Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 8 Issue 3 (2017), Pages 1-12.

NEW COMMON FIXED POINT THEOREMS FOR CYCLIC COMPATIBLE CONTRACTIONS

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Abstract. The aim of this paper is to introduce cyclic compatible Θ-contraction and establish some common fixed point theorems in the setting of a generating space of a b-quasi-metric family. We also apply a simple condition on the function Θ to extend the result of Samet et al. (J. Inequal. Appl. $2014, 38(2014)$). With this condition, we also prove a fixed point theorem for Suzuki type Θ-contractions which generalize various results of literature.

1. Introduction and Preliminaries

Banach's contraction principle [8] is one of the pivotal results of analysis. It establishes that, given a mapping T on a complete metric space (X, d) into itself and a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

holds for all $x, y \in X$. Then T has a unique fixed point in X.

Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1-34] and references therein). In 1962, Edelstein [12] established the following version of the Banach contraction principle for the compact metric space.

Theorem 1.1. [12] *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that*

$$d(Tx, Ty) < d(x, y) \tag{1.2}$$

holds for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

In 2008, Suzuki [33] proved generalized versions of Edelstein's results in compact metric space as follows.

Theorem 1.2. [33] *Let (X, d) be a compact metric space and let $T : X \to X$ be a self-mapping. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < d(x, y) \tag{1.3}$$

holds for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point in X.

2000 *Mathematics Subject Classification.* 46S40, 47H10.
Key words and phrases. generating space of b-quasi-metric family, cyclic compatible Θ-contraction, common fixed point.

Submitted January 18, 2017. Published April 11, 2017.
Communicated by Wasfi Shatanawi.

In 2003, Kirk et al.[22] introduced the notion of cyclic mapping in this way:

Definition 1.3. .[22]*Let (X, d) be a complete metric space and A and B be non-empty closed subsets of X and let $T : A \cup B \to A \cup B$. Then T is daid to be cyclic iff $T(A) \subseteq B$ and $T(B) \subseteq A$.*

He generalized the Banach contraction principle by using cyclic mapping in this way:

Theorem 1.4. [22] *Let (X, d) be a complete metric space and A and B be non-empty closed subsets of X and let T be cyclic mapping from $A \cup B$ to $A \cup B$. If there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.4}$$

for all $x \in A$ and $y \in B$, then T has a unique fixed point x^*. Moreover $x^* \in A \cap B$.

The concept of weakly compatible maps was introduced by Jungck [21].

Definition 1.5. *Let (X, d) be a complete metric space and S and T be two self mappings. Then S and T are said to be weakly compatible if they commute at their coincidence point x, that is $Sx = Tx$ implies $STx = TSx$.*

In 2015, Kumari et al. [23] discussed the cyclic contractions and proved some fixed point theorems on various generating spaces.

Definition 1.6. [23] *Let X be a non-empty set and $\{d_\alpha : \alpha \in (0, 1]\}$ a family of mappings $d_\alpha : X \times X \to \mathbb{R}^+$. Consider the following conditions for any $x, y, z \in X$ and $s \geq 1$:*

(d_1) the family of self distances are zero: $d_\alpha(x, x) = 0$;
(d_2) the family of distances are symmetric: $d_\alpha(x, y) = d_\alpha(y, x)$;
(d_3) the family of positive distances between distinct points: $d_\alpha(x, y) = d_\alpha(y, x) = 0$ implies $x = y$;
(d_4) for any $\alpha \in (0, 1]$ there exists $\beta \in (0, \alpha]$ such that $d_\alpha(x, z) \leq s[d_\beta(x, y) + d_\beta(y, z)]$;
(d_5) for any $x, y \in X$, $d_\alpha(x, y)$ is non-increasing and left continuous in α.
d_α is called:
(i) the generating space of the b-quasi-metric family (shortly, the G_{bq}-family) if d_α satisfies (d_1) trough (d_5);
(ii) the generating space of the b-dislocated metric family (shortly, the G_{bd}-family) if satisfies (d_2) trough (d_5);
(iii) the generating space of the b-dislocated-quasi-metric family (shortly, the G_{bdq}-family) if satisfies (d_3) trough (d_5).
Now we give some basic definitions of the generating space of a b-quasi-metric family.

Definition 1.7. [23]

(1) Let (X, d_α) be a G_{bq}-family and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ G_{bq}-converges to x in (X, d_α) if $\lim_{n\to\infty} d_\alpha(x_n, x) = 0$ for all $\alpha \in (0, 1]$.
(2) Let (X, d_α) be a G_{bq}-family and let $A \subseteq X$, $x \in X$. We say that x is a G_{bq}-limit point of A if there exists a sequence $\{x_n\}$ in $A - \{x\}$ such that $\lim_{n\to\infty} x_n = x$.
(3) A sequence $\{x_n\}$ in a G_{bq}-family is called a G_{bq}-Cauchy sequence if given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, we have $d_\alpha(x_n, x_m) < \varepsilon$ or $\lim_{n,m\to\infty} d_\alpha(x_n, x_m) = 0$ for all $\alpha \in (0, 1]$.

(4) A G_{bq}-family (X, d_α) is called complete if every G_{bq}-Cauchy sequence in X is G_{bq}-Convergent.

Remark. *Every G_{bq}-convergent sequence in a G_{bq}-family is G_{bq}-Cauchy.*

A similar argument can be found in [10].

If we take $s = 1$ then generating space of b-quasi-metric family becomes generating space of quasi-metric family as defined by Chang et al. [10].

Example 1.8. *Let (X, d) be a metric space. If we put d_α instead of d for all $\alpha \in (0, 1]$ and $x, y \in X$, then (X, d_α) is a generating space of quasi-metric family.*

Very recently, Jleli and Samet [19] introduced a new type of contraction called Θ-contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.9. [19] *Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;
(Θ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $T : X \to X$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \Longrightarrow \Theta(d(Tx, Ty)) \leq [\Theta(d(x, y))]^k. \tag{1.5}$$

Theorem 1.10. [19] *Let (X, d) be a complete metric space and $T : X \to X$ be a Θ-contraction, then T has a unique fixed point.*

They showed that any Banach contraction is a particular case of Θ-contraction while there are Θ-contractions which are not Banach contractions.To be consistent with Samet et al. [19], we denote by the Ψ set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions (Θ_1)-(Θ_3).

Hussain et al. [15] modified and extended the above result and proved the following fixed point theorem for generalized Θ-contractive condition in the setting of complete metric spaces.

In this paper, we use the following condition instead of the condition (Θ_3) in Definition 9.

(Θ_3') Θ is continuous on $(0, \infty)$.

We denote by Ω the set of all functions satisfying the conditions (Θ_1), (Θ_2),and (Θ_3').

Example 1.11. *Let $\Theta_1(t) = e^{\sqrt{t}}$, $\Theta_2(t) = e^{\sqrt{te^t}}$, $\Theta_3(t) = e^t$, $\Theta_4(t) = \cosh t$, $\Theta_5(t) = 1+\ln(1+t)$ and $\Theta_6(t) = e^{te^t}$ for all $t > 0$. Then $\Theta_1, \Theta_2, \Theta_3, .\Theta_4, \Theta_5, \Theta_6 \in \Omega$.*

Example 1.12. *Note that the conditions Θ_3 and Θ_3' are independent of each other. Indeed, for $p \geq 1$, $\Theta(t) = e^{t^p}$ satisfies the conditions (Θ_1) and (Θ_2) but it does not satisfy (Θ_3), while it satisfies the condition (Θ_3'). Therefore $\Omega \nsubseteq \Psi$. Again for $p > 1$, $m \in (0, \frac{1}{p})$ $\Theta(t) = 1 + t^m(1 + [t])$ where $[t]$ denotes the integral part of t, satisfies the conditions (Θ_1) and (Θ_2) but it does not satisfy (Θ_3'), while it satisfies the condition (Θ_3) for any $k \in (\frac{1}{p}, 1)$. Therefore $\Psi \nsubseteq \Omega$. Also, if we take $\Theta(t) = e^{\sqrt{t}}$, then $\Theta \in \Psi$ and $\Theta \in \Omega$. Therefore $\Psi \cap \Omega \neq \emptyset$.*

2. Main Results

Theorem 2.1. *Let (X, d_α) be a complete G_{bq}-family and A, B are non-empty closed subsets of (X, d_α). Suppose S and T are cyclic mappings from $A \cup B$ to $A \cup B$ such that $SX \subset TX$. If there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $x \in A$ such that*

$$\Theta(d_\alpha(S^n x, Sy)) \leq [\Theta(d_\alpha(S^{n-1}x, Ty))]^k \tag{2.1}$$

for $n \in \mathbb{N}$ and $y \in A$ with $Tx \neq Ty$. Then S and T have a point of coincidence in $A \cap B$. Moreover, if S and T are weakly compatible and one of S or T is continuous, then S and T have a unique common fixed point in $A \cap B$.

Proof. Fix $x \in A$. Since $SX \subset TX$, we may choose $x_0 \in X$ such that $Sx_0 = Tx_1$. Hence we define the sequence $\{x_n\}$ in X by $x_{n+1} = Sx_n = Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is the required point and we have nothing to prove. So without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From (Θ_4), we have

$$\begin{aligned} 1 &< \Theta(d_\alpha(x_n, x_{n+1})) = \Theta(d_\alpha(Sx_{n-1}, Sx_n)) \leq [\Theta(d_\alpha(S^0 x_{n-1}, Tx_n))]^k \\ &= [\Theta(d_\alpha(x_{n-1}, x_n))]^k = \Theta(d_\alpha(Sx_{n-2}, Sx_{n-1})) \leq [\Theta(d_\alpha(S^0 x_{n-2}, Tx_{n-1}))]^{k^2} \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq [\Theta(d_\alpha(x_0, x_1))]^{k^n} \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, so by taking limit as $n \to \infty$ in (2.2) we have,

$$\lim_{n\to\infty} \Theta(d_\alpha(x_n, x_{n+1})) = 1 \Longleftrightarrow \lim_{n\to\infty} d_\alpha(x_n, x_{n+1}) = 0. \tag{2.3}$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^\infty$ is not a Cauchy sequence, then we assume that there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^\infty$ and $\{q(n)\}_{n=1}^\infty$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d_\alpha(x_{p(n)}, x_{q(n)}) \geq \varepsilon.$$

Then

$$d_\alpha(x_{p(n)-1}, x_{q(n)}) < \frac{\varepsilon}{s} \tag{2.4}$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (2.4), we have

$$\begin{aligned} \varepsilon &\leq d_\alpha(x_{p(n)}, x_{q(n)}) \leq s[d_\alpha(x_{p(n)}, x_{p(n)-1}) + d_\alpha(x_{p(n)-1}, x_{q(n)})] \\ &\leq sd_\alpha(x_{p(n)-1}, x_{p(n)}) + \varepsilon. \end{aligned}$$

By taking the limit and using inequality (2.4), we get

$$\lim_{n\to\infty} d_\alpha(x_{p(n)}, x_{q(n)}) = \varepsilon. \tag{2.5}$$

From (2.3), we can choose a natural number $n_0 \in \mathbb{N}$ such that

$$d_\alpha(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4s} \text{ and } d_\alpha(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4s^2} \tag{2.6}$$

for all $n \geq n_0$. Next, we claim that $Tx_{p(n)} \neq Tx_{q(n)}$ for all $n \geq n_0$ that is

$$d_\alpha(x_{p(n)+1}, x_{q(n)+1}) = d_\alpha(Tx_{p(n)}, Tx_{q(n)}) > 0. \tag{2.7}$$

Arguing by contradiction, there exists $N_0 \geq n_0$ such that $d_\alpha(x_{p(n)+1}, x_{q(n)+1}) = 0$. It follows from (2.3), (2.6), and (2.7) that

$$\begin{aligned} \varepsilon &\leq d_\alpha(x_{p(n)}, x_{q(n)}) \leq s d_\alpha(x_{p(n)}, x_{p(n)+1}) + s^2 d_\alpha(x_{p(n)+1}, x_{q(n)+1}) + s^2 d_\alpha(x_{p(n)+1}, x_{q(n)}) \\ &\leq s\frac{\varepsilon}{4s} + 0 + s^2\frac{\varepsilon}{4s} = \frac{\varepsilon}{2} \end{aligned}$$

a contradiction. Thus the relation (2.6) holds. Then by the assumption, we get

$$\Theta(d_\alpha(Tx_{p(n)}, Tx_{q(n)})) \leq [\Theta(d_\alpha(x_{p(n)}, x_{q(n)}))]^k. \tag{2.8}$$

By taking limit as $n \to +\infty$ and using (Θ_3'), (2.5) and (2.8), we get

$$\Theta(\varepsilon) \leq [\Theta(\varepsilon)]^k$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d_α) there is a sequences $\{T^{2n}x_0\}$ in A and $\{T^{2n-1}x_0\}$ in B such that both converge to some $x^* \in X$ for all $\alpha \in (0, 1]$. Since A and B are closed subsets of X, so $x^* \in A$ and $x^* \in B$. Hence $x^* \in A \cup B$. As TX is closed, there exists z in X such that

$$Tz = x^*. \tag{2.9}$$

Since $Sx_n = Tx_{n+1}$, so by the same arguement given above, there exist sequences $\{S^{2n-1}x_0\}$ in A and $\{S^{2n-2}x_0\}$ in B such that both converge to some $x^* \in X$ for all $\alpha \in (0, 1]$. This means

$$\lim_{n\to\infty} d_\alpha(S^{2n-1}x_0, x^*) = 0 \text{ and } \lim_{n\to\infty} d_\alpha(S^{2n-2}x_0, x^*) = 0$$

for all $\alpha \in (0, 1]$. Consider

$$\begin{aligned} \Theta(d_\alpha(S^{2n-1}x_0, Sz)) &\leq [\Theta(d_\alpha(S^{2n-2}x_0, Tz))]^k \\ &= [\Theta(d_\alpha(S^{2n-2}x_0, x^*))]^k < \Theta(d_\alpha(S^{2n-2}x_0, x^*)). \end{aligned}$$

By (Θ_1), we get $d_\alpha(S^{2n-1}x_0, Sz) < d_\alpha(S^{2n-2}x_0, x^*)$. Taking the limit as $n \to \infty$, we get $d_\alpha(x^*, Sz) = 0$ for all $\alpha \in (0, 1]$. Thus

$$x^* = Sz. \tag{2.10}$$

From (2.9) and (2.10), it follows that $Tz = Sz = x^*$. Thus x^* is a point of coincidence for S and T. From the weakly compatibility definition, we get

$$Tx^* = Sx^*. \tag{2.11}$$

From (Θ_1) and (Θ_4), we can get T as continuous mapping. Therefore

$$\begin{aligned} d_\alpha(x^*, Tx^*) &= \lim_{n\to\infty} d_\alpha(T^{2n-1}x_0, T(T^{2n-1}x_0)) \\ &= \lim_{n\to\infty} d_\alpha(T^{2n-1}x_0, T^{2n}x_0) \\ &= d_\alpha(x^*, x^*) \\ &= 0. \end{aligned}$$

Thus $x^* = Tx^*$. From (2.11), we get $x^* = Tx^* = Sx^*$. Hence x^* is a common fixed point of S and T. Now we prove that the common fixed point is unique. We suppose on the contrary that there exist an other common fixed point x' of S and

T distinct from x^* that is $x^* = Tx^* = Sx^*$ and $x^{'} = Tx^{'} = Sx^{'}$ but $x^* = x^{'}$. Hence $d_\alpha(x^*, x^{'}) > 0$. Then from assumption of theorem, we obtain

$$\begin{aligned} \Theta(d_\alpha(x^*, x^{'})) &= \Theta(d_\alpha(Sx^*, Sx^{'})) \\ &\leq [\Theta(d_\alpha(S^0x^*, Tx^{'}))]^k \\ &= [\Theta(d_\alpha(x^*, x^{'}))]^k \end{aligned}$$

which is contradiction because $k \in (0, 1)$. Hence $x^* = x^{'}$. □

If we put $s = 1$ in the above theorem, we obtain the following corollary in the generating space of a quasi-metric family.

Corollary 2.2. *Let (X, d_α) be a complete G_q-family and A, B are non-empty closed subsets of (X, d_α). Suppose S and T are cyclic mappings from $A \cup B$ to $A \cup B$ such that $SX \subset TX$. If there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $x \in A$ such that*

$$\Theta(d_\alpha(S^nx, Sy)) \leq [\Theta(d_\alpha(S^{n-1}x, Ty))]^k$$

for $n \in \mathbb{N}$ and $y \in A$ with $Tx \neq Ty$. Then S and T have a point of coincidence in $A \cap B$. Moreover, if S and T are weakly compatible and one of S or T is continuous, then S and T have a unique common fixed point in $A \cap B$.

If we write d instead of d_α in the above theorem, we obtain the following corollary in complete b-metric space.

Corollary 2.3. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and A, B are non-empty closed subsets of (X, d). Suppose S and T are cyclic mappings from $A \cup B$ to $A \cup B$ such that $SX \subset TX$. If there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $x \in A$ such that*

$$\Theta(d(S^nx, Sy)) \leq [\Theta(d(S^{n-1}x, Ty))]^k$$

for $n \in \mathbb{N}$ and $y \in A$ with $Tx \neq Ty$. Then S and T have a point of coincidence in $A \cap B$. Moreover, if S and T are weakly compatible and one of S or T is continuous, then S and T have a unique common fixed point in $A \cap B$.

If we put $s = 1$ and d instead of d_α in the above theorem,we obtain the following corollary in a complete metric space.

Corollary 2.4. *Let (X, d) be a complete metric space and A, B are non-empty closed subsets of (X, d). Suppose S and T are cyclic mappings from $A \cup B$ to $A \cup B$ such that $SX \subset TX$. If there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $x \in A$ such that*

$$\Theta(d(S^nx, Sy)) \leq [\Theta(d(S^{n-1}x, Ty))]^k$$

for $n \in \mathbb{N}$ and $y \in A$ with $Tx \neq Ty$. Then S and T have a point of coincidence in $A \cap B$. Moreover, if S and T are weakly compatible and one of S or T is continuous, then S and T have a unique common fixed point in $A \cap B$.

3. Fixed point results for Suzuki type Θ-contraction

Theorem 3.1. *Let (X, d_α) be a complete G_{bq}-family and A, B are non-empty closed subsets of (X, d_α). Suppose S and T are cyclic mappings from $A \cup B$ to $A \cup B$ such that $SX \subset TX$. If there exist some $\Theta \in \Omega$, $k \in (0, 1)$ and $x \in A$ such that*

$$\frac{1}{2s} d_\alpha(x, Tx) < d_\alpha(x, y) \tag{3.1}$$

implies

$$\Theta(s^2 d_\alpha(S^n x, Sy)) \leq [\Theta(d_\alpha(S^{n-1}x, Ty))]^k \tag{3.2}$$

for $n \in \mathbb{N}$ *and* $y \in A$ *with* $Tx \neq Ty$. *Then* S *and* T *have a point of coincidence in* $A \cap B$. *Moreover, if* S *and* T *are weakly compatible and one of* S *or* T *is continuous, then* S *and* T *have a unique common fixed point in* $A \cap B$.

Proof. Fix $x \in A$. Since $SX \subset TX$, we may choose $x_0 = x \in X$ such that $Sx_0 = Tx_1$. Hence we define the sequence $\{x_n\}$ in X by $x_{n+1} = Sx_n = Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If there exist $n_0 \in \mathbb{N}$ such that, $x_{n_0} = x_{n_0+1}$. Then x_{n_0} is the required point and we have nothing to prove. So without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$

$$0 < d_\alpha(x_{n-1}, Tx_{n-1})$$

for all $n \in \mathbb{N}$. Therefore

$$\frac{1}{2s} d_\alpha(x_{n-1}, Tx_{n-1}) < d_\alpha(x_{n-1}, Tx_{n-1}) = d(x_{n-1}, x_n) \tag{3.3}$$

for all $n \in \mathbb{N}$. It follows from assumption of theorem that

$$\begin{aligned} \Theta(d_\alpha(x_n, x_{n+1})) &\leq \Theta(s^2 d_\alpha(x_n, x_{n+1})) = \Theta(s^2 d_\alpha(Sx_{n-1}, Sx_n)) \\ &\leq [\Theta(s^2 d_\alpha(S^0 x_{n-1}, Tx_n))]^k \\ &= [\Theta(d_\alpha(x_{n-1}, x_n))]^k. \end{aligned} \tag{3.4}$$

Continuing in this way, we obtain

$$\begin{aligned} 1 &< \Theta(d_\alpha(x_n, x_{n+1})) \leq \Theta(s^2 d_\alpha(Sx_{n-1}, Sx_n)) \leq [\Theta(s^2 d_\alpha(S^0 x_{n-1}, Tx_n))]^k \\ &= [\Theta(d_\alpha(x_{n-1}, x_n))]^k \leq \Theta(s^2 d_\alpha(Sx_{n-2}, Sx_{n-1})) \leq [\Theta(d_\alpha(S^0 x_{n-2}, Tx_{n-1}))]^{k^2} \\ & \quad . \\ & \quad . \\ & \quad . \\ &\leq [\Theta(d_\alpha(x_0, x_1))]^{k^n} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\Theta \in \Omega$, so by taking limit as $n \to \infty$ in (3.4) we have,

$$\lim_{n\to\infty} \Theta(d_\alpha(x_n, x_{n+1})) = 1 \Longleftrightarrow \lim_{n\to\infty} d_\alpha(x_n, x_{n+1}) = 0. \tag{3.5}$$

Now, we claim that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. We suppose on the contrary that $\{x_n\}_{n=1}^\infty$ is not a Cauchy sequence, then we assume that there exists $\varepsilon > 0$ and sequences $\{p(n)\}_{n=1}^\infty$ and $\{q(n)\}_{n=1}^\infty$ of natural numbers such that for $p(n) > q(n) > n$, we have

$$d_\alpha(x_{p(n)}, x_{q(n)}) \geq \varepsilon. \tag{3.6}$$

Then

$$d_\alpha(x_{p(n)}, x_{q(n)-1}) < \varepsilon \tag{3.7}$$

for all $n \in \mathbb{N}$. So, by triangle inequality and (3.6), we have

$$\varepsilon \leq d_\alpha(x_{p(n)}, x_{q(n)}) \leq s[d_\alpha(x_{p(n)}, x_{p(n)+1}) + d_\alpha(x_{p(n)+1}, x_{q(n)})].$$

By taking the upper limit as $n \to \infty$ and using (3.5), we get

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup d_\alpha(x_{p(n)+1}, x_{q(n)}). \tag{3.8}$$

From (3.4) and (Θ_1), we have

$$d_\alpha(x_n, x_{n+1}) < d_\alpha(x_{n-1}, x_n) \tag{3.9}$$

for all $n \in \mathbb{N}$. Suppose there exists a natural number $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2s} d_\alpha(x_{p(n_0)}, Tx_{p(n_0)}) > d_\alpha(x_{p(n_0)}, x_{q(n_0)-1}) \tag{3.10}$$

and

$$\frac{1}{2s} d_\alpha(x_{p(n_0)+1}, Tx_{p(n_0)+1}) > d_\alpha(x_{p(n_0)+1}, x_{q(n_0)-1}) \tag{3.11}$$

Then from (3.9),(3.10) and (3.11), we have

$$\begin{aligned} d_\alpha(x_{p(n_0)}, x_{p(n_0)+1}) &\leq s[d_\alpha(x_{p(n_0)}, x_{q(n_0)-1}) + d_\alpha(x_{p(n_0)+1}, x_{q(n_0)-1})] \\ &< s[\frac{1}{2s} d_\alpha(x_{p(n_0)}, Tx_{p(n_0)}) + \frac{1}{2s} d_\alpha(x_{p(n_0)+1}, Tx_{p(n_0)+1})] \\ &= \frac{1}{2}[d_\alpha(x_{p(n_0)}, x_{p(n_0)+1}) + d_\alpha(x_{p(n_0)+1}, x_{p(n_0)+2})] \\ &< \frac{1}{2}[d_\alpha(x_{p(n_0)}, x_{p(n_0)+1}) + [d_\alpha(x_{p(n_0)}, x_{p(n_0)+1})] \\ &= [d_\alpha(x_{p(n_0)}, x_{p(n_0)+1}) \end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{2s} d_\alpha(x_{p(n)}, Tx_{p(n)}) \leq d_\alpha(x_{p(n)}, x_{q(n)-1}) \tag{3.12}$$

or

$$\frac{1}{2s} d_\alpha(x_{p(n)+1}, Tx_{p(n)+1}) \leq d_\alpha(x_{p(n)+1}, x_{q(n)-1}) \tag{3.13}$$

for all $n \in \mathbb{N}$. First suppose that

$$\frac{1}{2s} d_\alpha(x_{p(n)}, Tx_{p(n)}) \leq d_\alpha(x_{p(n)}, x_{q(n)-1})$$

holds for all $n \in J$, where J is an infinite set. By (Θ_1), we get

$$\begin{aligned} \Theta(s^2.\frac{\varepsilon}{s}) &\leq \Theta(s^2. \lim_{n\to\infty} \sup d_\alpha(x_{p(n)+1}, x_{q(n)})) \\ &= \Theta(s^2. \lim_{n\to\infty} \sup d_\alpha(Sx_{p(n)}, Sx_{q(n)-1})) \\ &= \Theta(s^2. \lim_{n\to\infty} \sup d_\alpha(S^0x_{p(n)}, Tx_{q(n)-1})) \\ &\leq [\Theta(d_\alpha(x_{p(n)}, x_{q(n)-1}))]^k \end{aligned}$$

which implies by using (Θ_3), that

$$\Theta(s\varepsilon) \leq [\Theta(\varepsilon)]^k$$

a contradiction because $k \in (0, 1)$ and $s \geq 1$. Now, if J is a finite set, then we can assume that

$$\frac{1}{2s} d_\alpha(x_{p(n)+1}, Tx_{p(n)+1}) \leq d_\alpha(x_{p(n)+1}, x_{q(n)-1})$$

holds for all $n \in \mathbb{N}$. Further, from triangle inequality and using (3.6), we get

$$\begin{aligned} \varepsilon &\leq d_\alpha(x_{p(n)}, x_{q(n)}) \leq sd_\alpha(x_{p(n)}, x_{p(n)+2}) + sd_\alpha(x_{p(n)+2}, x_{q(n)}) \\ &\leq s^2 d_\alpha(x_{p(n)}, x_{p(n)+1}) + s^2 d_\alpha(x_{p(n)+1}, x_{p(n)+2}) + sd_\alpha(x_{p(n)+2}, x_{q(n)}) \end{aligned}$$

Taking the upper limit as $n \to \infty$ and using (3.5), we get

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup d_\alpha(x_{p(n)+2}, x_{q(n)}). \tag{3.14}$$

Also, from triangle inequality , we get

$$d_\alpha(x_{p(n)+1}, x_{q(n)-1}) \leq sd_\alpha(x_{p(n)+1}, x_{q(n)}) + sd_\alpha(x_{q(n)}, x_{q(n)-1})$$

Taking the upper limit as $n \to \infty$ and using (3.5), (3.7) and (3.9), we get

$$\lim_{n\to\infty} \sup d_\alpha(x_{p(n)+1}, x_{q(n)-1}) \leq s\varepsilon. \tag{3.15}$$

Thus from (3.14), we have

$$\begin{aligned}
\Theta(s^2.\frac{\varepsilon}{s}) &\leq \Theta(s^2.\lim_{n\to\infty} \sup d_\alpha(x_{p(n)+2}, x_{q(n)})) \\
&= \Theta(s^2.\lim_{n\to\infty} \sup d_\alpha(Sx_{p(n)+1}, Sx_{q(n)-1})) \\
&= \Theta(s^2.\lim_{n\to\infty} \sup d_\alpha(S^0x_{p(n)+1}, Tx_{q(n)-1})) \\
&\leq [\Theta(d_\alpha(x_{p(n)+1}, x_{q(n)-1}))]^k
\end{aligned}$$

which implies by using (Θ_3) and (3.15) that

$$\Theta(s\varepsilon) \leq [\Theta(s\varepsilon)]^k$$

a contradiction because $k \in (0,1)$. Thus $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d_α) there is a sequences $\{T^{2n}x_0\}$ in A and $\{T^{2n-1}x_0\}$ in B such that both converge to some $x^* \in X$ for all $\alpha \in (0,1]$. Since A and B are closed subsets of X, so $x^* \in A$ and $x^* \in B$. Hence $x^* \in A \cup B$. As TX is closed, there exists z in X such that

$$Tz = x^*. \tag{3.16}$$

Since $Sx_n = Tx_{n+1}$, so by the same arguement given above, there exist sequences $\{S^{2n-1}x_0\}$ in A and $\{S^{2n-2}x_0\}$ in B such that both converge to some $x^* \in X$ for all $\alpha \in (0,1]$. This means

$$\lim_{n\to\infty} d_\alpha(S^{2n-1}x_0, x^*) = 0 \text{ and } \lim_{n\to\infty} d_\alpha(S^{2n-2}x_0, x^*) = 0 \tag{3.17}$$

for all $\alpha \in (0,1]$. □

Next, we claim that

$$\frac{1}{2s} d_\alpha(x_{2n-1}, Tx_{2n-1}) < d_\alpha(x_{2n-1}, x^*) \text{ or } \frac{1}{2s} d_\alpha(Tx_{2n-1}, T^2x_{2n-1}) < d_\alpha(Tx_{2n-1}, x^*). \tag{3.18}$$

for all $n \in \mathbb{N}$. We suppose on the contrary that there exists $m \in \mathbb{N}$ such that

$$\frac{1}{2s} d_\alpha(x_{2m-1}, Tx_{2m-1}) \geq d_\alpha(x_{2m-1}, x^*) \text{ or } \frac{1}{2s} d_\alpha(Tx_{2m-1}, T^2x_{2m-1}) \geq d_\alpha(Tx_{2m-1}, x^*). \tag{3.19}$$

Therefore

$$\begin{aligned}
2sd_\alpha(x_{2m-1}, x^*) &\leq d_\alpha(x_{2m-1}, Tx_{2m-1}) \\
&\leq sd_\alpha(x_{2m-1}, x^*) + sd_\alpha(x^*, Tx_{2m-1})
\end{aligned}$$

which implies that

$$d_\alpha(x_{2m-1}, x^*) \leq d_\alpha(x^*, Tx_{2m-1}). \tag{3.20}$$

It follows from (3.9) and (3.20) that

$$\begin{aligned}
d_\alpha(Tx_{2m-1}, T^2x_{2m-1}) &< d_\alpha(x_{2m-1}, Tx_{2m-1}) \leq sd_\alpha(x_{2m-1}, x^*) + sd_\alpha(x^*, Tx_{2m-1}) \\
&\leq 2sd_\alpha(x_{2m-1}, x^*) \qquad (3.21)
\end{aligned}$$

It follows from (3.19) and (3.21) that $d_\alpha(Tx_{2m-1}, T^2x_{2m-1}) < d_\alpha(Tx_{2m-1}, T^2x_{2m-1})$. This is a contradiction. Hence (3.18) holds. If part (I) of (3.18) is true, then we have

$$\begin{aligned} 1 &< \Theta(d_\alpha(S^{2n-1}x_0, Sz)) = \Theta(d_\alpha(Sx_{2n-1}, Sz)) \\ &\leq [\Theta(d_\alpha(S^{2n-2}x_0, Tz))]^k \\ &= [\Theta(d_\alpha(S^{2n-2}x_0, x^*))]^k < \Theta(d_\alpha(S^{2n-2}x_0, x^*)). \end{aligned}$$

By (Θ_1), we get $d_\alpha(S^{2n-1}x_0, Sz) < d_\alpha(S^{2n-2}x_0, x^*)$. Taking the limit as $n \to \infty$, we get $d_\alpha(x^*, Sz) = 0$ for all $\alpha \in (0,1]$. Thus

$$x^* = Sz. \tag{3.22}$$

If part (II) of (3.18) is true, using a similar method to the above, we get $x^* = Sz$..From (3.16) and (3.22), it follows that $Tz = Sz = x^*$. Thus x^* is a point of coincidence for S and T. From the weakly compatibility definition, we get

$$Tx^* = Sx^*. \tag{3.23}$$

From (Θ_1) and (Θ_4), we can get T as continuous mapping. Therefore

$$\begin{aligned} d_\alpha(x^*, Tx^*) &= \lim_{n\to\infty} d_\alpha(T^{2n-1}x_0, T(T^{2n-1}x_0)) \\ &= \lim_{n\to\infty} d_\alpha(T^{2n-1}x_0, T^{2n}x_0) \\ &= d_\alpha(x^*, x^*) \\ &= 0. \end{aligned}$$

Thus $x^* = Tx^*$. From (3.23), we get $x^* = Tx^* = Sx^*$. Hence x^* is a common fixed point of S and T. Now we prove that the common fixed point is unique. We suppose on the contrary that there exist an other common fixed point $x^{'}$ of S and T distinct from x^* that is $x^* = Tx^* = Sx^*$ and $x^{'} = Tx^{'} = Sx^{'}$ but $x^* = x^{'}$. Hence $d_\alpha(x^*, x^{'}) > 0$. Then we have

$$0 = \frac{1}{2s} d_\alpha(x^*, Tx^*) < d_\alpha(x^*, x^{'})$$

from assumption of theorem, we obtain

$$\begin{aligned} \Theta(d_\alpha(x^*, x^{'})) &= \Theta(d_\alpha(Sx^*, Sx^{'})) \\ &\leq [\Theta(d_\alpha(S^0x^*, Tx^{'}))]^k \\ &= [\Theta(d_\alpha(x^*, x^{'}))]^k \end{aligned}$$

which is contradiction because $k \in (0,1)$. Hence $x^* = x^{'}$.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgement The authors would like to express their thanks to the referees for their helpful comments and suggestions. This article was funded by the Deanship of Scientific Research (DSR), University of Jeddah. Therefore second author acknowledges with thanks DSR, UJ for financial support.

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