ON STRONGLY DEFERRED CESÁRO MEAN OF DOUBLE SEQUENCES

Ş. SEZGEK, İ. DAĞADUR

Abstract. In this paper, the concepts of $\alpha$-strongly deferred Cesàro mean and $\alpha$-strongly Cesàro submethod for double sequences are defined and studied by using deferred double natural density of the subset of natural numbers. Also, we consider the case $\beta(n) = \lambda(n) - \lambda(n - 1)$, $\gamma(m) = \mu(m) - \mu(m - 1)$ for $\alpha$-strongly deferred Cesàro mean $D_{\beta,\gamma}^\alpha$ where $\lambda := \{\lambda(n)\}_{n=1}^\infty$ and $\mu := \{\mu(m)\}_{m=1}^\infty$ are strictly increasing sequences of positive integers with $\lambda(0) = 0$ and $\mu(0) = 0$. Finally, we obtain some inclusion results between $\alpha$-strongly Cesàro submethod and $\alpha$-strongly deferred Cesàro mean $D_{\beta,\gamma}^\alpha$ of the double sequences.

1. Introduction

The concept of statistical convergence was first introduced by Fast [8] and also independently by Buck [3] and Schoenberg [19] for real and complex sequences. Further, this concept was studied by Šalát [18], Fridy [9] and many others. Some equivalence results for Cesàro submethods have been studied by Goffman and Petersen [11], Armitage and Maddox [2] and Osikiewicz [15]. In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying $p(n) < q(n)$ and $\lim_{n \to \infty} q(n) = \infty$.

Let $0 < \alpha < \infty$ be a real number. A sequence $x = (x_k)$ is said to be $\alpha$-strongly deferred Cesàro summable to $L \in \mathbb{N}$ if

$$\lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - L|^\alpha = 0$$

1Corresponding Author

2000 Mathematics Subject Classification. 40A05, 40C05, 40B05, 40D25.

Key words and phrases. Statistical convergence, Deferred Cesàro mean, Double sequence, Deferred statistical convergence.

©2017 Ilirias Research Institute, Prishtinë, Kosovë.


Submitted by Mikail Et.
exist, where \( \{p(n)\} \) and \( \{q(n)\} \) are sequences of nonnegative integers satisfying the conditions \( p(n) < q(n) \) and \( \lim_{n \to \infty} q(n) = \infty \). 

In [21], the first study on double sequences was examined by Bromwich. And then it was investigated by many authors such as Hardy [12], Moricz [13], Tripathy [24], Başarır and Sonalcan [20]. The notion of regular convergence for double sequences was defined by Hardy [12]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [27]. The statistical and Cauchy convergence for double sequences were examined by Mursaleen and Edely [14] and Tripathy [23] in recent years. Many recent improvements containing the summability by four dimensional matrices might be found in [16]. We begin with some definitions and notations. By the convergence of a double sequence we mean the convergence in Pringsheim’s sense [17]. A double sequence \( x = (x_{kl}) \) is said to be convergent in the Pringsheim’s sense if for all \( \varepsilon > 0 \) there exists an \( n_0 = n_0(\varepsilon) \) such that \( |x_{kl} - L| < \varepsilon \) whenever \( k, l \geq n_0 \). In this case, we write \( \lim_{k,l \to \infty} x_{kl} = L \).

A double sequence \( x = (x_{kl}) \) is bounded if there exist a positive number \( M \) such that \( |x_{kl}| < M \) holds for all \((k, l) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2 \), i.e., if

\[
\|x\|_{(\infty, 2)} := \sup_{k,l} |x_{kl}| < \infty.
\]

We will denote the set of all bounded double sequences by \( \mathcal{M}_b \). Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. In [14], let \( K \subset \mathbb{N}^2 \) be a two-dimensional set of positive integers and let

\[
K(n, m) := \{(k, l) \in K : (k, l) \leq (n, m)\}
\]

where \((k, l) \leq (n, m)\) means that \( k \leq n, l \leq m \).

Then, the upper asymptotic density of the set \( K \subset \mathbb{N}^2 \) is defined as

\[
\delta^*(K(n, m)) := \limsup_{n,m \to \infty} \frac{|K(n, m)|}{nm},
\]

if the limit exists and finite. The vertical bars above indicate the cardinality of the set \( K(n, m) \). In case the sequence \( \left\{ \frac{|K(n, m)|}{nm} \right\} \) has a limit in Pringsheim’s sense then we say that \( K \) has a double natural density and is defined as

\[
\delta_2(K(n, m)) := \lim_{n,m \to \infty} \frac{|K(n, m)|}{nm}.
\]

Following Mursaleen [14] we say that a double sequence \( x = (x_{kl}) \) is statistically convergent to the number \( L \) if for each \( \epsilon > 0 \)

\[
\lim_{n,m \to \infty} \frac{1}{nm} |\{(k, l) : k \leq n, l \leq m, | x_{kl} - L | \geq \epsilon\}| = 0.
\]

In this case, we write \( st_2 - \lim_{k,l \to \infty} x_{kl} = L \) and we denote the set of all double statistically convergent sequences by \( st_2 \).

Let \( x = (x_{kl}) \) be a double sequence and \( p \) be a positive real number. Then the double sequence \( x \) is said to be strongly \( p \)-Cesàro summable to \( L \) if

\[
\lim_{n,m \to \infty} \frac{1}{nm} \sum_{k=1}^{n} \sum_{l=1}^{m} |x_{kl} - L|^p = 0.
\]

We denote the space of all strongly \( p \)-Cesàro summable double sequences by \( \omega_p^2 \).
It is clear that, for the function \( \beta(n) = q(n) - p(n), \) \( \gamma(m) = r(m) - t(m) \). Then deferred Cesàro mean \( D_{\beta, \gamma} \) of the double sequence \( x \) is defined by

\[
(D_{\beta, \gamma} x)_{nm} = \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} x_{kl}
\]

where \( \{p(n)\}, \{q(n)\}, \{r(m)\} \) and \( \{t(m)\} \) are sequences of nonnegative integers satisfying the conditions \( p(n) < q(n) \), \( t(m) < r(m) \) and \( \lim_{n \to \infty} q(n) = \infty \), \( \lim_{m \to \infty} r(m) = \infty \). We note that \( D_{\beta, \gamma} \) is clearly regular for any choice of \( \{p(n)\}, \{q(n)\}, \{r(m)\} \) and \( \{t(m)\} \).

Throughout this paper \( \beta(n) = q(n) - p(n), \gamma(m) = r(m) - t(m) \) are represented \( \beta \) and \( \gamma \) respectively.

Let \( x = (x_{kl}) \) be a double sequence and a real number \( L \). Then, the double sequence \( x \) is said to be \( D_{\beta, \gamma} \)-summable to \( L \) if

\[
\lim_{n,m \to \infty} \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} (x_{kl} - L) = 0
\]

exists and it is denoted by \( \lim_{n,m \to \infty} (D_{\beta, \gamma} x)_{nm} = L \).

Let \( K \) be a subset of \( \mathbb{N}^2 \) and denote the set

\[
\{ (k, l) : p(n) < k \leq q(n), \ t(m) < l \leq r(m), \ (k, l) \in K \}
\]

by \( K_{\beta, \gamma}(n, m) \). The deferred double natural density of \( K \) is defined by

\[
\delta^{(2)}_{D_{\beta, \gamma}}(K) := \lim_{n,m \to \infty} \frac{1}{\beta(n) \gamma(m)} |K_{\beta, \gamma}(n, m)|
\]

whenever the limit exists. The vertical bars indicate the cardinality of the set \( K_{\beta, \gamma}(n, m) \). Also, because of \( \delta^{(2)}_{D_{\beta, \gamma}}(K) \) does not exists for all \( K \subset \mathbb{N}^2 \), it is convenient to use upper deferred asymptotic density of \( K \), defining by

\[
\delta^{*}_{D_{\beta, \gamma}}(K) = \limsup_{n,m \to \infty} \frac{|\{(k, l) : p(n) + 1 \leq k \leq q(n), \ t(m) + 1 \leq l \leq r(m), \ (k, l) \in K\}|}{\beta(n) \gamma(m)}.
\]

It is clear that, for the function \( \delta^{*}_{D_{\beta, \gamma}}(K) \) the following axioms are hold:

i) if \( \delta^{(2)}_{D_{\beta, \gamma}}(K) \) exists, then \( \delta^{(2)}_{D_{\beta, \gamma}}(K) = \delta^{*}_{D_{\beta, \gamma}}(K) \),

ii) \( \delta^{(2)}_{D_{\beta, \gamma}}(K) \neq 0 \) if and only if \( \delta^{*}_{D_{\beta, \gamma}}(K) > 0 \) and

iii) The function \( \delta^{(2)}_{D_{\beta, \gamma}}(K) \) is monotone increasing.

A double sequence \( x = (x_{kl}) \) is said to be deferred statistically convergent to \( L \in \mathbb{N} \) if for every \( \varepsilon > 0 \),

\[
\lim_{n,m \to \infty} \frac{|\{(k, l) : p(n) + 1 \leq k \leq q(n), \ t(m) + 1 \leq l \leq r(m), \ |x_{kl} - L| \geq \varepsilon\}|}{\beta(n) \gamma(m)} = 0
\]

and it is denoted by \( st_2 - \lim_{n,m \to \infty} (D_{\beta, \gamma} x)_{nm} = L \).
2. \(\alpha\)-strongly deferred Cesáro mean

**Definition 2.1.** Let \( x = (x_{kl}) \) be a double sequence and \( 0 < \alpha < \infty \) be a real number. Then, the double sequence \( x = (x_{kl}) \) is said to be \(\alpha\)-strongly deferred Cesáro summable to \( L \in \mathbb{N} \) if

\[
\lim_{n,m \to \infty} \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^{\alpha} = 0
\]

exists and it is denoted by \( \lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L \).

**Theorem 2.1.** Let \( x = (x_{kl}) \) be a double sequence and \( 0 < \alpha < \infty \) be a real number. If \( \lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L \) then, the double sequence \( x \) is deferred statistical convergent to \( L \).

**Proof.** Assume that \( \lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L \) and denote set

\[
\{(k, l) : p(n) + 1 \leq k \leq q(n), t(m) + 1 \leq l \leq r(m), |x_{kl} - L| \geq \varepsilon\}
\]

by \( K(\varepsilon) \). Therefore, the inequality

\[
\frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^{\alpha} \geq \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^{\alpha}
\]

\[
\geq \varepsilon^{\alpha} \frac{1}{\beta(n) \gamma(m)} |K(\varepsilon)|
\]

holds. After taking limits when \( n, m \to \infty \) the proof of theorem is obtained. \( \square \)

Now we get following

**Corollary 2.2.** Let \( q(n) = n, r(m) = m \). If \( \lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L \), then

\[
st_{2} - \lim_{n,m \to \infty} x_{nm} = L.
\]

**Theorem 2.3.** Let a double sequence \( x = (x_{kl}) \) be bounded. If it is deferred statistical convergent to \( L \), then

\[
\lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L.
\]

**Proof.** We denote the complement of \( K(\varepsilon) \) by

\[
K^{c}(\varepsilon) := \{(k, l) : p(n) < k \leq q(n), t(m) < l \leq r(m), |x_{kl} - L| < \varepsilon\}.
\]

We assume \( x = (x_{kl}) \) is bounded and deferred statistical convergent to \( L \). By hypothesis, it is clear that there is a positive real number \( M \) such that \( |x_{kl} - L| \leq M \) for all \( k, l \in \mathbb{N} \). Therefore, we have
ON STRONGLY DEFERRED CESÀRO MEAN OF DOUBLE SEQUENCES

\[ \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^\alpha \]

\[ = \frac{1}{\beta(n) \gamma(m)} \left[ \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^\alpha + \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ (k,l) \notin K(\varepsilon)}}^{q(n)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ (k,l) \notin K(\varepsilon)}}^{r(m)} |x_{kl} - L|^\alpha \right] \]

\[ \leq \frac{1}{\beta(n) \gamma(m)} \left[ M^\alpha + \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ (k,l) \notin K(\varepsilon)}}^{q(n)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ (k,l) \notin K(\varepsilon)}}^{r(m)} \varepsilon^\alpha \right] \]

\[ \leq \frac{1}{\beta(n) \gamma(m)} [M^\alpha |K(\varepsilon)| + \varepsilon^\alpha |K^c(\varepsilon)|]. \]

Since

\[ \lim_{n,m \to \infty} \frac{|K(\varepsilon)|}{\beta(n) \gamma(m)} = 0 \quad \text{and} \quad \lim_{n,m \to \infty} \frac{|K^c(\varepsilon)|}{\beta(n) \gamma(m)} = 1, \]

this proves theorem. \( \square \)

Our next result is obtained from Theorem 2.3

**Corollary 2.4.** Let \( \{q(n)\}, \{r(m)\} \) be arbitrary strictly increasing sequences and \( \frac{p(n)}{\beta(n)} \) and \( \frac{t(m)}{\gamma(m)} \) are bounded sequence. If \( s_{2} - \lim_{n,m \to \infty} x_{nm} = L \) for a bounded double sequence \( x = (x_{kl}) \), then \( \lim_{n,m \to \infty} \left( D_{\beta,\gamma}^{\alpha} x \right)_{nm} = L. \)

Now we have

**Theorem 2.5.** The \( \alpha \)-strongly deferred convergent double sequence \( x = (x_{kl}) \) is an \( \alpha \)-strongly Cesàro convergent only if

\( \frac{p(n)}{\beta(n)} \) and \( \frac{t(m)}{\gamma(m)} \)

are bounded.

**Proof.** The technique that was used by Agnew in [1] can be applied for this proof. Let us assume that \( x = (x_{kl}) \) is an \( \alpha \)-strongly Cesàro convergent to \( L. \) In this case, the following equality

\[ \frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} |x_{kl} - L|^\alpha \]

is established.
holds. It can be said that the $\alpha$-strongly deferred convergence of the double sequence $x = (x_{kl})$ is the linear combination of the $\alpha$-strongly Cesàro convergence of the double sequence $x = (x_{kl})$. We can consider this linear combination as a matrix transformation. For the regularity of this matrix transformation the sequence

$$\left\{ \frac{(q(n) + p(n))}{\beta(n) \gamma(m)} (r(m) + t(m)) \right\}$$

must be bounded. For the boundedness of \([1]\)

$$\frac{p(n)}{q(n) - p(n)} \quad \text{and} \quad \frac{t(m)}{r(m) - t(m)}$$

must be bounded since

$$\frac{\frac{(q(n) + p(n))}{\beta(n) \gamma(m)} (r(m) + t(m))}{\frac{q(n)}{q(n) - p(n)} \frac{(q(n) - p(n))}{(r(m) - t(m))} (r(m) - t(m))} = \frac{(q(n) + p(n))}{q(n) - p(n)} \frac{(q(n) - p(n) + 2p(n))}{(r(m) - t(m) + 2t(m))}$$

$$= \left( 1 + \frac{2p(n)}{q(n) - p(n)} \right) \left( 1 + \frac{2t(m)}{r(m) - t(m)} \right).$$

The assertion completes the proof. $\square$
Now, in the following theorems $D_{\beta,\gamma}^\alpha$-convergence and $D_{\beta',\gamma'}^\alpha$-convergence of the double sequence $x = (x_{kl})$ are compared under the restriction

$$p(n) \leq p'(n) < q'(n) \leq q(n)$$ (2)

$$t(m) \leq t'(m) < r'(m) \leq r(m)$$ (3)

for all $n, m \in \mathbb{N}$.

**Theorem 2.6.** \{p'(n)\}, \{q'(n)\}, \{t'(m)\} and \{r'(m)\} be sequences of positive natural numbers satisfying (2), (3) and the sets \{k : p(n) < k \leq p'(n)\}, \{k : q'(n) < k \leq q(n)\}, \{k : t(m) < k \leq t'(m)\}, \{k : r'(m) < k \leq r(m)\} are finite for all $n, m \in \mathbb{N}$. Then, $D_{\beta',\gamma'}^\alpha$-convergence of a bounded double sequence implies $D_{\beta,\gamma}^\alpha$-convergence.

**Proof.** There is a positive real number $M$ in the assumption such that $|x_{kl} - L| \leq M$ which holds for all $k, l \in \mathbb{N}$. Therefore, we have

$$\frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n),r(m)} \sum_{l=t(m)+1}^{q(n),r(m)} |x_{kl} - L|^\alpha$$

$$= \frac{1}{\beta(n) \gamma(m)} \left( \sum_{k=p(n)+1}^{p'(n),t'(m)} \sum_{l=t'(m)+1}^{q'(n),r'(m)} + \sum_{k=q'(n)+1}^{q(n),t'(m)} \sum_{l=t'(m)+1}^{q(n),r'(m)} + \sum_{k=p'(n)+1}^{p(n),t(m)} \sum_{l=t(m)+1}^{q'(n),r(m)} + \sum_{k=q(n)+1}^{q(n),r(m)} \sum_{l=t(m)+1}^{q(n),r(m)} \right) |x_{kl} - L|^\alpha$$

$$+ \frac{1}{\beta(n) \gamma(m)} \left( \sum_{k=p(n)+1}^{p'(n),t'(m)} \sum_{l=t'(m)+1}^{q'(n),r'(m)} + \sum_{k=q'(n)+1}^{q(n),t'(m)} \sum_{l=t'(m)+1}^{q(n),r'(m)} + \sum_{k=p'(n)+1}^{p(n),t(m)} \sum_{l=t(m)+1}^{q'(n),r(m)} + \sum_{k=q(n)+1}^{q(n),r(m)} \sum_{l=t(m)+1}^{q(n),r(m)} \right) |x_{kl} - L|^\alpha$$

$$\leq \frac{8}{\beta'(n) \gamma'(m)} M^\alpha O(1) + \frac{1}{\beta'(n) \gamma'(m)} \sum_{k=p'(n)+1}^{p(n),r'(m)} \sum_{l=t'(m)+1}^{q'(n),r'(m)} |x_{kl} - L|^\alpha$$

If we take the limit, we obtain the double sequence $x = (x_{kl})$ which is a $D_{\beta,\gamma}^\alpha$-convergent.

**Theorem 2.7.** Let \{p'(n)\}, \{q'(n)\}, \{t'(m)\} and \{r'(m)\} be sequences of positive natural numbers satisfying (2), (3) and

$$\lim_{n, m \to \infty} \frac{\beta'(n) \gamma'(m)}{\beta(n) \gamma(m)} = d > 0.$$ 

Then, the $D_{\beta,\gamma}^\alpha$-convergence of the double sequence $x = (x_{kl})$ implies $D_{\beta',\gamma'}^\alpha$-convergence.
Proof. It is easy to see that the inequality
\[
\frac{1}{\beta(n) \gamma(m)} \sum_{k=p(n)+1}^{q(n),r(m)} |x_{kl} - L|^\alpha
\]
holds. After taking limit when \( n, m \to \infty \), we get that the double sequence \( x = (x_{kl}) \) is a \( D_{\beta',\gamma'}^{\alpha} \) convergent to \( L \). \( \square \)

Recall that if \( F \) is an infinite subset of \( \mathbb{N} \) and \( F \) as the range of a strictly increasing sequence of positive integers, say \( F = \{\lambda(n)\}_{n=1}^\infty \), the Cesàro submethod \( C_\lambda \) is defined as
\[
(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (n = 1, 2, \ldots)
\]
where \( \{x_k\} \) is a sequence of a real or complex numbers. Therefore, the \( C_\lambda \)-method yields a subsequence of the Cesàro method \( C_1 \), and hence it is regular for any \( \lambda \). \( C_\lambda \) is obtained by deleting a set of rows from Cesàro matrix. The basic properties of \( C_\lambda \)-method can be found in [2].

Now, the concept of \( \alpha \)-strongly Cesàro submethod for a double sequence is defined, and several theorems on this subject are given.

**Definition 2.2.** Let the index sequences \( \lambda(n) \) and \( \mu(m) \) are strictly increasing single sequences of positive integers and \( x = (x_{kl}) \) be a double sequence. Then, \( \alpha \)-strongly Cesàro submethod \( (C_{\lambda,\mu}^\alpha) \) is defined as
\[
(C_{\lambda,\mu}^\alpha x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{k=1, l=1}^{\lambda(n), \mu(m)} |x_{kl} - L|^\alpha.
\]
A double sequence \( x = (x_{kl}) \) is said to be \( C_{\lambda,\mu}^\alpha \)-summable to \( L \in \mathbb{N} \) if
\[
\lim_{n,m \to \infty} \frac{1}{\lambda(n)\mu(m)} \sum_{k=1, l=1}^{\lambda(n), \mu(m)} |x_{kl} - L|^\alpha = 0
\]
even exists and it is denoted by \( \lim_{n,m \to \infty} (C_{\lambda,\mu}^\alpha x)_{nm} = L \).

We now examine inclusion relationships between \( C_{\lambda,\mu}^\alpha \) and \( D_{\beta,\gamma}^\alpha \). Let us denote \( D_{\lambda,\mu} \) denote the \( \alpha \)-strongly deferred Cesàro mean \( D_{\beta,\gamma} \) in which \( p(n) = \lambda(n-1) \), \( q(n) = \lambda(n) \), \( t(m) = \mu(m-1) \) and \( r(m) = \mu(m) \).

**Theorem 2.8.** Let \( \{\lambda(n)\}_{n \in \mathbb{N}}, \{\mu(m)\}_{m \in \mathbb{N}} \) are increasing sequences of positive integers and \( \lambda(0) = 0, \mu(0) = 0 \). If a sequence \( x = (x_{kl}) \) is an \( D_{\lambda,\mu}^\alpha \)-convergent to \( L \), then it is an \( C_{\lambda,\mu}^\alpha \)-convergent to \( L \).
Proof. We are going to use the same technique used by [1]. Assume that the double sequence \(x = (x_{kl})\) is a \(D_{\lambda,\mu}^\alpha\)-convergent to \(L\). So, for any \(n, m \in \mathbb{N}\) and we have

\[
\frac{1}{\lambda(n) \mu(m)} \sum_{(k,l)=(1,1)}^{(n-1,m-1)} |x_{kl} - L|^\alpha
\]

\[
= \sum_{(i,j)=(0,0)}^{(n-1,m-1)} \left( \frac{1}{\lambda(n) \mu(m)} \sum_{(k,l)=(\lambda(i)+1,\mu(j)+1)}^{\lambda(i), \mu(j)} |x_{kl} - L|^\alpha \right)
\]

\[
= \sum_{(i,j)=(0,0)}^{(n-1,m-1)} \frac{\Delta \lambda(i)}{\lambda(n) \mu(m)} \frac{\Delta \mu(i)}{\lambda(i) \mu(i)} \sum_{k=\lambda(i)+1}^{\lambda(i+1), \mu(j+1)} \sum_{l=\mu(j)+1}^{l} |x_{kl} - L|^\alpha
\]

where \(\Delta \lambda(i) = \lambda(i + 1) - \lambda(i)\), \(\Delta \mu(i) = \mu(i + 1) - \mu(i)\),

\[
(D_{\lambda,\mu} x)_{ij} = \frac{1}{\Delta \lambda(i) \Delta \mu(i)} \sum_{k=\lambda(i)+1}^{\lambda(i+1), \mu(j+1)} \sum_{l=\mu(j)+1}^{l} |x_{kl} - L|^\alpha
\]

and

\[
b_{i,m} = \begin{cases} \frac{\Delta \lambda(i)}{\lambda(n) \mu(m)}, & i = 1, 2, \ldots n - 1 \text{ and } j = 1, 2, \ldots m - 1 \\ 0, & \text{otherwise}. \end{cases}
\]

Since the matrix regular and \(\lim_{i,j \to \infty} (D_{\lambda,\mu} x)_{ij} = 0\), then

\[
\lim_{n,m \to \infty} \frac{1}{\lambda(n) \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n), \mu(m)} |x_{kl} - L|^\alpha = 0.
\]

It means that the sequence \(x = (x_{kl})\) is a \(C_{\lambda,\mu}^\alpha\)-convergent to \(L\). \(\square\)

**Theorem 2.9.** Let \(\{\lambda(n)\}_{n \in \mathbb{N}}, \{\mu(m)\}_{m \in \mathbb{N}}\) are increasing sequences of positive integers and \(\lambda(0) = 0, \mu(0) = 0\). The \(C_{\lambda,\mu}^\alpha\)-convergent sequence can be a \(D_{\lambda,\mu}^\alpha\)-convergent only if

\[
\lim_{n \to \infty} \frac{\lambda(n)}{\lambda(n-1)} > 1 \quad \text{and} \quad \liminf_{m \to \infty} \frac{\mu(m)}{\mu(m-1)} > 1.
\]

If the accept that \(p(n) = \lambda(n-1), q(n) = \lambda(n), t(m) = \mu(m-1)\) and \(r(m) = \mu(m)\) in Theorem 2.5 then we have

\[
\frac{\lambda(n-1)}{\lambda(n) - \lambda(n-1)} = \frac{1}{\frac{\lambda(n)}{\lambda(n-1)} - 1}
\]

and

\[
\frac{\mu(m-1)}{\mu(m) - \mu(m-1)} = \frac{1}{\frac{\mu(m)}{\mu(m-1)} - 1}.
\]
The proof of Theorem 2.11 becomes clear with this fact. So, it is omitted here.

References

İ. Dağdurban
Department of Mathematics, Faculty of Sciences and Arts, Mersin University, Mersin, 33343 Turkey

E-mail address: ilhandagduran@yahoo.com