), (g,S)$ and (h,T) are weakly compatible pairs of mappings.*
(5) *$\psi\big(G(fx,gy,hz)\big) \le \psi\big(M(x,y,z)\big) - M(x,y,z)$ for all $x, y, z \in X$ where ψ is an altering distance function and*

$$\begin{aligned} M(x,y,z) = \max\{&G(Rx,Sy,Tz), G(fx,Sy,Tz), G(Rx,gy,Tz), \\ &G(Rx,Sy,hz), G(fx,gy,Tz), G(fx,Sy,hz), G(Rx,gy,hz)\}. \end{aligned}$$

Then f, g, h, R, S and T have a unique common fixed point in X.

As in the above corollary, the following results follows from Theorem (3.1) by taking $\psi(t) = t$.

Corollary 3.2. *Let (X, G) be a complete G-metric space and $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(f, R) and (g, S) satisfy E.A property.*
(2) *$f(X) \subseteq T(X), g(X) \subseteq R(X)$ and $h(X) \subseteq S(X)$.*
(3) *$R(X)$ is a closed subspace of X.*
(4) *$(f, R), (g, S)$ and (h, T) are weakly compatible pairs of mappings.*
(5) *$G(fx, gy, hz) \leq M(x, y, z) - \phi\big(M(x, y, z)\big)$ for all $x, y, z \in X$ where ϕ is an altering distance function and*

$$M(x,y,z) = \max\{G(Rx,Sy,Tz), G(fx,Sy,Tz), G(Rx,gy,Tz), G(Rx,Sy,hz), G(fx,gy,Tz), G(fx,Sy,hz), G(Rx,gy,hz)\}.$$

Then f, g, h, R, S and T have a unique common fixed point in X.

By specifying $\psi(t) = t$ and $\phi(t) = \frac{t}{k}$ with $k > 1$ in Theorem (3.1), we get the following corollaries.

Corollary 3.3. *Let (X, G) be a complete G-metric space and $f, g, h, R, S, T : X \to X$ be self mappings such that*

(1) *(f, R) and (g, S) satisfy E.A property.*
(2) *$f(X) \subseteq T(X), g(X) \subseteq R(X)$ and $h(X) \subseteq S(X)$.*
(3) *$R(X)$ is a closed subspace of X.*
(4) *$(f, R), (g, S)$ and (h, T) are weakly compatible pairs of mappings.*
(5) *$G(fx, gy, hz) \leq \frac{k-1}{k} M(x, y, z)$ for all $x, y, z \in X$, where k is a positive integer and*

$$M(x,y,z) = \max\{G(Rx,Sy,Tz), G(fx,Sy,Tz), G(Rx,gy,Tz), G(Rx,Sy,hz), G(fx,gy,Tz), G(fx,Sy,hz), G(Rx,gy,hz)\}.$$

Then f, g, h, R, S and T have a unique common fixed point in X.

By taking $f = g$ and $R = S$ in Theorem (3.1), we get the following result.

Corollary 3.4. *Let (X, G) be a complete G-metric space and $g, h, S, T : X \to X$ be self mappings such that*

(1) *(g, S) satisfy E.A property.*
(2) *$g(X) \subseteq T(X)$ and $h(X) \subseteq S(X)$.*
(3) *$S(X)$ is a closed subspace of X.*
(4) *(g, S) and (h, T) are weakly compatible pairs of mappings.*
(5) *$\psi\big(G(fx, gy, hz)\big) \leq \psi\big(M(x, y, z)\big) - \phi\big(M(x, y, z)\big)$ for all $x, y, z \in X$ where ψ, ϕ are altering distance functions and*

$$M(x,y,z) = \max\{G(Sx,Sy,Tz), G(gx,Sy,Tz), G(Sx,gy,Tz), G(Sx,Sy,hz), G(gx,gy,Tz), G(gx,Sy,hz), G(Sx,gy,hz)\}.$$

Then g, h, S and T have a unique common fixed point in X.

Example 3.1. *Let $X = [0, \infty)$. Define the function $G : X \times X \times X \to [0, \infty)$ as follows:*

$$G(x,y,z) = \begin{cases} 0, & \textit{if}, x = y = z; \\ \max\{x,y,z\}, & \textit{otherwise}, \end{cases}$$

Then, clearly that (X,G) is a complete G-metric space. Also, define the mappings f,g,h,R,S and T by

$$fx=\frac{x}{16},\ g(x)=\frac{x}{18},\ h(x)=\frac{x}{20},$$

$$R(x)=\frac{x}{4},\ S(x)=\frac{x}{3}\ \text{and}\ T(x)=\frac{x}{6}$$

for all $x\in X$. Further, define $\psi(t)=2t$ and $\phi(t)=\frac{t}{2}$ for all $t\in[0,\infty)$. Then f,g,h,R,S and T have unique common fixed point.

Proof. We will show that the mappings f,g,h,R,S and T satisfy the conditions of Theorem (3.1).

(1) (f,R) and (g,S) satisfy E.A property. In fact, $\left\{\frac{1}{n}\right\}$ is one of the required sequence for both pairs.

(2) $f(X)\subseteq T(X)$, $g(X)\subseteq S(X)$ and $h(X)\subseteq R(X)$. In fact, $f(X)=g(X)=S(X)=R(X)=T(X)=[0,\infty)$.

(3) $R(X)=[0,\infty)$ is a closed subspace of X.

(4) $(f,R),(g,S)$ and (h,T) are weakly compatible pairs of mappings. In fact, the only coincident point for f and R is 0 and $f(R(0))=R(f(0))=0$. Similarly for the other two pairs.

(5) We shall show that the above mappings satisfy the contractive condition in (3.1). Note that

$$\begin{aligned}\psi(G(fx,gy,hz)) &= \psi(\max\{\frac{x}{16},\frac{y}{18},\frac{z}{20}\})\\ &= 2\max\{\frac{x}{16},\frac{y}{18},\frac{z}{20}\}\\ &= \max\{\frac{x}{8},\frac{y}{9},\frac{z}{10}\}. \qquad (3.26)\end{aligned}$$

And from (3.2),

$$M(x,y,z)=\max\left\{\begin{array}{c}\max\{\frac{x}{4},\frac{y}{3},\frac{z}{6}\},\max\{\frac{x}{16},\frac{y}{3},\frac{z}{6}\},\max\{\frac{x}{4},\frac{y}{18},\frac{z}{6}\},\max\{\frac{x}{4},\frac{y}{3},\frac{z}{20}\}\\ \max\{\frac{x}{16},\frac{y}{18},\frac{z}{6}\},\max\{\frac{x}{16},\frac{y}{3},\frac{z}{20}\},\max\{\frac{x}{4},\frac{y}{18},\frac{z}{20}\}\end{array}\right\}.$$

One can easily see that each element of

$$\left\{\begin{array}{c}\max\{\frac{x}{16},\frac{y}{3},\frac{z}{6}\},\max\{\frac{x}{4},\frac{y}{18},\frac{z}{6}\},\max\{\frac{x}{4},\frac{y}{3},\frac{z}{20}\}\\ \max\{\frac{x}{16},\frac{y}{18},\frac{z}{6}\},\max\{\frac{x}{16},\frac{y}{3},\frac{z}{20}\},\max\{\frac{x}{4},\frac{y}{18},\frac{z}{20}\}\end{array}\right\}$$

is less than or equal to

$$\max\left\{\frac{x}{4},\frac{y}{3},\frac{z}{6}\right\}.$$

Thus,

$$M(x,y,z)=\max\left\{\frac{x}{4},\frac{y}{3},\frac{z}{6}\right\}.$$

And so,

$$\begin{aligned}\psi(M(x,y,z))-\phi(M(x,y,z)) &= 2\max\left\{\frac{x}{4},\frac{y}{3},\frac{z}{6}\right\}-\frac{1}{2}\max\left\{\frac{x}{4},\frac{y}{3},\frac{z}{6}\right\}\\ &= \max\left\{\frac{3x}{8},\frac{y}{2},\frac{z}{2}\right\}. \qquad (3.27)\end{aligned}$$

Thus, from (3.26) and (3.27), we get that

$$\begin{aligned} \psi(G(fx, gy, hz)) &= \max\left\{\frac{x}{8}, \frac{y}{9}, \frac{z}{10}\right\} \\ &\leq \max\left\{\frac{3x}{8}, \frac{y}{2}, \frac{z}{2}\right\} \\ &= \psi(M(x,y,z)) - \phi(M(x,y,z)). \end{aligned}$$

Therefore, the self mappings f, g, h, R, S and T satisfies all conditions of Theorem (3.1), and $x = 0$ is the unique common fixed point of f, g, h, R, S and T. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] B. Alvarez, *The Cauchy problem for a nonlocal perturbation of the KdV equation*, Differential Integral Equations **16 10** (2003) 1249–1280.

[2] H. Aydi, E. Karapinar and W. Shatanawi, *Coupled fixed point results for (ψ, ϕ)-weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl. **62**, (2011) 4449–4460.

[3] S. Banach: *Sur les operations dans les ensembles abstraits et.leur application aux equations integrals"*, Fund. Math., **3**(1922) 133–181.

[4] J.Harjani and K. Sadarangni, *Fixed point theorems for weakly contraction mappings in partially ordered sets*, Nonlinear Anal., **71** (2009) 3403 - 3410.

[5] M.S. Khan, M. Swalesh and S. Sessa, *Fixed points theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984) 1–9.

[6] Z. Mustafa, J.R. Roshan, and V. Parvaneh, *Coupled coincidence point results for (ψ, ϕ) - weakly contractive mappings in partially ordered G-metric spaces*, Fixed Point Theory Appl. **206** (2013). doi:10.1186/1687-1812-2013-206

[7] M. Sarwar, Abdullah, S.I. Ali Shah, *Fixed point theorem satisfying some rational type contraction in G-metric spaces*, J. Adv. Math. Stud.,**9 2**(2016) 320-329.

[8] H.K.Nashine, B.Samet, *Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal., **74** (2011) 2201–2209.

[9] F. Yan, Y. Su, and Q. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl., **152** (2012).

[10] Z. Mustafa, J. R. Roshan and V. Parvaneh, *Existence of a tripled coincidence point in ordered G-metric spaces and applications to a system of integral equations*,Journal of Inequalities and Applications, **2013:453** (2013).

[11] A.Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered Gb-metric spaces*, Filomat, **28 6** (2014)1087-1101.

[12] M. Abbas and B.E. Rhoades, *Common fixed point results for noncomuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput, **215** (2009) 262-269.

[13] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear and Convex Analsis, **7** (2006) 289-297.

[14] Z. Mustafa , H.Obiedat and F. Awawdeh, *Some common fixed point theorems for mapping on complete G-metric spaces*, Fixed Point Theory Appl, **2008** , Article ID **189870** 12 pages.

[15] Z. Mustafa, *Some New Common Fixed Point Theorems Under Strict Contractive Conditions in G- Metric Spaces*, Journal of Applied Mathematics, **2012**, Article ID 248937, 21 pages, doi:10.1155/2012/248937.

[16] W. Shatanawi, *Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces*, Fixed Point Theory Appl., **2010** (2010), Article ID **181650**, 9 pages.

[17] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces.* Far East J. Math. Sci., **4** (1996) 199–215.

[18] M. Aamri and D.EI Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002) 181–188.

[19] R.P. Pant, *R-weak commutativity and common fixed points*, Soochow J. Math., **25** (1999)37–42.
[20] R.P. Pant, *Common fixed point of contractive maps*, J. Math. Anal. Appl., **226** (1998) 251–258.
[21] K. Abodayeh, W. Shatanawi, A. Bataihah, A.H. Ansari, *Some Fixed Point and Common Fixed Point Results Through Ω-Distance Under Nonlinear Contractions*, GU J Sci **30(1)** (2017) 293-302.
[22] K. Abodayeh, W. Shatanawi, A. Bataiha, *Fixed Point Theorem Through Ω-distance of Suzuki Type Contraction Condition*, Gazi University Journal of Science, **29(1)** (2016) 129-133.
[23] W. Shatanawi, A. Bataihah, A. Pitea, *Fixed and common fixed point results for cyclic mappings of Ω-distance*, Journal of Nonlinear Science and Applications, **9** (2016) 727–735.
[24] W. Shatanawi, A. Pitea, *Fixed and coupled fixed point theorems of omega-distance for non-linear contraction*, Fixed Point Theory and Applications, **2013:275**, (2013) 116.
[25] W. Shatanawi, A. Pitea, *Ω-Distance and coupled fixed point in G-metric spaces*, Fixed Point Theory and Applications, **2013:208** (2013)1–15.
[26] W. Shatanawi, *Common fixed point result for two self-maps in G-metric Spaces*, Matematicki Vesnik, **65,2** (2013) 143–150.
[27] W. Shatanawi and Mihai Postolache, *Some fixed point results for a G-weak contraction in G-metric spaces*, Abstract and Applied Analysis, **2012** (2012), Article ID 815870, 19 pages doi:10.1155/2012/815870

Abdullah, Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, Pakistan
E-mail address: abdullahibad474@gmail.com

Muhammad Sarwar, Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, Pakistan
E-mail address: sarwarswati@gmail.com

Zead Mustafa, Department of Mathematics, Statistics and Physics, Qatar University, Doha-Qatar
E-mail address: zead@qu.edu.qa

Department of Mathematics, The Hashemite University, Zarqa- Jordan
E-mail address: zmagablh@hu.edu.jo

M.M.M. Jaradat, Department of Mathematics, Statistics and Physics, Qatar University,Doha-Qatar
E-mail address: mmjst4@qu.edu.qa