A COMMON FIXED POINT THEOREM FOR GENERALIZED
$(\psi, \varphi)$-WEAK CONTRACTIONS OF SUZUKI TYPE

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Abstract. The purpose of this paper is to prove a conjecture proposed by
Singh et al. in [Filomat 29:7 (2015), 1481-1490].

1. Introduction

The Banach contraction principle is a basic and remarkable result in fixed point
theory. Over the years, it has been extended in many different directions and
spaces, see [1-24] and the references therein. In 1974, Ćirić [8] established a fixed
point theorem which is one of the most important results in various generalizations
of this principle. Since then, many extensions and generalizations of Ćirić’s result
have appeared in the literature; see for instance [9, 10, 14, 15, 16, 22, 23].

Recently, Suzuki [20] obtained a powerful generalization of Banach contraction
theorem. Using the idea of the Suzuki contraction, various fixed point results
have been extended in many directions; see for instance [12, 17, 18, 19, 20, 21].
Particularly, Singh et al. [19] give a weakly contractive version of Suzuki type and
Ćirić type contractions and generalize some results of Dorić [11], Zhang et al. [24]
and others. A question and a new conjecture were proposed by Singh et al. in [19].

In this paper, we give a proof of the conjecture of Singh et al. [19] and answer
their question.

2. Main results

Now we begin to state our result, which is the conjecture arisen in [19].

Theorem 2.1. Let $X$ be a complete metric space and $S, T : X \to X$ such that for
every $x, y \in X$,

$$\frac{1}{2} \min \{d(x, Sx), d(y, Ty)\} \leq d(x, y)$$

implies

$$\psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \quad (2.1)$$

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where

(i) \( \psi : [0, \infty) \to [0, \infty) \) is a continuous and monotone nondecreasing function with \( \psi(t) = 0 \) if and only if \( t = 0 \).

(ii) \( \varphi : [0, \infty) \to [0, \infty) \) is a lower semi-continuous function with \( \varphi(t) = 0 \) if and only if \( t = 0 \), and

\[
m(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\}.
\]

Then \( S \) and \( T \) have a unique common fixed point.

**Proof.** Let \( x_0 \in X \) be given. Construct a sequence \( \{x_n\} \) in \( X \) such that \( x_{2n-1} = Sx_{2n-2} \) and \( x_{2n} = Tx_{2n-1} \), \( n = 1, 2, \cdots \). The following fact will be used in the sequel.

\[
\frac{1}{2} \min\{d(x, Sx), d(y, Ty)\} \leq d(x, y)
\]

if and only if

\[
\frac{1}{2} d(x, Sx) \leq d(x, y), \text{ or } \frac{1}{2} d(y, Ty) \leq d(x, y).
\]

Next, we always assume that \( x_n \neq x_{n-1} \) for all \( n \in \mathbb{N} \). If not, the existence of the common fixed point can be proved.

Indeed, if \( x_{2n} = x_{2n-1} \) for some \( n \in \mathbb{N} \), we can prove \( x_{2n-1} = x_{2n-2} \) is a common fixed point of \( T \) and \( S \). In fact, using (2.1), we obtain that

\[
\frac{1}{2} d(x_{2n-1}, Tx_{2n-1}) = \frac{1}{2} d(x_{2n-1}, x_{2n}) = 0 \leq d(x_{2n}, x_{2n-1})
\]

implies

\[
\psi(d(Sx_{2n}, Tx_{2n-1})) \leq \psi(m(x_{2n}, x_{2n-1})) - \varphi(m(x_{2n}, x_{2n-1})), \quad (2.2)
\]

where

\[
m(x_{2n}, x_{2n-1}) = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1}), \frac{d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Sx_{2n})}{2}\}
\]

\[
= \max\{0, d(x_{2n}, Sx_{2n}), 0, \frac{0 + d(x_{2n}, Sx_{2n})}{2}\}
\]

\[
= d(x_{2n}, Sx_{2n}).
\]

Thus (2.2) becomes that

\[
\psi(d(Sx_{2n}, x_{2n})) \leq \psi(d(Sx_{2n}, x_{2n})) - \varphi(d(Sx_{2n}, x_{2n})),
\]

which implies \( \varphi(d(Sx_{2n}, x_{2n})) \leq 0 \). By the property of \( \varphi \), we get that \( d(Sx_{2n}, x_{2n}) = 0 \) and \( Sx_{2n} = x_{2n} \). From \( Tx_{2n-1} = x_{2n} = x_{2n-1} \), it follows that \( Sx_{2n-1} = Tx_{2n-1} = x_{2n-1} \), that is \( x_{2n-1} \) is a common fixed point of \( T \) and \( S \). Similarly, if \( x_{2n-1} = x_{2n-2} \) for some \( n \in \mathbb{N} \), then \( x_{2n-2} \) is a common fixed point of \( T \) and \( S \).

Now, we divide our proof in the following five steps.

**Step 1.** We prove

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \quad (2.3)
\]

and

\[
\lim_{n \to \infty} d(x_n, x_{n+2}) = 0. \quad (2.4)
\]

Notice that, for any \( n \in \mathbb{N} \),

\[
\frac{1}{2} d(x_{2n-1}, Tx_{2n-1}) = \frac{1}{2} d(x_{2n-1}, x_{2n}) = d(x_{2n}, x_{2n-1}).
\]
Therefore by (2.4), we have
\[ \psi(d(Sx_{2n}, Tx_{2n-1})) \leq \psi(m(x_{2n}, x_{2n-1})) - \varphi(m(x_{2n}, x_{2n-1})) \]  
(2.5)
where
\[ m(x_{2n}, x_{2n-1}) = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, Sx_{2n}), d(x_{2n-1}, Tx_{2n-1}), \]
\[ \frac{d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Sx_{2n})}{2} \]
\[ = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n-1}, x_{2n}), \]
\[ \frac{d(x_{2n-1}, x_{2n+1})}{2} \]
\[ = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1})\}. \]

If \( m(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n+1}) \), then (2.5) becomes
\[ \psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n+1})) - \varphi(d(x_{2n}, x_{2n+1})), \]
which implies \( \varphi(d(x_{2n}, x_{2n+1})) \leq 0 \) and \( d(x_{2n}, x_{2n+1}) = 0 \). This is a contradiction with \( x_{2n} \neq x_{2n+1} \). Consequently, \( m(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n+1}) \) and (2.5) becomes that
\[ \psi(d(x_{2n}, x_{2n+1})) \leq \psi(d(x_{2n}, x_{2n-1})) - \varphi(d(x_{2n}, x_{2n-1})). \]  
(2.6)

Similarly, we can deduce that
\[ \psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_{n}, x_{n+1})) - \varphi(d(x_{n}, x_{n+1})). \]  
(2.7)
Combining (2.6) and (2.7), we see that
\[ \psi(d(x_{n+1}, x_{n})) \leq \psi(d(x_{n}, x_{n-1})), \]  
(2.8)
for any \( n \in \mathbb{N} \). Since \( \varphi(d(x_{n}, x_{n-1})) > 0 \), we have that
\[ \psi(d(x_{n+1}, x_{n})) < \psi(d(x_{n}, x_{n-1})). \]

Hence from the property of \( \psi \) we conclude that
\[ d(x_{n+1}, x_{n}) < d(x_{n}, x_{n-1}). \]

Thus the sequence \( \{d(x_{n+1}, x_{n})\} \) is monotone nonincreasing and bounded below, which implies that
\[ \lim_{n \to \infty} d(x_{n+1}, x_{n}) = r \]
for some \( r \geq 0 \). We claim that \( r = 0 \). In fact, taking upper limits as \( n \to \infty \) on each side of (2.8), we see that
\[ \psi(r) \leq \psi(r) - \varphi(r). \]

Consequently, we get that \( \varphi(r) \leq 0 \) and \( r = 0 \). This means that \( \lim_{n \to \infty} d(x_{n+1}, x_{n}) = 0 \) and (2.3) holds. Using the triangular inequality, we see that
\[ d(x_{n}, x_{n+2}) \leq d(x_{n}, x_{n+1}) + d(x_{n+1}, x_{n+2}), \]
and by (2.3) we get that (2.4) holds.

Step 2. We prove the following claim.
Claim: for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( m > n > N \) with \( m - n \equiv 1 \) mod 2 then \( d(x_{m}, x_{n}) < \varepsilon \).

Suppose, to the contrary that there exists \( \varepsilon_0 > 0 \) such that for any \( N \in \mathbb{N} \) we can find \( m > n > N \) with \( m - n \equiv 1 \) mod 2 satisfying \( d(x_{m}, x_{n}) > \varepsilon_0 \). Using (2.3) and (2.4), corresponding to this \( \varepsilon_0 \), we find \( N_0 \) such that \( n > N_0 \) implies
\[ d(x_{n}, x_{n+1}) < \varepsilon_0 \] and \( d(x_{n}, x_{n+2}) < \varepsilon_0 \).

Next, by induction we can get two subsequences \(\{x_{m_k}\}\) and \(\{x_{n_k}\}\) of \(\{x_n\}\) such that

\[
d(x_{m_k}, x_{n_k}) \geq \varepsilon_0, \quad d(x_{m_k-2}, x_{n_k}) < \varepsilon_0 \quad \text{and} \quad m_k - n_k \equiv 1 \mod 2. \quad (2.10)
\]

Indeed, if we take \(N = N_0\), we can find that \(l_1 > n_1 > N_0\) with \(l_1 - n_1 \equiv 1 \mod 2\) such that \(d(x_{l_1}, x_{n_1}) \geq \varepsilon_0\). Due to \(2.9\), we can choose a \(n_1 \in \{n_1 + 3, n_1 + 5, \cdots, l_1\}\) in such a way that it is the smallest integer satisfying \(d(x_{m_1}, x_{n_1}) \geq \varepsilon_0\). Then we obtain

\[
d(x_{m_1}, x_{n_1}) \geq \varepsilon_0, \quad d(x_{m_1-2}, x_{n_1}) < \varepsilon_0 \quad \text{and} \quad m_1 - n_1 \equiv 1 \mod 2.
\]

If we take \(N = m_1\), we can find that \(l_2 > n_2 > m_1\) with \(l_2 - n_2 \equiv 1 \mod 2\) such that \(d(x_{l_2}, x_{n_2}) \geq \varepsilon_0\). Similar to the choice of \(m_1\), we can choose a \(m_2 \in \{n_2 + 3, n_2 + 5, \cdots, l_2\}\) such that

\[
d(x_{m_2}, x_{n_2}) \geq \varepsilon_0, \quad d(x_{m_2-2}, x_{n_2}) < \varepsilon_0 \quad \text{and} \quad m_2 - n_2 \equiv 1 \mod 2.
\]

Continuing this process, we obtain two subsequences \(\{x_{m_k}\}\) and \(\{x_{n_k}\}\) of \(\{x_n\}\) satisfying \(2.10\).

Observe that

\[
\varepsilon_0 \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-2}) + d(x_{m_k-2}, x_{n_k}) \leq d(x_{m_k}, x_{m_k-2}) + \varepsilon_0.
\]

Then by \(2.4\) we have

\[
\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \varepsilon_0.
\]

From this and \(2.3\), it follows that

\[
\lim_{k \to \infty} d(x_{m_k+1}, x_{n_k}) = \varepsilon_0.
\]

By a similar way, we get

\[
\lim_{k \to \infty} d(x_{n_k+1}, x_{m_k}) = \varepsilon_0
\]

and

\[
\lim_{k \to \infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon_0.
\]

Now, using \(m_k - n_k \equiv 1 \mod 2\), we consider the following two cases.

Case i. If \(m_k = 2p_k - 1\) and \(n_k = 2q_k\) for some \(p_k, q_k \in \mathbb{N}\). From \(2.9\) and \(2.10\), we see that

\[
\frac{1}{2} d(x_{m_k}, Tx_{m_k}) = \frac{1}{2} d(x_{m_k}, x_{m_k+1}) < \frac{1}{2} \varepsilon_0 < \varepsilon_0 \leq d(x_{n_k}, x_{m_k}).
\]

Therefore by \(2.1\) we have

\[
\psi(d(x_{m_k+1}, x_{n_k+1})) = \psi(d(Sx_{n_k}, Tx_{m_k})) \leq \psi(m(x_{n_k}, x_{m_k})) - \varphi(m(x_{n_k}, x_{m_k})),
\]

where

\[
m(x_{n_k}, x_{m_k}) = \max\left\{d(x_{n_k}, x_{m_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Sx_{n_k})\right\}
\]

\[
\frac{d(x_{n_k}, T x_{m_k}) + d(x_{m_k}, S x_{n_k})}{2}
\]

\[
= \max\left\{d(x_{n_k}, x_{m_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1})\right\}
\]

\[
\frac{d(x_{n_k}, x_{m_k+1}) + d(x_{m_k}, x_{n_k+1})}{2}.
\]
Then
\[ \lim_{k \to \infty} m(x_{n_k}, x_{m_k}) = \max\{\varepsilon_0, 0, \frac{\varepsilon_0 + \varepsilon_0}{2}\} = \varepsilon_0. \]

Thus, taking the upper limits as \( k \to \infty \) in (2.11), we obtain \( \psi(\varepsilon_0) \leq \psi(\varepsilon_0) - \varphi(\varepsilon_0) \), and therefore \( \varphi(\varepsilon_0) = 0 \), which is a contradiction with \( \varepsilon_0 > 0 \).

Case ii. If \( m_k = 2p_k \) and \( n_k = 2q_k - 1 \) for some \( p_k, q_k \in \mathbb{N} \). From (2.9) and (2.10), we see that
\[ \frac{1}{2} d(x_{m_k}, \psi(x_{m_k})) = \frac{1}{2} d(x_{m_k}, x_{m_k+1}) < \frac{1}{2} \varepsilon_0 < \varepsilon_0 \leq d(x_{m_k}, x_{n_k}). \]

Therefore by (2.1) we have
\[ \psi(d(x_{m_k+1}, x_{n_k+1})) = \psi(d(Sx_{m_k}, Tx_{n_k})) \leq \psi(m(x_{m_k}, x_{n_k}))-\varphi(m(x_{m_k}, x_{n_k})), \]

where
\[ m(x_{m_k}, x_{n_k}) = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, \psi(x_{m_k})), d(x_{n_k}, \psi(x_{n_k})), \frac{d(x_{m_k}, \psi(x_{m_k}))+d(x_{n_k}, \psi(x_{n_k}))}{2}\} \]
\[ = \max\{d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), \frac{d(x_{m_k}, x_{m_k+1})+d(x_{n_k}, x_{n_k+1})}{2}\}. \]

Similar to the proof of Case i, we see that this case is impossible.

Thus we proved the Claim.

Step 3. We prove that \( \{x_n\} \) is a Cauchy sequence.

Let \( \varepsilon > 0 \) be given. Using the Claim, we find \( N_1 \in \mathbb{N} \) such that if \( m > n > N_1 \) with \( m - n \equiv 1 \mod 2 \) then
\[ d(x_n, x_m) < \frac{\varepsilon}{2}. \]

On the other hand, Using (2.3), we also find \( N_2 \in \mathbb{N} \) such that for any \( n > N_2 \) we have
\[ d(x_n, x_{n+1}) < \frac{\varepsilon}{2}. \]

Let \( m, n > N = \max\{N_1, N_2\} \) with \( m > n \). We get the following two cases.

(1) If \( m - n \equiv 1 \mod 2 \), then
\[ d(x_m, x_n) < \frac{\varepsilon}{2} < \varepsilon. \]

(2) If \( m - n \equiv 0 \mod 2 \), then
\[ d(x_m, x_n) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_n) < \frac{\varepsilon}{2} < \varepsilon. \]

Thus we obtain \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists \( z \in X \) such that
\[ \lim_{n \to \infty} d(x_n, z) = 0. \]

Step 4. We prove that \( z \) is a common fixed point of \( T \) and \( S \).

We claim that
\[ \frac{1}{2} d(x_{2n}, Sx_{2n}) \leq d(x_{2n}, z) \quad \text{or} \quad \frac{1}{2} d(x_{2n+1}, Tx_{2n+1}) \leq d(x_{2n+1}, z). \]
Otherwise, we have
\[
d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, z) + d(z, x_{2n+1})
\]
\[
< \frac{1}{2} d(x_{2n}, x_{2n+1}) + \frac{1}{2} d(x_{2n+1}, x_{2n+2})
\]
\[
< \frac{1}{2} d(x_{2n}, x_{2n+1}) + \frac{1}{2} d(x_{2n}, x_{2n+1})
\]
\[
= d(x_{2n}, x_{2n+1}).
\]

This is a contraction. Then there exists a subsequence \( \{n_k\} \) of \( \{n\} \) such that
\[
\frac{1}{2} d(x_{2n_k}, Sx_{2n_k}) \leq d(x_{2n_k}, z) \quad \text{or} \quad \frac{1}{2} d(x_{2n_k+1}, Tx_{2n_k+1}) \leq d(x_{2n_k+1}, z).
\]

We consider the following two cases.

Case i. If \( \frac{1}{2} d(x_{2n_k}, Sx_{2n_k}) \leq d(x_{2n_k}, z) \), then, by (2.1), we get
\[
\psi(d(x_{2n_k+1}, Tz)) = \psi(d(Sx_{2n_k}, Tz)) \leq \psi(m(x_{2n_k}, z)) - \varphi(m(x_{2n_k}, z)),
\]  
(2.12)

where
\[
m(x_{2n_k}, z) = \max\{d(x_{2n_k}, z), d(x_{2n_k}, Sx_{2n_k}), d(z, Tz), \frac{d(x_{2n_k}, Tz) + d(Sx_{2n_k}, z)}{2}\}
\]
\[
= \max\{d(x_{2n_k}, z), d(x_{2n_k}, x_{2n_k+1}), d(z, Tz), \frac{d(x_{2n_k}, Tz) + d(x_{2n_k+1}, z)}{2}\}.
\]

Then
\[
\lim_{k \to \infty} m(x_{2n_k}, z) = \max\{d(z, z), 0, d(z, Tz), \frac{d(z, Tz) + d(z, z)}{2}\} = d(z, Tz).
\]

Taking upper limits as \( k \to \infty \) in (2.12),
\[
\psi(d(z, Tz)) \leq \psi(d(z, Tz)) - \varphi(d(z, Tz)),
\]

which implies \( d(z, Tz) = 0 \) and \( Tz = z \).

Notice that
\[
\frac{1}{2} d(z, Tz) = \frac{1}{2} d(z, z) = 0 \leq d(z, z).
\]

Using (2.1), we get
\[
\psi(d(Sz, z)) = \psi(d(Sz, Tz)) \leq \psi(m(z, z)) - \varphi(m(z, z))
\]
\[
= \psi(d(z, Sz)) - \varphi(d(z, Sz)),
\]

which implies \( d(z, Sz) = 0 \) and \( Sz = z \). Thus, in this case, we obtain that \( z \) is a common fixed point of \( T \) and \( S \).

Case ii. If \( \frac{1}{2} d(x_{2n_k+1}, Tx_{2n_k+1}) \leq d(x_{2n_k+1}, z) \), then, by (2.1), we get
\[
\psi(d(x_{2n_k+2}, Sz)) = \psi(d(Sz, Tx_{2n_k+1})) \leq \psi(m(z, x_{2n_k+1})) - \varphi(m(z, x_{2n_k+1})),
\]  
(2.13)

where
\[
m(z, x_{2n_k+1}) = \max\{d(z, x_{2n_k+1}), d(z, Sz), d(x_{2n_k+1}, Tx_{2n_k+1}), \frac{d(z, Tz_{2n_k+1}) + d(x_{2n_k+1}, Sz)}{2}\}
\]
\[
= \max\{d(z, x_{2n_k+1}), d(z, Sz), d(x_{2n_k+1}, x_{2n_k+2}), \frac{d(z, x_{2n_k+2}) + d(x_{2n_k+2}, Sz)}{2}\}.
\]

Similar to the proof of Case i, we conclude that \( Tz = Sz = z \).
Step 5. We prove the uniqueness of the common fixed point $z$.
Suppose that $y$ is another common fixed point of $T$ and $S$. Then
\[ \frac{1}{2} d(z, Tz) = 0 \leq d(y, z) \]
implies
\[ \psi(d(y, z)) = \psi(d(Sy, Tz)) \leq \psi(m(y, z)) - \varphi(m(y, z)) = \psi(d(y, z)) - \varphi(d(y, z)). \]
This leads to $d(y, z) = 0$ and $y = z$.
This completes the proof of Theorem 2.1. □

If we take $S = T$ in Theorem 2.1, we obtain the following result.

**Corollary 2.2.** [19, Theorem 2.1] Let $X$ be a complete metric space and $T : X \to X$ such that for every $x, y \in X$,
\[ \frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } \psi(d(Tx, Ty)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \]
where $\psi$ and $\varphi$ are defined as in Theorem 2.1 and
\[ m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}. \]
Then $T$ has a unique fixed point.

From Theorem 2.1 we obtain the following result which is a generalization of [11, Theorem 2.2].

**Corollary 2.3.** Let $X$ be a complete metric space and $S, T : X \to X$ such that for every $x, y \in X$,
\[ \psi(d(Sx, Ty)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \quad (2.14) \]
where $\psi$, $\varphi$ and $m(x, y)$ are defined as in Theorem 2.1. Then $S$ and $T$ have a unique common fixed point.

The next is an example which can apply Theorem 2.1 but not Corollary 2.3.

**Example 2.4.** Let $X = \{(1, 1, 1), (4, 1, 0), (1, 4, 0), (4, 5, 1), (5, 4, 1)\}$ be endowed with the usual metric. Set
\[ a = (1, 1, 1), b = (4, 1, 0), c = (1, 4, 0), d = (4, 5, 1), e = (5, 4, 1). \]
Define $S : X \to X$ and $T : X \to X$ by
\[ Sa = Sb = Sc = a, Sd = b, Se = c; Ta = Tb = Tc = a,Td = c, Te = b. \]
Then $T$ does not satisfy the condition (2.14) of Corollary 2.3 at $x = d, y = d$ and $x = e, y = e$. Choose $\psi(t) = \frac{2}{3}t$ and $\varphi(t) = \frac{1}{6}t$. It is easily verified that $T$ satisfies all the conditions of Theorem 2.1. In fact, $a = (1, 1, 1)$ is a unique common fixed point of $S$ and $T$.

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