Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 3 (2017), Pages 43-53.

ON STRONGLY DEFERRED CESÀRO MEAN OF DOUBLE SEQUENCES

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Abstract. In this paper, the concepts of α-strongly deferred Cesàro mean and α-strongly Cesàro submethod for double sequences are defined and studied by using deferred double natural density of the subset of natural numbers. Also, we consider the case $\beta(n) = \lambda(n) - \lambda(n-1)$, $\gamma(m) = \mu(m) - \mu(m-1)$ for α-strongly deferred Cesàro mean $D^{\alpha}_{\beta,\gamma}$ where $\lambda := \{\lambda(n)\}_{n=1}^{\infty}$ and $\mu = \{\mu(m)\}_{n=1}^{\infty}$ are strictly increasing sequences of positive integers with $\lambda(0) = 0$ and $\mu(0) = 0$. Finally, we obtain some inclusion results between α-strongly Cesàro submethod and α-strongly deferred Cesàro mean $D^{\alpha}_{\beta,\gamma}$ of the double sequences.

1. Introduction

The concept of statistical convergence was first introduced by Fast [8] and also independently by Buck [3] and Schoenberg [19] for real and complex sequences. Further, this concept was studied by Šalát[18], Fridy [9] and many others. Some equivalence results for Cesàro submethods have been studied by Goffman and Petersen [11], Armitage and Maddox [2] and Osikiewicz [15]. In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

Let $0 < \alpha < \infty$ be a real number. A sequence $x = (x_k)$ is said to be α-strongly deferred Cesàro summable to $L \in \mathbb{N}$ if

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - L|^{\alpha} = 0$$

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2000 *Mathematics Subject Classification.* 40A05, 40C05, 40B05, 40D25.

Key words and phrases. Statistical convergence, Deferred Cesáro mean, Double sequence, Deferred statistical convergence.

Submitted January 13, 2017. May 4, 2017.
Submitted by Mikail Et.

exist, where $\{p(n)\}$, $\{q(n)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$ [25].

In [21], the first study on double sequences was examined by Bromwich. And than it was investigated by many authors such as Hardy [12], Moricz [13], Tripathy [24], Başarır and Sonalcan [20]. The notion of regular convergence for double sequences was defined by Hardy [12]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [27]. The statistical and Cauchy convergence for double sequences were examined by Mursaleen and Edely [14] and Tripathy [23] in recent years. Many recent improvements containing the summability by four dimensional matrices might be found in [16]. We begin with some definitions and notations. By the convergence of a double sequence we mean the convergence in Pringsheim's sense [17]. A double sequence $x = (x_{kl})$ is said to be convergent in the Pringsheim's sense if for all $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{kl} - L| < \varepsilon$ whenever $k, l \geq n_0$ [17]. In this case, we write $P - \lim_{k,l\to\infty} x_{kl} = L$. A double sequence $x = (x_{kl})$ is bounded if there exist a positive number M such that $|x_{kl}| < M$ holds for all $(k,l) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$, i.e., if

$$\| x \|_{(\infty,2)} := \sup_{k,l} |x_{kl}| < \infty.$$

We will denote the set of all bounded double sequences by $\mathcal{M}_u$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. In [14], let $K \subset \mathbb{N}^2$ be a two-dimensional set of positive integers and let

$$K(n,m) := \{(k,l) \in K \ : \ (k,l) \leqslant (n,m)\}$$

where $(k,l) \leqslant (n,m)$ means that $k \leq n, \ l \leqslant m$.

Then, the upper asymptoic density of the set $K \subset \mathbb{N}^2$ is defined as

$$\delta^{*(2)}(K(n,m)) := \lim \sup_{n,m\to\infty} \frac{|K(n,m)|}{nm},$$

if the limit exists and finite. The vertical bars above indicate the cardinality of the set $K(n,m)$. In case the sequence $\left(\frac{|K(n,m)|}{nm}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K(n,m)) := \lim_{n,m\to\infty} \frac{|K(n,m)|}{nm}.$$

Following Mursaleen [14] we say that a double sequence $x = (x_{kl})$ is statistically convergent to the number L if for each $\epsilon > 0$

$$\lim_{n,m\to\infty} \frac{1}{nm} |\{(k,l) : k \leq n, \ l \leq m, | \, x_{kl} - L \, | \geq \epsilon\}| = 0.$$

In this case, we write $st_2 - \lim_{k,l\to\infty} x_{kl} = L$ and we denote the set of all double statistically convergent sequences by st_2.

Let $x = (x_{kl})$ be a double sequence and p be a positive real number. Then the double sequence x is said to be strongly p-Cesàro summable to L if

$$\lim_{n,m} \frac{1}{nm} \sum_{k=1}^{n} \sum_{l=1}^{m} |x_{kl} - L|^p = 0.$$

We denote the space of all strongly p-Cesàro summable double sequences by ω_p^2 [5].

Definition 1.1. ([6]) Let $x = (x_{kl})$ be a double sequence and $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$. Then deferred Cesàro mean $D_{\beta,\gamma}$ of the double sequence x is defined by

$$(D_{\beta,\gamma}x)_{nm} = \frac{1}{\beta(n)\ \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} x_{kl} = \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} x_{kl}$$

where $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$, $t(m) < r(m)$ and $\lim_{n\to\infty} q(n) = \infty$, $\lim_{m\to\infty} r(m) = \infty$. We note that $D_{\beta,\gamma}$ is clearly regular for any choice of $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$.

Throughout this paper $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$ are represented β and γ respectively.

Let $x = (x_{kl})$ be a double sequence and a real number L. Then, the double sequence x is said to be $D_{\beta,\gamma}$-summable to L if

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} (x_{kl} - L) = 0$$

exists and it is denoted by $\lim\limits_{n,m\to\infty} (D_{\beta,\gamma}x)_{nm} = L$ [6].

Let K be a subset of $\mathbb{N}^2$ and denote the set

$$\{(k,l) : p(n) < k \le q(n),\ t(m) < l \le r(m),\ (k,\ l) \in K\}$$

by $K_{\beta,\gamma}(n,m)$. The deferred double natural density of K is defined by

$$\delta^{(2)}_{D_{\beta,\gamma}}(K) := \lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} |K_{\beta,\gamma}(n,\ m)|$$

whenever the limit exists. The vertical bars indicate the cardinality of the set $K_{\beta,\gamma}(n,m)$. Also, because of $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ does not exists for all $K \subset \mathbb{N}^2$, it is convenient to use upper deferred asymptotic density of K, defining by

$$\delta^{*(2)}_{D_{\beta,\gamma}}(K) = \limsup_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ (k,l) \in K\}|}{\beta(n)\ \gamma(m)}.$$

It is clear that, for the function $\delta^{*(2)}_{D_{\beta,\gamma}}(K)$ the following axioms are hold:

i) if $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ exists, then $\delta^{(2)}_{D_{\beta,\gamma}}(K) = \delta^{*(2)}_{D_{\beta,\gamma}}(K)$,

ii) $\delta^{(2)}_{D_{\beta,\gamma}}(K) \neq 0$ if and only if $\delta^{*(2)}_{D_{\beta,\gamma}}(K) > 0$ and

iii) The function $\delta^{*(2)}_{D_{\beta,\gamma}}(K)$ is monotone increasing.

A double sequence $x = (x_{kl})$ is said to be deferred statistically convergent to $L \in \mathbb{N}$ if for every $\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} = 0$$

and it is denoted by $st_2 - \lim\limits_{n,m\to\infty} (D_{\beta,\gamma}x)_{nm} = L$ [6].

2. α-strongly deferred Cesáro mean

Definition 2.1. Let $x = (x_{kl})$ be a double sequence and $0 < \alpha < \infty$ be a real number. Then, the double sequence $x = (x_{kl})$ is said to be α-strongly deferred Cesàro summable to $L \in \mathbb{N}$ if

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L|^{\alpha} = 0$$

exists and it is denoted by $\lim_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x\right)_{nm} = L$.

Theorem 2.1. *Let $x = (x_{kl})$ be a double sequence and $0 < \alpha < \infty$ be a real number. If $\lim_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x\right)_{nm} = L$ then, the double sequence x is deferred statistical convergent to L.*

Proof. Assume that $\lim_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x\right)_{nm} = L$ and denote set

$$\{(k,l) : p(n) + 1 \leq k \leq q(n), t(m) + 1 \leq l \leq r(m), |x_{kl} - L| \geq \varepsilon\}$$

by $K(\varepsilon)$. Therefore, the inequality

$$\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L|^{\alpha} \geq \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ (k,l)\in K(\varepsilon)}}^{q(n),\ r(m)} |x_{kl} - L|^{\alpha}$$

$$\geq \quad \varepsilon^{\alpha} \frac{1}{\beta(n)\ \gamma(m)} |K(\varepsilon)|$$

holds. After taking limits when $n, m \to \infty$ the proof of theorem is obtained. □

Now we get following

Corollary 2.2. *Let $q(n) = n$, $r(m) = m$. If $\lim_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x\right)_{nm} = L$, then $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Theorem 2.3. *Let a double sequence $x = (x_{kl})$ be bounded. If it is deferred statistical convergent to L, then $\lim_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x\right)_{nm} = L$.*

Proof. We denote the complement of $K(\varepsilon)$ by

$$K^c(\varepsilon) := \{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ |x_{kl} - L| < \varepsilon\}.$$

We assume $x = (x_{kl})$ is bounded and deferred statistical convergent to L. By hypothesis, it is clear that there is a positive real number M such that $|x_{kl} - L| \leq M$ for all $k, l \in \mathbb{N}$. Therefore, we have

$$
\begin{aligned}
&\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl}-L|^{\alpha} \\
= \ & \frac{1}{\beta(n)\ \gamma(m)} \left[\sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ (k,l)\in K(\varepsilon)}}^{q(n),\ r(m)} |x_{kl}-L|^{\alpha} + \sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ (k,l)\notin K(\varepsilon)}}^{q(n),\ r(m)} |x_{kl}-L|^{\alpha} \right] \\
\leq \ & \frac{1}{\beta(n)\ \gamma(m)} \left[\sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ (k,l)\in K(\varepsilon)}}^{q(n),\ r(m)} M^{\alpha} + \sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ (k,l)\notin K(\varepsilon)}}^{q(n),\ r(m)} \varepsilon^{\alpha} \right] \\
\leq \ & \frac{1}{\beta(n)\ \gamma(m)} [M^{\alpha}|K(\varepsilon)| + \varepsilon^{\alpha}|K^{c}(\varepsilon)|].
\end{aligned}
$$

Since

$$
\lim_{n,m\to\infty} \frac{|K(\varepsilon)|}{\beta(n)\ \gamma(m)} = 0 \quad \text{and} \quad \lim_{n,m\to\infty} \frac{|K^{c}(\varepsilon)|}{\beta(n)\ \gamma(m)} = 1,
$$

this proves theorem. □

Our next result is obtained from Theorem 2.3.

Corollary 2.4. *Let $\{q(n)\}$, $\{r(m)\}$ be arbitrary strictly increasing sequences and*

$$
\left\{ \frac{p(n)}{\beta(n)} \right\} \quad \text{and} \quad \left\{ \frac{t(m)}{\gamma(m)} \right\}
$$

are bounded sequence. If $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ for a bounded double sequence $x = (x_{kl})$, then $\lim\limits_{n,m\to\infty} \left(D^{\alpha}_{\beta,\gamma} x \right)_{nm} = L$.

Now we have

Theorem 2.5. *The α-strongly deferred convergent double sequence $x = (x_{kl})$ is an α-strongly Cesàro convergent only if*

$$
\left\{ \frac{p(n)}{\beta(n)} \right\} \quad \text{and} \quad \left\{ \frac{t(m)}{\gamma(m)} \right\}
$$

are bounded.

Proof. The technique that was used by Agnew in [1] can be applied for this proof. Let us assume that $x = (x_{kl})$ is an α-strongly Cesàro convergent to L. In this case, the following equality

$$
\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} |x_{kl}-L|^{\alpha}
$$

$$= \frac{1}{\beta(n)\ \gamma(m)} \left[\sum_{(k,l)=(1,1)}^{q(n),r(m)} - \sum_{\substack{k=1 \\ l=t(m)+1}}^{p(n),r(m)} - \sum_{\substack{k=p(n)+1 \\ l=1}}^{q(n),t(m)} - \sum_{(k,l)=(1,1)}^{p(n),t(m)} \right] |x_{kl} - L|^{\alpha}$$

$$= \frac{1}{\beta(n)\ \gamma(m)} \left[\sum_{(k,l)=(1,1)}^{q(n),r(m)} - \sum_{(k,l)=(1,1)}^{p(n),r(m)} - \sum_{(k,l)=(1,1)}^{q(n),t(m)} + \sum_{(k,l)=(1,1)}^{p(n),t(m)} \right] |x_{kl} - L|^{\alpha}$$

$$\begin{aligned} &= \frac{1}{\beta(n)\ \gamma(m)} \left[\sum_{(k,l)=(1,1)}^{q(n),r(m)} - \sum_{(k,l)=(1,1)}^{p(n),r(m)} - \sum_{(k,l)=(1,1)}^{q(n),t(m)} + \sum_{(k,l)=(1,1)}^{p(n),t(m)} \right] |x_{kl} - L|^{\alpha} \\ &= \frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{q(n)\ r(m)} \sum_{(k,l)=(1,1)}^{q(n),r(m)} |x_{kl} - L|^{\alpha} \right) \\ &\quad - \frac{p(n)\ r(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{p(n)\ r(m)} \sum_{(k,l)=(1,1)}^{p(n),r(m)} |x_{kl} - L|^{\alpha} \right) \\ &\quad - \frac{q(n)\ t(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{q(n)\ t(m)} \sum_{(k,l)=(1,1)}^{q(n),t(m)} |x_{kl} - L|^{\alpha} \right) \\ &\quad + \frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{p(n)\ t(m)} \sum_{(k,l)=(1,1)}^{p(n),t(m)} |x_{kl} - L|^{\alpha} \right) \end{aligned}$$

holds. It can be said that the α-strongly deferred convergence of the double sequence $x = (x_{kl})$ is the linear combination of the α-strongly Cesàro convergence of the double sequence $x = (x_{kl})$. We can consider this linear combination as a matrix transformation. For the regularity of this matrix transformation the sequence

$$\left\{ \frac{(q(n) + p(n))\ \ (r(m) +\ t(m))}{\beta(n)\ \gamma(m)} \right\} \tag{1}$$

must be bounded. For the boundedness of (1)

$$\frac{p(n)}{q(n) - p(n)} \quad \text{and} \quad \frac{t(m)}{r(m) - t(m)}$$

must be bounded since

$$\begin{aligned} &\frac{(q(n) + p(n))\ \ (r(m) +\ t(m))}{\beta(n)\ \gamma(m)} \\ = &\frac{(q(n) + p(n))\ \ (r(m) +\ t(m))}{(q(n) - p(n))\ (r(m) - t(m))} \\ = &\frac{(q(n) - p(n) + 2p(n))\ \ (r(m) -\ t(m) + 2t(m))}{(q(n) - p(n))\ (r(m) - t(m))} \\ = &\left(1 + \frac{2p(n)}{q(n) - p(n)} \right) \left(1 + \frac{2t(m)}{r(m) - t(m)} \right). \end{aligned}$$

The assertion completes the proof. □

Now, in the following theorems $D^{\alpha}_{\beta,\gamma}$-convergence and $D^{\alpha}_{\beta',\gamma'}$-convergence of the double sequence $x=(x_{kl})$ are compared under the restriction

$$p(n) \leq p'(n) < q'(n) \leq q(n) \tag{2}$$

$$t(m) \leq t'(m) < r'(m) \leq r(m) \tag{3}$$

for all $n,m \in \mathbb{N}$.

Theorem 2.6. *$\{p'(n)\}$, $\{q'(n)\}$, $\{t'(m)\}$ and $\{r'(m)\}$ be sequences of positive natural numbers satisfying* (2), (3) *and the sets $\{k: \ p(n) < k \leq p'(n)\}$, $\{k : q'(n) < k \leq q(n)\}$, $\{k : t(m) < k \leq t'(m)\}$, $\{k : r'(m) < k \leq r(m)\}$ are finite for all $n,m \in \mathbb{N}$. Then, $D^{\alpha}_{\beta',\gamma'}$-convergence of a bounded double sequence implies $D^{\alpha}_{\beta,\gamma}$-convergence.*

Proof. There is a positive real number M in the assumption such that $|x_{kl}-L| \leq M$ which holds for all $k,l \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
&\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} |x_{kl}-L|^{\alpha}\\
= \ &\frac{1}{\beta(n)\ \gamma(m)} \left(\sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{p'(n),t'(m)} + \sum_{\substack{k=p'(n)+1\\ l=t'(m)+1}}^{q'(n),r'(m)} + \sum_{\substack{k=q'(n)+1\\ l=r'(m)+1}}^{q(n),r(m)} + \sum_{\substack{k=p'(n)+1\\ l=t(m)+1}}^{q'(n),t'(m)} \right) |x_{kl}-L|^{\alpha}\\
+ \ &\frac{1}{\beta(n)\ \gamma(m)} \left(\sum_{\substack{k=p(n)+1\\ l=t'(m)+1}}^{p'(n),r'(m)} + \sum_{\substack{k=p(n)+1\\ l=r'(m)+1}}^{p'(n),r(m)} + \sum_{\substack{k=q'(n)+1\\ l=t(m)+1}}^{q(n),t'(m)} + \sum_{\substack{k=p'(n)+1\\ l=r'(m)+1}}^{q'(n),r(m)} + \sum_{\substack{k=q'(n)+1\\ l=t'(m)+1}}^{q(n),r'(m)} \right) |x_{kl}-L|^{\alpha}\\
\leq \ &\frac{8}{\beta'(n)\ \gamma'(m)} M^{\alpha} O(1) + \frac{1}{\beta'(n)\ \gamma'(m)} \sum_{\substack{k=p'(n)+1\\ l=t'(m)+1}}^{q'(n),r'(m)} |x_{kl}-L|^{\alpha}
\end{aligned}$$

If we take the limit, we obtain the double sequence $x=(x_{kl})$ which is a $D^{\alpha}_{\beta,\gamma}$-convergent. □

Theorem 2.7. *Let $\{p'(n)\}$, $\{q'(n)\}$, $\{t'(m)\}$ and $\{r'(m)\}$ be sequences of positive natural numbers satisfying* (2), (3) *and*

$$\lim_{n,m\to\infty} \frac{\beta'(n)\ \gamma'(m)}{\beta(n)\ \gamma(m)} = d > 0.$$

Then, the $D^{\alpha}_{\beta,\gamma}$-convergence of the double sequence $x=(x_{kl})$ implies $D^{\alpha}_{\beta',\gamma'}$-convergence.

Proof. It is easy to see that the inequality

$$\begin{aligned} & \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} |x_{kl} - L|^{\alpha} \\ \geq & \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p'(n)+1 \\ l=t'(m)+1}}^{q'(n),r'(m)} |x_{kl} - L|^{\alpha} \\ \geq & \frac{\beta'(n)\ \gamma'(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{\beta'(n)\ \gamma'(m)} \sum_{\substack{k=p'(n)+1 \\ l=t'(m)+1}}^{q'(n),r'(m)} |x_{kl} - L|^{\alpha} \right) \end{aligned}$$

holds. After taking limit when $n, m \to \infty$, we get that the double sequence $x = (x_{kl})$ is a $D^{\alpha}_{\beta',\gamma'}$ convergent to L. □

Recall that if F is an infinite subset of $\mathbb{N}$ and F as the range of a strictly increasing sequence of positive integers, say $F = \{\lambda(n)\}_{n=1}^{\infty}$, the Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \qquad (n = 1, 2, \ldots)$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesàro matrix. The basic properties of C_λ-method can be found in [2].

Now, the concept of α-strongly Cesàro submethod for a double sequence is defined, and several theorems on this subject are given.

Definition 2.2. Let the index sequences $\lambda(n)$ and $\mu(m)$ are strictly increasing single sequences of positive integers and $x = (x_{kl})$ be a double sequence. Then, $\alpha-$strongly Cesàro submethod $\left(C^{\alpha}_{\lambda,\mu}\right)$ is defined as

$$(C_{\lambda,\mu} x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{k=1,l=1}^{\lambda(n),\mu(m)} |x_{kl} - L|^{\alpha}.$$

A double sequence $x = (x_{kl})$ is said to be $C^{\alpha}_{\lambda,\mu}-$summable to $L \in \mathbb{N}$ if

$$\lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \sum_{k=1,l=1}^{\lambda(n),\mu(m)} |x_{kl} - L|^{\alpha} = 0$$

exists and it is denoted by $\lim_{n,m\to\infty} \left(C^{\alpha}_{\lambda,\mu} x\right)_{nm} = L$.

We now examine inclusion relationships between $C^{\alpha}_{\lambda,\mu}$ and $D^{\alpha}_{\beta,\gamma}$. Let us denote $D_{\lambda,\mu}$ denote the α-strongly deferred Cesàro mean $D_{\beta,\gamma}$ in which $p(n) = \lambda(n-1)$, $q(n) = \lambda(n)$, $t(m) = \mu(m-1)$ and $r(m) = \mu(m)$.

Theorem 2.8. *Let $\{\lambda(n)\}_{n\in\mathbb{N}}$, $\{\mu(m)\}_{m\in\mathbb{N}}$ are increasing sequences of positive integers and $\lambda(0) = 0$, $\mu(0) = 0$. If a sequence $x = (x_{kl})$ is an $D^{\alpha}_{\lambda,\mu}$-convergent to L, then it is an $C^{\alpha}_{\lambda,\mu}$-convergent to L.*

Proof. We are going to use the same technique used by [1]. Assume that the double sequence $x = (x_{kl})$ is a $D^{\alpha}_{\lambda,\mu}$-convergent to L. So, for any $n, m \in \mathbb{N}$ and we have

$$\begin{aligned}
&\frac{1}{\lambda(n)\ \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n),\ \mu(m)} |x_{kl} - L|^{\alpha} \\
&= \sum_{(i,j)=(0,0)}^{(n-1,m-1)} \left(\frac{1}{\lambda(n)\ \mu(m)} \sum_{(k,l)=(\lambda(i)+1,\mu(j)+1)}^{\lambda(i+1),\ \mu(j+1)} |x_{kl} - L|^{\alpha} \right) \\
&= \sum_{(i,j)=(0,0)}^{(n-1,m-1)} \frac{\Delta\lambda(i)\ \Delta\mu(i)}{\lambda(n)\ \mu(m)\ \Delta\lambda(i)\ \Delta\mu(i)} \sum_{\substack{k=\lambda(i)+1 \\ l=\mu(j)+1)}}^{\lambda(i+1),\ \mu(j+1)} |x_{kl} - L|^{\alpha} \\
&= \sum_{(i,j)=(0,0)}^{(n-1,m-1)} b_{nmij} \left(D_{\lambda,\mu} x\right)_{ij}
\end{aligned}$$

where $\Delta\lambda(i) = \lambda(i+1) - \lambda(i),\ \Delta\mu(i) = \mu(j+1) - \mu(i),$

$$\left(D_{\lambda,\mu} x\right)_{ij} = \frac{1}{\Delta\lambda(i)\ \Delta\mu(i)} \sum_{\substack{k=(\lambda(i)+1 \\ l=\mu(j)+1)}}^{\lambda(i+1),\ \mu(j+1)} |x_{kl} - L|^{\alpha}$$

and

$$b_{nmij} = \begin{cases} \frac{\Delta\lambda(i)\ \Delta\mu(i)}{\lambda(n)\ \mu(m)}, & i = 1, 2, \ldots n-1 \text{ and } j = 1, 2, \ldots m-1 \\ 0, & \text{otherwise.} \end{cases}$$

Since the matrix regular and $\lim_{i,j\to\infty} \left(D_{\lambda,\mu} x\right)_{ij} = 0$, then

$$\lim_{n,m\to\infty} \frac{1}{\lambda(n)\ \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n),\ \mu(m)} |x_{kl} - L|^{\alpha} = 0.$$

It means that the sequence $x = (x_{kl})$ is a $C^{\alpha}_{\lambda,\mu}$-convergent to L. □

Theorem 2.9. *Let $\{\lambda(n)\}_{n\in\mathbb{N}}$, $\{\mu(m)\}_{m\in\mathbb{N}}$ are increasing sequences of positive integers and $\lambda(0) = 0,\ \mu(0) = 0$. The $C^{\alpha}_{\lambda,\mu}$-convergent sequence can be a $D^{\alpha}_{\lambda,\mu}$-convergent only if*

$$\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1 \quad \text{and} \quad \liminf_{m\to\infty} \frac{\mu(m)}{\mu(m-1)} > 1.$$

If the accept that $p(n) = \lambda(n-1)$, $q(n) = \lambda(n)$, $t(m) = \mu(m-1)$ and $r(m) = \mu(m)$ in Theorem 2.5 then we have

$$\frac{\lambda(n-1)}{\lambda(n) - \lambda(n-1)} = \frac{1}{\frac{\lambda(n)}{\lambda(n-1)} - 1}$$

and

$$\frac{\mu(m-1)}{\mu(m) - \mu(m-1)} = \frac{1}{\frac{\mu(m)}{\mu(m-1)} - 1}.$$

The proof of Theorem 2.11 becomes clear with this fact. So, it is omitted here.

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