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COMPLEX VALUED S_b-METRIC SPACES

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Abstract. Inspired by the concept of S_b-metric spaces which was introduced by Souayah and Mlaiki in [8]. In this paper, we define complex valued S_b-metric spaces. Along with the topology of this space, we prove very interesting fixed point theorems. Examples are also given as a support of our results.

1. Introduction

In 2011, Azam et. al. [1] introduced the concepts of complex valued metric spaces and proved some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. After, the establishment of complex valued metric spaces, Rouzkard et. al. [4] established some common fixed point theorems satisfying certain rational expressions in these spaces to generalize the result of [1]. Subsequently Sintunavarat et. al. [5, 10] obtained common fixed point results by replacing the constant of contractive condition to control functions. In 2012, Sitthikul et. al. [6] prove some fixed point results by generalising the contractive conditions in the context of complex valued metric spaces and Rao et. al. [7] introduced the complex valued b-metric spaces. Many researchers have contributed with different approaches in this space, which can be found in [11, 12, 13, 14, 15, 16, 17, 18, 19]. Moreover, Shin Min Kang et. al. [9] introduced the notion of complex valued G-metric spaces and proved contraction principle in this space. In 2014, Nabil M. Mlaiki [29] introduced the complex valued S-Metric space and proved the existence and the uniqueness of a common fixed point of two self mappings in this space. Recently, Ozgur [3] introduced the concept of complex valued G_b-metric spaces and proved Banach contraction principle and Kannan's fixed point theorem in this newly space.

Note that, complex valued metric spaces form a special class of cone metric space, but our contraction which has a product and quotient of S_b-metric spaces cannot be extended to a cone metric.

Recently, Nizar Souayah and Nabil Mlaiki in [8], introduced the concept of S_b-metric spaces as a generalization of metric space and subsequently S-metric space introduced by Sedghi et. al. [2]. For more results on generalisation of metric spaces

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one can see results in [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38].

2. Preliminaries

In this section we recall some properties of S_b-metric spaces.

Definition 2.1. [8] *Let X be a nonempty set and let $b \geq 1$ be a given number. A function $S : X^3 \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, t \in X$; the following conditions hold:*

(i): $S(x, y, z) = 0$ *if and only if* $x = y = z$,
(ii): $S(x, x, y) = S(y, y, x)$ *for all* $x, y \in X$,
(iii): $S(x, y, z) \leq b[S(x, x, t) + S(y, y, t) + S(z, z, t)]$

the pair (X, S) is called an S_b-metric space.

Remark 1. *Note that the class of S_b-metric spaces is larger than the class of S-metric spaces. Indeed, every S-metric space is an S_b-metric space with $b = 1$. However, the converse is not always true.*

Definition 2.2. [8] *Let (X, S) be an S_b-metric space and $\{x_n\}$ be a sequence in X. Then*

(1) *a sequence $\{x_n\}$ is called convergent if and only if there exists $z \in X$ such that $S(x_n, x_n, z) \to 0$ as $n \to \infty$. In this case we write $\lim_{n\to\infty} x_n = z$.*
(2) *a sequence $\{x_n\}$ is called a Cauchy sequence if and only if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.*
(3) *(X, S) is said to be a complete S_b-metric space if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that*
$$\lim_{n,m\to\infty} S(x_n, x_n, x_m) = \lim_{n,m\to\infty} S(x_n, x_n, x) = S(x, x, x).$$

The concept of complex valued metric space was initiated by Azam et.al. [1]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

If follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied :

(C_1)**:** $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C_2)**:** $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C_3)**:** $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(C_4)**:** $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one (C_2), (C_3) and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.
(2) If $0 \lesssim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
(3) If $z_1 \lesssim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

In the next section, we introduce complex valued S_b-metric spaces a long with our main result.

3. Complex Valued S_b-metric spaces

First, we give the definition of the complex valued S_b-metric space.

Definition 3.1. *Let X be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S : X^3 \to \mathbb{C}$ satisfies:*

(CS_b1): $0 \prec S(x, y, z)$ *for all* $x, y, z \in X$ *with* $x \neq y \neq z \neq x$,
(CS_b2): $S(x, y, z) = 0 \Leftrightarrow x = y = z$,
(CS_b3): $S(x, x, y) = S(y, y, x)$, *for all* $x, y \in X$,
(CS_b4): $S(x, y, z) \precsim b(S(x, x, a) + S(y, y, a) + S(z, z, a))$ *for all* $x, y, z, a \in X$.

Then, S is called a complex valued S_b-metric and (X, S) is called a complex valued S_b-metric space.

Definition 3.2. *Let (X, S) be a complex valued S_b-metric space, let $\{x_n\}$ be a sequence in X.*

(i): *$\{x_n\}$ is a complex valued S_b-convergent to x if for every $a \in \mathbb{C}$ with $0 < a$, there exists $k \in \mathbb{N}$ such that $S(x_n, x_n, x) \prec a$ or $S(x, x, x_n) \prec a$ for all $n \succeq k$ and denoted by $\lim_{n\to\infty} x_n = x$.*
(ii): *A sequence $\{x_n\}$ is called complex valued S_b-Cauchy if for every $a \in \mathbb{C}$ with $0 < a$,there exists $k \in \mathbb{N}$ such that $S(x_n, x_n, x_m) \prec a$ for each $m, n \geq k$.*
(iii): *If every complex valued S_b-Cauchy sequence is complex valued S_b-convergent in (X, S), then (X, S) is said to be complex valued S_b-complete.*

Proposition 3.1. *Let (X, S) be a complex valued S_b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued S_b-convergent to x if and only if $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$ or $|S(x, x, x_n)| \to 0$ as $n \to \infty$.*

Proof. ($\Rightarrow$) Assume that $\{x_n\}$ is a complex valued S_b-convergent to x and let

$$a = \tfrac{\varepsilon}{\sqrt{2}} + i\tfrac{\varepsilon}{\sqrt{2}}$$

where $\varepsilon > 0$ is a real number. Then $0 \prec a \in \mathbb{C}$ and there is a natural number k such that $S(x_n, x_n, x) \prec a$ for all $n \geq k$. Thus, $|S(x_n, x_n, x)| < |a| = \varepsilon$ for all $n \geq k$ and so $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.

($\Leftarrow$) Suppose that $|S(x_n, x_n, x)| \to 0$ as $n \to \infty$. For a given $a \in \mathbb{C}$ with $0 \prec a$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec a.$$

Considering δ, we have a natural number k such that $|S(x_n, x_n, x)| < \delta$ for all $n \geq k$. This means that $S(x_n, x_n, x) \prec a$ for all $n \geq k$ i.e $\{x_n\}$ is complex valued S_b-convergent to x.

Similarly we can prove for the other condition as

$$S(x, x, x_n) \precsim bS(x_n, x_n, x).$$

□

Theorem 3.2. *Let (X, S) be a complex valued S_b-metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$, the following are equivalent*

(1): *$\{x_n\}$ is a complex valued S_b-convergent to x.*
(2): *$|S(x_n, x_n, x)| \to 0$ as $n \to \infty$.*

Proof. (1) $\Rightarrow$ (2). It is cleared from proposition 3.3.
(2) $\Rightarrow$ (1). □

Theorem 3.3. *Let (X, S) be a complex valued S_b-metric space and $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is complex valued S_b-Cauchy sequence if and only if $|S(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.*

Proof. $(\Rightarrow)$ Let $\{x_n\}$ be complex valued S_b-Cauchy sequence and

$$b = \tfrac{\varepsilon}{\sqrt{2}} + i\tfrac{\varepsilon}{\sqrt{2}}$$

where $\varepsilon > 0$ is a real number. Then $0 \prec b \in \mathbb{C}$ and there is a natural number k such that $|S(x_n, x_m, x_l)| \prec b$ for $n, m, l \geq k$. Therefore, we get $|S(x_n, x_m, x_l)| < |b| = \varepsilon$ for $n, m, l \geq k$ and the required result.

$(\Leftarrow)$ Assume that $|S(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$. Then there exists a real number $r > 0$ such that for $z \in \mathbb{C}$

$$|z| < r \text{ implies } z \prec b$$

where $b \in \mathbb{C}$ with $0 \prec b$. For this r there is a natural number k such that $|S(x_n, x_m, x_l)| \prec r$ for all $n, m, l \geq k$. This means that $S(x_n, x_m, x_l) \prec b$ for all $n, m, l \geq k$. Hence $\{x_n\}$ is complex valued S_b-Cauchy sequence. □

Theorem 3.4. *Let (X, S) be a complete complex valued S_b-metric space with coefficient $b > 1$ and $f : X \to X$ be a mapping satisfying*

$$S(fx, fy, fz) \precsim kS(x, y, z) \tag{1}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{b}]$. Then f has a unique fixed point.

Proof. Let f satisfy (1), $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = f^n x_0$. From(1), we obtain

$$S(x_n, x_n, x_{n+1}) \precsim kS(x_{n-1}, x_{n-1}, x_n) \tag{2}$$

Using again (1), we have

$$S(x_{n-1}, x_{n-1}, x_n) \precsim kS(x_{n-2}, x_{n-2}, x_{n-1})$$

and by (2), we get

$$S(x_n, x_n, x_{n+1}) \precsim k^2 S(x_{n-2}, x_{n-2}, x_{n-1})$$

If we continue in this way, we find

$$S(x_n, x_n, x_{n+1}) \precsim k^n S(x_0, x_0, x_1) \tag{3}$$

Using (CS_b4) and (3) for all $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned}
S(x_n, x_n, x_m) &\precsim 2bS(x_n, x_n, x_{n+1}) + bS(x_{n+1}, x_{n+1}, x_m) \\
&\precsim 2bS(x_n, x_n, x_{n+1}) + 2b^2 S(x_{n+1}, x_{n+1}, x_{n+2}) + b^2 S(x_{n+2}, x_{n+2}, x_m) \\
&\precsim 2bS(x_n, x_n, x_{n+1}) + 2b^2 S(x_{n+1}, x_{n+1}, x_{n+2}) + 2b^3 S(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + b^3 S(x_{n+3}, x_{n+3}, x_m) \\
&\precsim 2bS(x_n, x_n, x_{n+1}) + 2b^2 S(x_{n+1}, x_{n+1}, x_{n+2}) + \dots \\
&\quad + b^{m-n} S(x_{m-1}, x_{m-1}, x_m) \\
&\prec 2b\{S(x_n, x_n, x_{n+1}) + bS(x_{n+1}, x_{n+1}, x_{n+2}) + b^2 S(x_{n+2}, x_{n+2}, x_{n+3}) \\
&\quad + \dots + b^{m-n-1} S(x_{m-1}, x_{m-1}, x_m)\} \\
&\precsim 2b(k^n + bk^{n+1} + b^2 k^{n+2} + \dots + b^{m-n-1} k^{m-1}) S(x_0, x_0, x_1) \\
&\precsim 2bk^n (1 + bk + b^2 k^2 + \dots + b^{m-n-1} k^{m-n-1}) S(x_0, x_0, x_1) \\
&\precsim \frac{2bk^n}{1 - bk} S(x_0, x_0, x_1).
\end{aligned}$$

Thus, we obtain

$$|S(x_n, x_n, x_m)| \leq \frac{2bk^n}{1 - bk} |S(x_0, x_0, x_1)|.$$

Since, $k \in [0, \frac{1}{b})$ where $b > 1$, taking limit as $m, n \to \infty$, then

$$\frac{2bk^n}{1 - bk} |S(x_0, x_0, x_1)| \to 0.$$

This means that

$$|S(x_n, x_n, x_m)| \to 0.$$

By (CS_b3) we have

$$S(x_n, x_m, x_l) \precsim b(S(x_n, x_n, x_m) + S(x_l, x_l, x_m))$$

for all $n, m, l \in \mathbb{N}$. Thus

$$|S(x_n, x_m, x_l)| \leq |bS(x_n, x_n, x_m)| + |bS(x_l, x_l, x_m)|.$$

If we take limit as $n, m, l \to \infty$, we obtain $|S(x_n, x_m, x_l)| \to 0$. So $\{x_n\}$ is complex valued S_b-Cauchy sequence by theorem 3.4. Completeness of (X, S) gives us that there is an element $u \in X$ such that $\{x_n\}$ is complex valued S_b-convergent to u.

To prove $fu = u$, we will assume the contrary. From (1), we obtain

$$S(x_{n+1}, x_{n+1}, fu) \precsim kS(x_n, x_n, u)$$

and

$$|S(x_{n+1}, x_{n+1}, fu)| \leq k|S(x_n, x_n, u)|.$$

If we take the limit as $n \to \infty$, we get

$$|S(u, u, fu)| \leq k|S(u, u, u)|,$$

which is a contradiction since $k \in [0, \frac{1}{b})$. As a result, $fu = u$.

Finally, we prove the uniqueness. Let $w \neq u$ be another fixed point of f. Using (1),

$$S(z, z, w) = S(fz, fz, fw) \precsim kS(z, z, w)$$

and

$$|S(z,z,w)| \leq k|S(z,z,w)|.$$

Since $k \in [0, \frac{1}{b})$, we have $|S(z,z,w)| \leq 0$. Thus, $u = w$ and so u is a unique fixed point of f. $\square$

Example 1. *Let $X = [-1,1]$ and $S : X^3 \to \mathbb{C}$ be defined as follows:*

$$S(x,y,z) = |x-y| + |y-z| + |z-x|$$

for all $x,y,z \in X$. (X,S) is complex valued S-metric space.
Define

$$S_*(x,y,z) = S(x,y,z)^2$$

S_* is a complex valued S_b-metric with $b = 2$. If we define $f : X \to X$ as $fx = \frac{x}{3}$, then f satisfies the following condition for all $x,y,z \in X$:

$$\begin{aligned} S(fx,fy,fz) &= S(\frac{x}{3},\frac{y}{3},\frac{z}{3}) \\ &= \frac{1}{3}S(x,y,z) \\ &\precsim kS(x,y,z) \end{aligned}$$

where $k \in [\frac{1}{3}, \frac{1}{b})$, $b > 1$. Thus $x = 0$ is the unique fixed point of f in X.

We will prove Kannan's fixed point theorem for complex valued S_b-metric spaces.

Theorem 3.5. *Let (X,S) be a complete complex valued S_b-metric space and the mapping $f : X \to X$ satisfies for every $x,y \in X$*

$$S(fx,fx,fy) \precsim \alpha(S(x,x,fx) + S(y,y,fy)) \tag{4}$$

where $\alpha \in [0, \frac{1}{2})$. Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. We defined a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is a S_b-Cauchy sequence. If $x_n = x_{n+1}$, then x_n is the fixed point of f. Thus, suppose that $x_n \neq x_{n+1}$ for all $n \leq 0$. Setting $S(x_n,x_n,x_{n+1}) = S_n$, it follows from (4) that

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &= S(fx_{n-1},fx_{n-1},fx_n) \\ &\precsim \alpha(S(x_{n-1},x_{n-1},fx_{n-1}) + S(x_n,x_n,fx_n)) \\ &\precsim \alpha(S(x_{n-1},x_{n-1},x_n) + S(x_n,x_n,fx_{n+1})) \\ &\precsim \alpha(S_{n-1} + S_n) \\ \Rightarrow S_n &\precsim \frac{\alpha}{1-\alpha}S_{n-1} = \beta S_{n-1} \end{aligned}$$

where $\beta = \frac{\alpha}{1-\alpha} < 1$ as $\alpha \in [0, \frac{1}{2})$. If we repeat this process, then we get

$$S_n \precsim \beta^n S_0. \tag{5}$$

We can also suppose that x_0 is not a periodic point.
If $x_n = x_0$, then we have

$$S_0 \precsim \beta^n S_0.$$

Since $\beta < 1$, then $1 - \beta^n < 1$ and

$$\begin{aligned} (1-\beta^n)|S_0| &\leq 0 \\ \Rightarrow |S_0| &= 0. \end{aligned}$$

If follows that x_0 is a fixed point of f. Therefore in the sequel of proof we can assume $f^n x_0 \neq x_0$ for $n = 1, 2, 3, \ldots$. From inequality (4), we obtain

$$\begin{aligned} S(f^n x_0, f^n x_0, f^{n+m} x_0) &\precsim \alpha[S(f^{n-1} x_0, f^{n-1} x_0, f^{n+m} x_0) + S(f^{n+m-1} x_0, f^{n+m-1} x_0, f^{n+m} x_0)] \\ &\precsim \alpha[\beta^{n-1} S(x_0, x_0, f x_0) + \beta^{n+m-1} S(x_0, x_0, f x_0)]. \end{aligned}$$

So, $|S(x_n, x_n, x_{n+m})| \to 0$ as $n \to \infty$. It implies that $\{x_n\}$ is a S_b-Cauchy in X. By the completeness of X, there exists $u \in X$ such that $x_n \to u$. From (CS_b4), we get

$$\begin{aligned} S(fu, fu, u) &\precsim 2bS(fu, fu, f^{n+1} x_0) + b^2 S(f^{n+1} x_0, f^{n+1} x_0, u) \\ &\precsim 2b(\alpha(S(u, u, fu) + S(f^n x_0, f^n x_0, f^{n+1} x_0))) + b^2 S(f^{n+1} x_0, f^{n+1} x_0, u) \\ &\precsim 2b\alpha S(u, u, fu) + 2b\alpha S(f^n x_0, f^n x_0, f^{n+1} x_0) + b^2 S(f^{n+1} x_0, f^{n+1} x_0, u) \end{aligned}$$

Letting $n \to \infty$, since $b\alpha < 1$ and $x_n \to u$, we have

$$|S(fu, fu, u)| \to 0 \text{ i.e } u = fu.$$

Now we show that f has a unique fixed point. For this, assume that there exists another point v in X such that $v = fv$. Now,

$$\begin{aligned} S(v, v, u) &\precsim S(fv, fv, fu) \\ &\precsim \alpha[S(v, v, fv) + S(u, u, fu)] \\ &\precsim \alpha[S(v, v, v) + S(u, u, u)] \\ &\precsim 0. \end{aligned}$$

Hence we conclude that $u = v$. □

Now, as mentioned in the introduction, we prove the existence and uniqueness of a common fixed point for two self mapping on complex valued S_b-metric spaces under a contraction that has the product and the quotient of complex valued S_b-metric spaces, which cannot be extended to cone metric spaces.

Theorem 3.6. *Let (X, S) be a complete complex valued S_b-metric space and f, g be two self mappings on X satisfying the following contraction condition:*

$$S(fx, fx, gy) \precsim \alpha S(x, x, y) + \frac{\beta S(x, x, fx) S_b(y, y, gy)}{b(2S(x, x, gy) + S(y, y, fx) + S(x, x, y))} \quad (6)$$

For all $x, y \in X$ such that $x \neq y$, $S(x, x, gy) + S(y, y, fx) + S(x, x, y) \neq 0$ where α, β are two nonnegative real numbers with $\alpha + \beta < 1$ or $S(fx, fx, gy) = 0$ if $S(x, x, gy) + S(y, y, fx) + S(x, x, y) = 0$. Then f, g have a unique common fixed point.

Proof. First, note that if $b = 1$, then we have the case of complex S-metric spaces, and this theorem was proved in [33].

Hence, we may assume that $b > 1$. Let $x_0 \in X$ and let $x_{2k+1} = fx_{2k}$, $x_{2k+2} = gx_{2k+1}$, $k \in \{0, 1, 2, \cdots\}$. Thus,

$$\begin{aligned}
S(x_{2k+1}, x_{2k+1}, x_{2k+2}) &= S(fx_{2k}, fx_{2k}, gx_{2k+1}) \\
&\precsim \alpha S(x_{2k}, x_{2k}, x_{2k+1}) \\
&+ \frac{\beta S(x_{2k}, x_{2k}, fx_{2k}) S(x_{2k+1}, x_{2k+1}, gx_{2k+1})}{b(2S(x_{2k}, x_{2k}, gx_{2k+1}) + S(x_{2k+1}, x_{2k+1}, fx_{2k}) + S(x_{2k}, x_{2k}, x_{2k+1}))} \\
&= \alpha S(x_{2k}, x_{2k}, x_{2k+1}) \\
&+ \frac{\beta S(x_{2k}, x_{2k}, x_{2k+1}) S(x_{2k+1}, x_{2k+1}, x_{2k+2})}{b(2S(x_{2k}, x_{2k}, x_{2k+2}) + S(x_{2k+1}, x_{2k+1}, x_{2k+1}) + S(x_{2k}, x_{2k}, x_{2k+1}))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|S(x_{2k+1}, x_{2k+1}, x_{2k+2})| &\leq \alpha |S(x_{2k}, x_{2k}, x_{2k+1})| \\
&+ \frac{\beta |S(x_{2k}, x_{2k}, x_{2k+1})| |S(x_{2k+1}, x_{2k+1}, x_{2k+2})|}{b|2S(x_{2k}, x_{2k}, x_{2k+2}) + S(x_{2k}, x_{2k}, x_{2k+1})|}.
\end{aligned}$$

By conditions (CS_b3) and (CS_b4) in the definition of the complex S_b-metric spaces, we deduce,

$$\begin{aligned}
|S(x_{2k+1}, x_{2k+1}, x_{2k+2})| = |S(x_{2k+2}, x_{2k+2}, x_{2k+1})| &\leq b|2S(x_{2k+2}, x_{2k+2}, x_{2k}) + S(x_{2k+1}, x_{2k+1}, x_{2k})| \\
&= b|2S(x_{2k}, x_{2k}, x_{2k+2}) + S(x_{2k}, x_{2k}, x_{2k+1})|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{|S(x_{2k+1}, x_{2k+1}, x_{2k+2})|}{b} \leq |S(x_{2k+1}, x_{2k+1}, x_{2k+2})| &\leq \alpha |S(x_{2k}, x_{2k}, x_{2k+1})| + \beta |S(x_{2k}, x_{2k}, x_{2k+1})| \\
&= (\alpha + \beta) |S(x_{2k}, x_{2k}, x_{2k+1})|
\end{aligned}$$

Hence, we deduce that

$$|S(x_{2k+1}, x_{2k+1}, x_{2k+2})| \leq \frac{\alpha + \beta}{b} |S(x_{2k}, x_{2k}, x_{2k+1})|$$

Using the same argument, we obtain

$$|S(x_{2k+2}, x_{2k+2}, x_{2k+3})| = \frac{\alpha + \beta}{b} |S(x_{2k+1}, x_{2k+1}, x_{2k+2})|.$$

Now, let $\eta = \dfrac{\alpha + \beta}{b}$, note that $\eta < 1$ and that is due to the fact that $\alpha + \beta < 1$, and $b \geq 1$. Therefore,

$$|S(x_{n+1}, x_{n+1}, x_{n+2})| \leq \frac{\alpha + \beta}{b} |S(x_n, x_n, x_{n+1}| \leq \cdots \leq (\frac{\alpha + \beta}{b})^{n+1} |S(x_0, x_0, x_1)|.$$

Hence, for any $m > n$ we have:

$$\begin{aligned}
|S(x_n, x_n, x_m)| &\leq 2[b(|S(x_n, x_n, x_{n+1})| + b^2|S(x_{n+1}, x_{n+1}, x_{n+2})| \\
&\quad + \cdots + b^{m-n}|S(x_{m-1}, x_{m-1}, x_m)|)] \\
&\leq 2[b(\frac{\alpha+\beta}{b})^n|S(x_0, x_0, x_1)| + b^2(\frac{\alpha+\beta}{b})^{n+1}|S(x_0, x_0, x_1)| \\
&\quad + \cdots + b^{m-n}(\frac{\alpha+\beta}{b})^{m-1}|S(x_0, x_0, x_1)| \\
&\leq 2[\eta^{n-1}|S(x_0, x_0, x_1)| + \eta^{n-1}|S(x_0, x_0, x_1)| \\
&\quad + \cdots + \eta^{n-1}|S(x_0, x_0, x_1)| \\
&\leq 2[(\frac{1}{b})^{n-1} + (\frac{1}{b})^{n-1} + \cdots + (\frac{1}{b})^{n-1}]|S(x_0, x_0, x_1)| \\
&\leq 2[(\frac{1}{b}) + (\frac{1}{b})^2 + \cdots + (\frac{1}{b})^{n-1}]|S(x_0, x_0, x_1)| \\
&\leq 2\frac{(\frac{1}{b})^n}{1-(\frac{1}{b})}|S(x_0, x_0, x_1)|.
\end{aligned}$$

Therefore, $|S(x_n, x_n, x_m)| \leq 2\frac{(\frac{1}{b})^n}{1-(\frac{1}{b})}|S(x_0, x_0, x_1)| \to 0$, as $m, n \to \infty$ and hence $\{x_n\}$ is an S_b-cauchy sequence.

Since, X is complete, then $\{x_n\}$ converge to some $v \in X$. We claim that v is the unique fixed common point of f and g. Assume that $fv \neq v$. Thus, $0 \prec z = S(v, v, fv)$. Therefore,

$$\begin{aligned}
z &\precsim b[2S(v, v, x_{2k+2}) + S(x_{2k+2}, x_{2k+2}, fv)] \\
&\precsim b[2S(v, v, x_{2k+2}) + S(gx_{2k+1}, gx_{2k+1}, fv)] \\
&\precsim 2bS(v, v, x_{2k+2}) + b\alpha S(x_{2k+1}, x_{2k+1}, v) \\
&\quad + b\frac{\beta S(v, v, fv)S(x_{2k+1}, x_{2k+1}, gx_{2k+1})}{b(2S(v, v, gx_{2k+1}) + S(x_{2k+1}, x_{2k+1}, fv) + S(v, v, x_{2k+1}))} \\
&\precsim 2bS(v, v, x_{2k+2}) + b\alpha S(x_{2k+1}, x_{2k+1}, v) \\
&\quad + \frac{\beta zS(x_{2k+1}, x_{2k+1}, gx_{2k+1})}{(2S(v, v, x_{2k+2}) + S(x_{2k+1}, x_{2k+1}, fv) + S(v, v, x_{2k+1}))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|z| &\leq 2b|S(v, v, x_{2k+2})| + b\alpha|S(x_{2k+1}, x_{2k+1}, v)| \\
&\quad + \frac{\beta|z||S(x_{2k+1}, x_{2k+1}, gx_{2k+1})|}{|b(2S(v, v, x_{2k+2}) + S(x_{2k+1}, x_{2k+1}, fv) + S(v, v, x_{2k+1}))|}.
\end{aligned}$$

Note that the right hand side of the above inequality goes to 0 as $k \to \infty$, which contradict our assumption about z. Thus, $fv = v$ and similarly one can show that $gv = v$. Therefore, f and g have a common fixed point. Now, to show uniqueness assume there exist another common fixed point of f and g say w. Hence,

$$\begin{aligned}
S(v, v, w) &= S(fv, fv, gw) \\
&\precsim \alpha S(v, v, w) + \frac{\beta S(v, v, fv)S(w, w, gw)}{b(2S(v, v, gw) + S(w, w, fv) + S(v, v, w))} \\
&= \alpha S(v, v, w).
\end{aligned}$$

Which implies that $|S(v,v,w)| = \alpha|S(v,v,w)|$, but given the fact that $\alpha < 1$ we deduce that $S(v,v,w) = 0$ and thus $v = w$ as desired.

Now, to finish the proof of our theorem, assume that for all natural numbers k if we have:

$$S(x_{2k}, x_{2k}, gx_{2k+1}) + S(x_{2k+1}, x_{2k+1}, fx_{2k}) + S(x_{2k}, x_{2k}, x_{2k+1}) = 0,$$

then $S(fx_{2k}, fx_{2k}, gx_{2k+1}) = 0$, which implies $x_{2k} = fx_{2k} = x_{2k+1} = gx_{2k+1} = x_{2k+2}$. Therefore, $x_{2k+1} = fx_{2k} = x_{2k}$, hence there exist n_1, m_1 such that $n_1 = fm_1 = m_1$. Similarly, there exist n_2, m_2 such that $n_2 = gm_2 = m_2$. We know that

$$S(m_1, m_1, gm_2) + S(m_2, m_2, fm_1) + S(m_1, m_1, m_2) = 0.$$

We deduce that $S(fm_1, fm_1, gm_2) = 0$, which implies that $n_1 = fm_1 = gm_2 = n_2$. Therefore, $n_1 = fm_1 = fn_1$, similarly we get $n_2 = gm_2 = gn_2$. Since $n_1 = n_2$ we deduce that $fn_1 = gn_1 = n_1$. Thus, n_1 is a common fixed point of f and g. To show uniqueness assume there exist u, v common fixed points of $f$and g. We know that

$$S(u, u, gv) + S_b(v, v, fu) + S(u, u, v) = 0.$$

Thus, $S(u,u,v) = S(fu, fu, gv) = 0$ which implies that $u = v$ as required.

□

In closing, the author would like to bring to reader's attention that in complex valued spaces, same results can be obtained for more than two mappings if we add some controlled functions to the contractions.

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