Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 7 Issue 6(2016), Pages 89-96.

# CLASSIFICATIONS OF $(\alpha, \beta)$-FUZZY IDEALS IN $BCK/BCI$-ALGEBRAS

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Abstract. Classifications of $(\alpha, \beta)$-fuzzy ideals of $BCK/BCI$-algebras are discussed. Relations between $(\in, \in\vee q)$-fuzzy ideals and $(q, \in\vee q)$-fuzzy ideals are established. Given a special set, so called $t$-$q$-set, conditions for the $t$-$q$-set to be an ideal are considered. The notions of an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal are introduced, and conditions for a fuzzy set to be an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal are provided.

## 1. Introduction

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [8], played a vital role to generate some different types of fuzzy subgroups, called $(\alpha, \beta)$-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. In $BCK/BCI$-algebras, the concept of $(\alpha, \beta)$-fuzzy ideals, which is studied in the papers [3], [4], [5] and [9], is also important and useful generalization of the well-known concepts, called fuzzy ideals.

In this paper, we classify $(\alpha, \beta)$-fuzzy ideals of $BCK/BCI$-algebras. We establish relations between $(\in, \in\vee q)$-fuzzy ideals and $(q, \in\vee q)$-fuzzy ideals. Given a special set, so called $t$-$q$-set, we discuss conditions for the $t$-$q$-set to be a ideal. We introduce the notions of an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal, and provide conditions for a fuzzy set to be an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal.

## 2. Preliminaries

By a *$BCI$-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a $BCI$-algebra $X$ satisfies the axiom

2010 *Mathematics Subject Classification.* 06F35, 03G25,06D72.
*Key words and phrases.* (Fuzzy) ideal; $(\alpha, \beta)$-fuzzy ideal; $t$-$q$-set; $(\in, q)^{\max}$-fuzzy ideal.

Submitted November 2, 2016. Published November 22, 2016.
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(a5) $0 * x = 0$ for all $x \in X$,

then we say that $X$ is a *BCK-algebra*. A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $A$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies:

(I1) $0 \in A$,
(I2) $(\forall x \in X)\ (\forall y \in A)\ (x * y \in A \implies x \in A)$.

We refer the reader to the books [2] and [6] for further information regarding $BCK/BCI$-algebras.

A fuzzy set $\mu$ in a set $X$ of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu [8] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. To say that $x_t \in \mu$ (resp. $x_t\ q\ \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mu$. To say that $x_t \in\vee q\ \mu$ (resp. $x_t \in\wedge q\ \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$.

A fuzzy set $\mu$ in a $BCK/BCI$-algebra $X$ is called a *fuzzy subalgebra* of $X$ if it satisfies:

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \tag{2.1}$$

for all $x, y \in X$.

**Proposition 2.1** ([4]). *Let $X$ be a $BCK/BCI$-algebra. A fuzzy set $\mu$ in $X$ is a fuzzy subalgebra of $X$ if and only if the following assertion is valid.*

$$x_t \in \mu,\ y_s \in \mu \implies (x * y)_{\min\{t,s\}} \in \mu \tag{2.2}$$

*for all $x, y \in X$ and $t, s \in (0, 1]$.*

A fuzzy set $\mu$ in a $BCK/BCI$-algebra $X$ is called a *fuzzy ideal* of $X$ if it satisfies:

$$\mu(0) \geq \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \tag{2.3}$$

for all $x, y \in X$.

**Proposition 2.2** ([3]). *Let $X$ be a $BCK/BCI$-algebra. A fuzzy set $\mu$ in $X$ is a fuzzy ideal of $X$ if and only if the following assertions are valid.*

$$x_t \in \mu \implies 0_t \in \mu, \tag{2.4}$$

$$(x * y)_t \in \mu,\ y_s \in \mu \implies x_{\min\{t,s\}} \in \mu \tag{2.5}$$

*for all $x, y \in X$ and $t, s \in (0, 1]$.*

## 3. Classifications of $(\alpha, \beta)$-fuzzy ideals

In what follows, let $X$ denote a $BCK/BCI$-algebra unless otherwise specified.

**Definition 3.1** ([4]). A fuzzy set $\mu$ in $X$ is said to be an $(\alpha, \beta)$-*fuzzy subalgebra* of $X$, where $\alpha \neq \in\wedge q$, if it satisfies the following condition:

$$x_{t_1} \alpha \mu,\ y_{t_2} \alpha \mu \Rightarrow (x * y)_{\min\{t_1, t_2\}} \beta \mu. \tag{3.1}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

**Definition 3.2** ([3])**.** A fuzzy set $\mu$ in $X$ is said to be an $(\alpha, \beta)$-*fuzzy ideal* of $X$, where $\alpha \neq\, \in\wedge\, q$, if it satisfies the following conditions:

$$x_t \alpha \mu \;\Rightarrow\; 0_t \beta \mu, \tag{3.2}$$

$$(x * y)_{t_1} \alpha \mu,\; y_{t_2} \alpha \mu \;\Rightarrow\; x_{\min\{t_1, t_2\}} \beta \mu \tag{3.3}$$

for all $x, y \in X$ and $t, t_1, t_2 \in (0, 1]$.

In a similar way to $(\alpha, \beta)$-fuzzy subalgebras (see [7]), we have twelve different types of $(\alpha, \beta)$-fuzzy ideals in $X$, that is, $(\alpha, \beta)$ is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in\wedge q)$, $(\in, \in\vee q)$, $(q, \in)$, $(q, q)$, $(q, \in\wedge q)$, $(q, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, and $(\in\vee q, \in\vee q)$. Clearly, we have relations among these types which are described in the following theorems.

**Theorem 3.1.** *We have the following relations:*

$$\begin{array}{ccccc}
(\in, \in) & \Longleftarrow & (\in, \in\wedge q) & \Longrightarrow & (\in, q) \\
& \searrow & \Downarrow & \swarrow & \\
& & (\in, \in\vee q) & & \\
& & \Uparrow & & \\
& & (\in\vee q, \in\vee q) & & \\
& \nearrow & \Uparrow & \nwarrow & \\
(\in\vee q, \in) & \Longleftarrow & (\in\vee q, \in\wedge q) & \Longrightarrow & (\in\vee q, q)
\end{array} \tag{3.4}$$

*and*

$$\begin{array}{ccccc}
(q, \in) & \Longleftarrow & (q, \in\wedge q) & \Longrightarrow & (q, q) \\
& \searrow & \Downarrow & \swarrow & \\
& & (q, \in\vee q) & &
\end{array} \tag{3.5}$$

**Theorem 3.2.** *If there exists $x \in X$ such that $\mu(x) > 0.5$, then we have the following relation:*

$$\begin{array}{ccccc}
(\in\wedge q, \in) & \Longleftarrow & (\in\wedge q, \in\wedge q) & \Longrightarrow & (\in\wedge q, q) \\
& \searrow & \Downarrow & \swarrow & \\
& & (\in\wedge q, \in\vee q) & & \\
& & \Uparrow & & \\
& & (\in, \in\vee q) & &
\end{array} \tag{3.6}$$

We investigate relations between $(\in, \in\vee q)$-fuzzy ideals and $(q, \in\vee q)$-fuzzy ideals.

**Theorem 3.3.** *Every $(q, \in\vee q)$-fuzzy ideal is an $(\in, \in\vee q)$-fuzzy ideal.*

*Proof.* Let $\mu$ be a $(q, \in \vee q)$-fuzzy ideal of $X$ and let $x \in X$ and $t \in (0,1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t$. Assume that $0_t \overline{\in \vee q}\, \mu$. Then $\mu(0) < t$ and $\mu(0) + t \leq 1$, which imply that $\mu(0) < 0.5$. Hence $\mu(0) < \min\{t, 0.5\}$, and so

$$\begin{aligned} 1 - \mu(0) &> 1 - \min\{t, 0.5\} \\ &= \max\{1 - t, 0.5\} \\ &\geq \max\{1 - \mu(x), 0.5\}. \end{aligned}$$

It follows that

$$1 - \mu(0) \geq \delta > \max\{1 - \mu(x), 0.5\} \tag{3.7}$$

for some $\delta \in (0,1]$. The right inequality in (3.7) induces $\mu(x) + \delta > 1$, i.e., $x_\delta\, q\, \mu$ and $\delta > 0.5$. Since $\mu$ is a $(q, \in \vee q)$-fuzzy ideal of $X$, we have $0_\delta \in \vee q\, \mu$. On the other hand, the left inequality in (3.7) implies $\mu(0) + \delta \leq 1$, that is, $0_\delta\, \overline{q}\, \mu$. Thus $0_\delta \overline{\in \vee q}\, \mu$, a contradiction. Hence $0_t \in \vee q\, \mu$. Let $x, y \in X$ and $t_1, t_2 \in (0,1]$ be such that $(x * y)_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x * y) \geq t_1$ and $\mu(y) \geq t_2$. Suppose $x_{\min\{t_1,t_2\}} \overline{\in \vee q}\, \mu$. Then

$$\mu(x) < \min\{t_1, t_2\}, \tag{3.8}$$

$$\mu(x) + \min\{t_1, t_2\} \leq 1. \tag{3.9}$$

It follows that

$$\mu(x) < 0.5. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$\mu(x) < \min\{t_1, t_2, 0.5\}$$

and so

$$\begin{aligned} 1 - \mu(x) &> 1 - \min\{t_1, t_2, 0.5\} \\ &= \max\{1 - t_1, 1 - t_2, 0.5\} \\ &\geq \max\{1 - \mu(x * y), 1 - \mu(y), 0.5\}. \end{aligned}$$

Hence there exists $\delta \in (0,1]$ such that

$$1 - \mu(x) \geq \delta > \max\{1 - \mu(x * y), 1 - \mu(y), 0.5\}. \tag{3.11}$$

From the right inequality in (3.11), we have $\mu(x * y) + \delta > 1$ and $\mu(y) + \delta > 1$, that is, $(x * y)_\delta\, q\, \mu$ and $y_\delta\, q\, \mu$. Since $\mu$ is a $(q, \in \vee q)$-fuzzy ideal of $X$, it follows that $x_\delta = x_{\min\{\delta,\delta\}} \in \vee q\, \mu$. But, from the left inequality in (3.11), we get $\mu(x) + \delta \leq 1$, that is, $x_\delta\, \overline{q}\, \mu$, and $\mu(x) \leq 1 - \delta < 1 - 0.5 = 0.5 < \delta$, i.e., $x_\delta \overline{\in}\, \mu$. Hence $x_\delta \overline{\in \vee q}\, \mu$, a contradiction. Therefore $x_{\min\{t_1,t_2\}} \in \vee q\, \mu$, and thus $\mu$ is an $(\in, \in \vee q)$-fuzzy ideal of $X$. □

Regarding $(\alpha,\beta)$-fuzzy ideals, Theorem 3.3 and (3.5) in Theorem 3.1 induces the the following relations.

$$\begin{array}{ccccc} (q,\in) & \Longleftarrow & (q,\in\wedge q) & \Longrightarrow & (q,q) \\ & \searrow & \Downarrow & \swarrow & \\ & & (q,\in\vee q) & & \\ & & \Downarrow & & \\ & & (\in,\in\vee q) & & \end{array} \tag{3.12}$$

We know that the $(\in,\in\vee q)$-fuzzy ideal has a characterization as follows.

**Lemma 3.4** ([3])**.** *A fuzzy set $\mu$ in $X$ is an $(\in,\in\vee q)$-fuzzy ideal of $X$ if and only if it satisfies:*

(1) $(\forall x\in X)(\mu(0)\geq\min\{\mu(x),0.5\})$,
(2) $(\forall x,y\in X)(\mu(x)\geq\min\{\mu(x*y),\mu(y),0.5\})$.

In general, an $(\in,\in\vee q)$-fuzzy ideal may not be a $(q,\in\vee q)$-fuzzy ideal as seen in the following example.

**Example 3.3.** Consider a $BCI$-algebra $X=\{0,a,b,c\}$ with the following Cayley table:

| $*$ | 0 | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| 0 | 0 | $a$ | $b$ | $c$ |
| $a$ | $a$ | 0 | $c$ | $b$ |
| $b$ | $b$ | $c$ | 0 | $a$ |
| $c$ | $c$ | $b$ | $a$ | 0 |

Let $\mu$ be a fuzzy set in $X$ defined by $\mu(0)=0.6$, $\mu(a)=0.7$ and $\mu(b)=\mu(c)=0.3$. Then it is easy to show that $\mu$ satisfies two conditions in Lemma 3.4. Hence $\mu$ is an $(\in,\in\vee q)$-fuzzy ideal of $X$. Note that $(c*a)_{0.71}=b_{0.71}\,q\,\mu$ and $a_{0.41}\,q\,\mu$, but $c_{\min\{0.71,\,0.41\}}=c_{0.41}\,\overline{\in\vee q}\,\mu$. Hence $\mu$ is not a $(q,\in\vee q)$-fuzzy ideal of $X$.

We provide conditions for an $(\in,\in\vee q)$-fuzzy ideal to be a $(q,\in\vee q)$-fuzzy ideal.

**Theorem 3.5.** *Assume that every fuzzy point has the value $t$ in $(0,0.5]$. Then every $(\in,\in\vee q)$-fuzzy ideal of $X$ is a $(q,\in\vee q)$-fuzzy ideal of $X$.*

*Proof.* Let $\mu$ be an $(\in,\in\vee q)$-fuzzy ideal of $X$. Assume that $x_t\,q\,\mu$ for any $x\in X$ and $t\in(0,0.5]$. Then $\mu(x)>1-t\geq t$, and so $x_t\in\mu$. Thus $0_t\in\vee q\,\mu$. Now, let $x,y\in X$ and $t_1,t_2\in(0,0.5]$ be such that $(x*y)_{t_1}\,q\,\mu$ and $y_{t_2}\,q\,\mu$. Then $\mu(x*y)>1-t_1\geq t_1$ and $\mu(y)>1-t_2\geq t_2$, that is, $(x*y)_{t_1}\in\mu$ and $y_{t_2}\in\mu$. Since $\mu$ is an $(\in,\in\vee q)$-fuzzy ideal of $X$, it follows that $x_{\min\{t_1,t_2\}}\in\vee q\,\mu$. Consequently, $\mu$ is a $(q,\in\vee q)$-fuzzy ideal of $X$. □

For a fuzzy set $\mu$ in $X$ and $t\in(0,1]$, consider the $q$-set and $\in\vee q$-set with respect to $t$ (briefly, $t$-$q$-set and $t$-$\in\vee q$-set, respectively) as follows:

$$X_q^t:=\{x\in X\mid x_t\,q\,\mu\}\text{ and }X_{\in\vee q}^t:=\{x\in X\mid x_t\in\vee q\,\mu\}.$$

Note that, for any $t,r\in(0,1]$, if $t\geq r$ then every $r$-$q$-set is contained in the $t$-$q$-set, that is, $X_q^r\subseteq X_q^t$, and every $r$-$\in\vee q$-set is contained in the $t$-$\in\vee q$-set, that is, $X_{\in\vee q}^r\subseteq X_{\in\vee q}^t$. Obviously, $X_{\in\vee q}^t=U(\mu;t)\cup X_q^t$.

**Theorem 3.6.** *If $\mu$ is an $(\in, \in)$-fuzzy ideal of $X$, then the t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0,1]$ whenever it is nonempty.*

*Proof.* Since $X_q^t \neq \emptyset$ for all $t \in (0,1]$, there exists $x \in X$ such that $x_t \, q \, \mu$, that is, $\mu(x) + t > 1$. It follows that $\mu(0) + t \geq \mu(x) + t > 1$, that is, $0_t \, q \, \mu$. Hence $0 \in X_q^t$. Let $x, y \in X$ be such that $x * y \in X_q^t$ and $y \in X_q^t$. Then $(x * y)_t \, q \, \mu$ and $y_t \, q \, \mu$, that is, $\mu(x * y) + t > 1$ and $\mu(y) + t > 1$. It follows that

$$\begin{aligned} \mu(x) + t &\geq \min\{\mu(x * y), \mu(y)\} + t \\ &= \min\{\mu(x * y) + t, \mu(y) + t\} > 1 \end{aligned}$$

and so that $x_t \, q \, \mu$. Hence $x \in X_q^t$, and therefore $X_q^t$ is an ideal of $X$. □

**Theorem 3.7.** *If $\mu$ is a $(q, \in \vee \, q)$-fuzzy ideal of $X$, then the t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0.5, 1]$ whenever it is nonempty.*

*Proof.* Assume that $X_q^t \neq \emptyset$ for all $t \in (0.5, 1]$. Then there exists $x \in X$ such that $x_t \, q \, \mu$, which implies that $0_t \in \vee \, q \, \mu$, that is, $0_t \in \mu$ or $0_t \, q \, \mu$. If $0_t \, q \, \mu$, then $0 \in X_q^t$. If $0_t \in \mu$, then $\mu(0) \geq t > 1 - t$ since $t > 0.5$. Hence $0_t \, q \, \mu$, and thus $0 \in X_q^t$. Now, let $x, y \in X$ be such that $y \in X_q^t$ and $x * y \in X_q^t$ for all $t \in (0.5, 1]$. Then $(x * y)_t \, q \, \mu$ and $y_t \, q \, \mu$. Since $\mu$ is a $(q, \in \vee \, q)$-fuzzy ideal of $X$, it follows that $x_t \in \vee \, q \, \mu$, i.e., $x_t \in \mu$ or $x_t \, q \, \mu$. If $x_t \, q \, \mu$, then $x \in X_q^t$. If $x_t \in \mu$, then $\mu(x) \geq t > 1 - t$ since $t > 0.5$. Hence $x_t \, q \, \mu$, and so $x \in X_q^t$. Therefore $X_q^t$ is an ideal of $X$. □

**Definition 3.4.** A fuzzy set $\mu$ in $X$ is said to be an $(\in, q)^{\max}$-*fuzzy ideal* of $X$ over $(0,1]$ if it satisfies the following condition:

$$x_t \in \mu \;\Rightarrow\; 0_t \, q \, \mu, \tag{3.13}$$

$$(x * y)_{t_1} \in \mu, \; y_{t_2} \in \mu \;\Rightarrow\; x_{\max\{t_1, t_2\}} \, q \, \mu. \tag{3.14}$$

for all $x, y \in X$ and $t, t_1, t_2 \in (0,1]$.

Obviously, every $(\in, q)$-fuzzy ideal is an $(\in, q)^{\max}$-fuzzy ideal over $(0,1]$.

**Theorem 3.8.** *For a fuzzy set $\mu$ in $X$, if the nonempty t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0.5, 1]$, then $\mu$ is an $(\in, q)^{\max}$-fuzzy ideal of $X$ over* $(0.5, 1]$.

*Proof.* Let $x \in X$ and $t \in (0.5, 1]$ be such that $x_t \in \mu$. Then $\mu(x) \geq t > 1 - t$, and so $x_t \, q \, \mu$. Thus $x \in X_q^t$, that is, $X_q^t \neq \emptyset$. Since $X_q^t$ is an ideal of $X$ for all $t \in (0.5, 1]$, we have $0 \in X_q^t$. Hence $0_t \, q \, \mu$. Let $x, y \in X$ and $t_1, t_2 \in (0.5, 1]$ be such that $(x * y)_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x * y) \geq t_1 > 1 - t_1$ and $\mu(y) \geq t_2 > 1 - t_2$, that is, $(x * y)_{t_1} \, q \, \mu$ and $y_{t_2} \, q \, \mu$. It follows that $x * y, y \in X_q^{\max\{t_1, t_2\}}$ and $\max\{t_1, t_2\} \in (0.5, 1]$. By hypothesis, we have $x \in X_q^{\max\{t_1, t_2\}}$ and so $x_{\max\{t_1, t_2\}} \, q \, \mu$. Therefore $\mu$ is an $(\in, q)^{\max}$-fuzzy ideal of $X$ over $(0.5, 1]$. □

**Corollary 3.9.** *For a fuzzy set $\mu$ in $X$, if the nonempty t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0.5, 1]$, then $\mu$ is an $(\in, q)^t$-fuzzy ideal of $X$ over* $(0.5, 1]$.

**Definition 3.5.** A fuzzy set $\mu$ in $X$ is said to be a $(q, \in)^{\max}$-*fuzzy ideal* of $X$ over $(0,1]$ if it satisfies the following condition:

$$x_t \, q \, \mu \;\Rightarrow\; 0_t \in \mu, \tag{3.15}$$

$$(x * y)_{t_1} \, q \, \mu, \; y_{t_2} \, q \, \mu \;\Rightarrow\; x_{\max\{t_1, t_2\}} \in \mu. \tag{3.16}$$

for all $x, y \in X$ and $t, t_1, t_2 \in (0,1]$.

Obviously, every $(q, \in)^{\max}$-fuzzy ideal over $(0, 1]$ is a $(q, \in)$-fuzzy ideal.

**Theorem 3.10.** *For a fuzzy set $\mu$ in $X$, if the nonempty t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0, 0.5]$, then $\mu$ is a $(q, \in)^{\max}$-fuzzy ideal of $X$ over $(0, 0.5]$.*

*Proof.* Assume that $x_t\, q\, \mu$ for all $x \in X$ and $t \in (0, 0.5]$. Then $x \in X_q^t$, and so $X_q^t \neq \emptyset$. Thus $0 \in X_q^t$ because $X_q^t$ is an ideal of $X$ for all $t \in (0, 0.5]$. It follows that $\mu(0) > 1 - t \geq t$ and so that $0_t \in \mu$. Let $x, y \in X$ and $t_1, t_2 \in (0, 0.5]$ be such that $(x * y)_{t_1}\, q\, \mu$ and $y_{t_2}\, q\, \mu$. Then $x * y \in X_q^{t_1}$ and $y \in X_q^{t_2}$. It follows that $x * y,\, y \in X_q^{\max\{t_1, t_2\}}$ and $\max\{t_1, t_2\} \in (0, 0.5]$. Thus $x \in X_q^{\max\{t_1, t_2\}}$. Hence $\mu(x) + \max\{t_1, t_2\} > 1$, and so $\mu(x) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\}$. Therefore $x_{\max\{t_1, t_2\}} \in \mu$. Consequently, $\mu$ is a $(q, \in)^{\max}$-fuzzy ideal of $X$ over $(0, 0.5]$. □

**Corollary 3.11.** *Let $\mu$ be a fuzzy set in $X$ and $t \in (0, 0.5]$. If the t-q-set $X_q^t$ is an ideal of $X$, then $\mu$ is a $(q, \in)^t$-fuzzy ideal of $X$ over $(0, 0.5]$.*

**Theorem 3.12.** *If $\mu$ is a $(q, \in\vee q)$-fuzzy ideal of $X$, then the nonempty t-q-set $X_q^t$ is an ideal of $X$ for all $t \in (0.5, 1]$.*

*Proof.* Assume that $X_q^t \neq \emptyset$ for all $t \in (0.5, 1]$ and let $x \in X_q^t$. Then $x_t\, q\, \mu$, which implies $0_t \in\vee q\, \mu$. Hence $0_t \in \mu$ or $0_t\, q\, \mu$. If $0_t\, q\, \mu$, then $0 \in X_q^t$. If $0_t \in \mu$, then $\mu(0) \geq t > 1 - t$ since $t > 0.5$. Thus $0 \in X_q^t$. Let $x, y \in X$ be such that $x * y \in X_q^t$ and $y \in X_q^t$. Then $(x * y)_t\, q\, \mu$ and $y_t\, q\, \mu$. Since $\mu$ is a $(q, \in\vee q)$-fuzzy ideal of $X$, we have $x_t \in\vee q\, \mu$, that is, $x_t \in \mu$ or $x_t\, q\, \mu$. If $x_t\, q\, \mu$, then $x \in X_q^t$. If $x_t \in \mu$, then $\mu(x) \geq t > 1 - t$ since $t > 0.5$. Hence $x_t\, q\, \mu$, and so $x \in X_q^t$. Therefore $X_q^t$ is an ideal of $X$. □

## Conclusions

We have twelve different types of $(\alpha, \beta)$-fuzzy ideals in $BCK/BCI$-algebras, that is, $(\alpha, \beta)$ is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in\wedge q)$, $(\in, \in\vee q)$, $(q, \in)$, $(q, q)$, $(q, \in\wedge q)$, $(q, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, and $(\in\vee q, \in\vee q)$. In this paper, we have described relations among these types. We have investigated relationships between $(\in, \in\vee q)$-fuzzy ideals and $(q, \in\vee q)$-fuzzy ideals. We have provided conditions for an $(\in, \in\vee q)$-fuzzy ideal to be a $(q, \in\vee q)$-fuzzy ideal. We have introduced the notions of an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal, and have provided conditions for a fuzzy set to be an $(\in, q)^{\max}$-fuzzy ideal and a $(q, \in)^{\max}$-fuzzy ideal.

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