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A MONOTONICITY RESULT FOR THE $q-$FRACTIONAL OPERATOR

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Abstract. In this article we prove that if the $q-$fractional operator $({}_q\nabla^{\alpha}_{qa}y)(t)$ of order $0 < \alpha \leq 1$, $0 < q < 1$ and starting at some $qa \in T_q = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $a > 0$ is positive such that $y(a) \geq 0$, then $y(t)$ is $c_q(\alpha)-$increasing, $c_q(\alpha) = \frac{1-q^{\alpha}}{1-q}q^{1-\alpha}$. Conversely, if y(t) is increasing and $y(a) \geq 0$, then $({}_q\nabla^{\alpha}_{qa}y)(t) \geq 0$. As an application, we proved a $q-$fractional version of the Mean-Value Theorem.

1. Introduction and Preliminaries

Fractional calculus [25, 28, 30] has recently occupied the minds of many researchers either theoretically or in different fields of applications [17, 26]. The theory of $q-$fractional calculus was initiated in early of fifties of last century [9, 11, 22, 6, 7, 8]. Then, this theory has started to be developed in the last decade or so ([3]-[5],[10, 11, 23, 29]). For some recent interesting extensions and generalization in the theory of $q-$polynomials and functions we guide the reader to [18, 31, 32]. For the preliminaries about $q-$ fractional calculus given here shortly, we refer the reader to the survey [19] and the recent book [10]. On the other hand the theory of discrete fractional calculus started to develop rapidly specially in the last decade ([1, 2],[12]-[16], [20, 21, 27]). Very recently, some monotonicity results have been reported for fractional difference type operators of order $0 < \alpha \leq 1$ [13], and of order $1 < \alpha < 2$ [20, 24]. Motivated by what mentioned above and the fact that monotonicity results are of interest in usual calculus itself we obtain some monotonicity results for the $q-$fractional type operators of order $0 < \alpha \leq 1$ in Section 2. An application is also given in Section 3 by giving a $q-$fractional mean value theorem version.

For $0 < q < 1$, let T_q be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}.$$

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For a function $f : T_q \to \mathbb{R}$, the nabla $q-$derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \in T_q - \{0\} \tag{1.1}$$

The nabla $q-$integral of f is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i) \tag{1.2}$$

and for $0 \leq a \in T_q$

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s$$

Alternatively, let $a = q^{n_0} \in T_q$ where $n < n_0$, the nabla $q-$integral of f is given by

$$\int_a^t f(s)\nabla_q s = (1-q) \sum_{i=n}^{n_0-1} q^i f(q^i) \tag{1.3}$$

From the theory of $q-$calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q(f(t)g(t)) = f(qt)\nabla_q g(t) + (\nabla_q f(t))g(t) \tag{1.4}$$

The $q-$factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \tag{1.5}$$

More generally, when α is a not a positive integer, the $q-$factorial fractional function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t}q^i}{1 - \frac{s}{t}q^{i+\alpha}} \tag{1.6}$$

The following properties will be used inside the proofs of the main results.

- $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t - q^\beta s)_q^\gamma$
- $(at - as)_q^\beta = a^\beta (t-s)_q^\beta$
- The nabla $q-$derivative of the $q-$factorial function with respect to t is
$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-s)_q^{\alpha-1}$$
- The nabla $q-$derivative of the $q-$factorial function with respect to s is
$$\nabla_q (t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q}(t-qs)_q^{\alpha-1}$$
where $\alpha, \beta, \gamma \in \mathbb{R}$.

Moreover, the $q-$fractional integral of order $\alpha \neq 0, -1, -2, ...$ is defined by

$${}_qI_0^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s. \tag{1.7}$$

Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the $\alpha-$order Liouville-Caputo (left) $q-$fractional derivative of a function f is defined by [3]

$${}_qC_a^\alpha f(t) \triangleq {}_qI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s)\nabla_q s \tag{1.8}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the greatest integer less than or equal to α. If $\alpha \in \mathbb{N}$, then ${}_qC_a^\alpha f(t) \triangleq \nabla_q^n f(t)$.
Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the $\alpha-$order Riemann (left) $q-$fractional derivative of a function f is defined by

$$ {}_q\nabla_a^\alpha f(t) \triangleq \nabla_q^n \ {}_qI_a^{(n-\alpha)} f(t) \tag{1.9}$$

The following identity is useful to transform Caputo $q-$fractional difference equation into $q-$fractional integrals.
Assume $\alpha > 0$ and f is defined in suitable domains. Then [3]

$$ {}_qI_a^\alpha \ {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \tag{1.10}$$

and if $0 < \alpha \leq 1$ then

$$ {}_qI_a^\alpha \ {}_qC_a^\alpha f(t) = f(t) - f(a) \tag{1.11}$$

Also, the the identity ${}_qI_a^\alpha \ {}_q\nabla_a^\alpha f(t) = f(t)$ is known. The following identity is essential to solve linear $q-$fractional equations

$$ {}_qI_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \ \ (0 \leq a < x < b) \tag{1.12}$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$.
For more about $q-$Gamma functions and other $q-$calculus concepts we refer, for example. to [19] The following lemma is used in the proof of the main result. The proof follows from (1.6).

Lemma 1.1.
- $(1-q^i)_q^{-\alpha} = \frac{1-q^i}{1-q^{i-\alpha}} (1-q^{i+1})_q^{-\alpha}$
- $(q^{n+1}-1)_q^{-\alpha} = \frac{1-q^{-1-n}}{q^\alpha - q^{-1-n}} (q^n-1)_q^{-\alpha}$
- $(q^m - q^n)_q^{-\alpha} = \frac{1-q^{n-m}}{q^\alpha - q^{n-m}} (q^{m-1} - q^n)_q^{-\alpha}$
- $(q^m - q^{n-1})_q^{-\alpha} = \frac{1-q^{-m+n-1}}{1-q^{-m+n-1-\alpha}} (q^m - q^n)_q^{-\alpha}$

Definition 1.2. *Fix $\alpha \geq 0$ and define*

$$ c_q(\alpha) = \frac{1-q^\alpha}{1-q} q^{1-\alpha} $$

Definition 1.3. *Let $y : T_q \to R$ be a function. y is called a $c_q(\alpha)-$increasing on T_q, if*

$$ y(q^{n-1}) \geq c_q(\alpha) y(q^n) \text{ for all } q^n \in T_q. $$

Definition 1.4. *Let $y : T_q \to R$ be a function. y is called a $c_q(\alpha)-$decreasing on T_q, if*

$$ y(q^{n-1}) \leq c_q(\alpha) y(q^n) \text{ for all } q^n \in T_q. $$

Notice that if $\alpha \geq 1$, then $c_q(\alpha) \geq 1$ and if $0 \leq \alpha \leq 1$, then $0 \leq c_q(\alpha) \leq 1$. Hence, if y is increasing (decreasing) on T_q and and $0 < \alpha < 1$ then y is $c_q(\alpha)-$increasing (decreasing) on T_q. Also, if y is $c_q(\alpha)-$increasing (decreasing) on T_q and $\alpha > 1$, then, y is increasing (decreasing) on T_q. If $\alpha = 1$ then y is $c_q(\alpha)-$increasing (decreasing) if and only if y is increasing (decreasing).

Our main Theorems that we will present their proofs in Section 2 are

Theorem 1.5. *Let $y : T_q \to \mathbb{R}$ be a function satisfying $y(a) \geq 0$, $a = q^{n_0} > 0$. Suppose that*

$$ {}_q\nabla^{\alpha}_{aq} y(t) \geq 0 \text{ for each } t = q^n, n < n_0. $$

Then, y is $c_q(\alpha)-$increasing on $\{t \in T_q : t \geq a = q^{n_0}\}$.

And conversely,

Theorem 1.6. *Let $y : T_q \to \mathbb{R}$ be a function satisfying $y(q^{n_0}) \geq 0$, $a = q^{n_0}$. Suppose that y is an increasing function on T_q. Then,*

$$ {}_q\nabla^{\alpha}_{aq} y(t) \geq 0 \text{ for each } t = q^n, n < n_0. $$

2. Main Results

In this Section we present the proof of our main results stated in Theorem 1.5 and Theorem 1.6 and their direct consequences.

Proof of Theorem 1.5: Let ${}_q\nabla^{\alpha}_{aq} y(t) \geq 0$ for each $t \in T_q$, $\alpha \in (0,1)$, then

$$ {}_q\nabla^{\alpha}_{aq} y(t) = \nabla_q \nabla^{-(1-\alpha)}_{aq} y(t) = \nabla_q \left[\frac{1}{\Gamma_q(1-\alpha)} \int_{aq}^{t} (t-qs)^{-\alpha}_q y(s) \nabla_q(s) \right] \geq 0 $$

Let $s(t) = \frac{1-q}{\Gamma_q(1-\alpha)} \sum_{i=n_0}^{n} q^i (q^n - q^{i+1})^{-\alpha}_q y(q^i)$. Since $\nabla_q s(t) \geq 0$, $s(t)$ is an increasing function on T_q. This implies that

$$ \begin{aligned} s(q^{n_0-1}) - s(q^{n_0}) &= \frac{(1-q)q^{n_0-1}}{\Gamma_q(1-\alpha)} [(q^{n_0-1} - q^{n_0})^{-\alpha}_q y(q^{n_0-1}) \\ &+ q(q^{n_0-1} - q^{n_0+1})^{-\alpha}_q y(q^{n_0}) - q(q^{n_0} - q^{n_0+1})^{-\alpha}_q y(q^{n_0})] \\ &= \frac{(1-q)q^{n_0-1}}{\Gamma_q(1-\alpha)} [q^{\alpha}(q^{n_0} - q^{n_0+1})^{-\alpha}_q y(q^{n_0-1}) \\ &+ q\frac{q^{\alpha}-q}{1-q}(q^{n_0} - q^{n_0+1})^{-\alpha}_q y(q^{n_0}) - q(q^{n_0} - q^{n_0+1})^{-\alpha}_q y(q^{n_0})] \\ &\geq 0 \end{aligned} $$

therefore, we have

$$ q[\frac{q^{\alpha}-q}{1-q} - 1] y(q^{n_0}) + q^{\alpha} y(q^{n_0-1}) \geq 0 $$

which implies that

$$ y(q^{n_0-1}) \geq \frac{1-q^{\alpha}}{1-q} q^{1-\alpha} y(q^{n_0}) $$

Now, we assume that the hypothesis is true for $n = k$. i.e.

$$ y(q^{n_0-k}) \geq \frac{1-q^{\alpha}}{1-q} q^{1-\alpha} y(q^{n_0-k+1}) $$

hence,we have

$$ y(q^{n_0-k}) \geq c_q(\alpha) y(q^{n_0-k+1}) \geq c^2_q(\alpha) y(q^{n_0-k+2}) \geq ... \geq c^{k-1}_q(\alpha) y(q^{n_0-1}) \geq c^k_q(\alpha) y(q^{n_0}) \tag{2.1} $$

We want to prove that

$$ y(q^{n_0-k-1}) \geq c_q(\alpha) y(q^{n_0-k}). \tag{2.2} $$

We start by calculating,

$$
\begin{aligned}
& s(q^{n_0-k-1}) - s(q^{n_0-k}) \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)}\left[\sum_{i=n_0-k-1}^{n_0} q^i(q^{n_0-k-1}-q^{i+1})_q^{-\alpha}y(q^i) - \sum_{i=n_0-k}^{n_0} q^i(q^{n_0-k}-q^{i+1})_q^{-\alpha}y(q^i)\right] \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)}[q^{n_0-k-1}(q^{n_0-k-1}-q^{n_0-k})_q^{-\alpha}y(q^{n_0-k-1}) \\
+ \; & q^{n_0-k}(q^{n_0-k-1}-q^{n_0-k+1})_q^{-\alpha}y(q^{n_0-k}) - q^{n_0-k}(q^{n_0-k}-q^{n_0-k+1})_q^{-\alpha}y(q^{n_0-k}) \\
+ \; & q^{n_0-k+1}(q^{n_0-k-1}-q^{n_0-k+2})_q^{-\alpha}y(q^{n_0-k+1}) - q^{n_0-k+1}(q^{n_0-k}-q^{n_0-k+2})_q^{-\alpha}y(q^{n_0-k+1}) \\
+ \; & ..... \\
+ \; & q^{n_0-1}(q^{n_0-k-1}-q^{n_0})_q^{-\alpha}y(q^{n_0-1}) - q^{n_0-1}(q^{n_0-k}-q^{n_0})_q^{-\alpha}y(q^{n_0-1}) \\
+ \; & q^{n_0}(q^{n_0-k-1}-q^{n_0+1})_q^{-\alpha}y(q^{n_0}) - q^{n_0}(q^{n_0-k}-q^{n_0+1})_q^{-\alpha}y(q^{n_0})] \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)}[q^{n_0-k-1}(q^{n_0-k-1}-q^{n_0-k})_q^{-\alpha}y(q^{n_0-k-1}) \\
+ \; & q^{n_0-k}(q^{n_0-k}-q^{n_0-k+1})_q^{-\alpha}y(q^{n_0-k})(\frac{q^\alpha-q}{1-q}-1) \\
+ \; & q^{n_0-k+1}(q^{n_0-k}-q^{n_0-k+2})_q^{-\alpha}y(q^{n_0-k+1})(\frac{q^\alpha-q^2}{1-q^2}-1) \\
+ \; & ..... \\
+ \; & q^{n_0-1}(q^{n_0-k}-q^{n_0})_q^{-\alpha}y(q^{n_0-1})(\frac{q^\alpha-q^k}{1-q^k}-1) \\
+ \; & q^{n_0}(q^{n_0-k}-q^{n_0+1})_q^{-\alpha}y(q^{n_0})(\frac{q^\alpha-q^{k+1}}{1-q^{k+1}}-1)]
\end{aligned}
$$

Since $s(t)$ is increasing, we get

$$
\begin{aligned}
& q^{n_0-k-1}(q^{n_0-k-1}-q^{n_0-k})_q^{-\alpha}y(q^{n_0-k-1}) + q^{n_0-k}(q^{n_0-k}-q^{n_0-k+1})_q^{-\alpha}y(q^{n_0-k})(\frac{q^\alpha-1}{1-q}) \\
\geq \; & q^{n_0-k+1}(q^{n_0-k}-q^{n_0-k+2})_q^{-\alpha}y(q^{n_0-k+1})(\frac{1-q^\alpha}{1-q^2}) \\
+ \; & ..... \\
+ \; & q^{n_0-1}(q^{n_0-k}-q^{n_0})_q^{-\alpha}y(q^{n_0-1})(\frac{1-q^\alpha}{1-q^k}) \\
+ \; & q^{n_0}(q^{n_0-k}-q^{n_0+1})_q^{-\alpha}y(q^{n_0})(\frac{1-q^\alpha}{1-q^{k+1}})
\end{aligned}
$$

Using the induction assumption (2.1), we get

$$\begin{aligned}
& q^{n_0-k-1}(q^{n_0-k}-q^{n_0-k+1})_q^{-\alpha}\left[q^{\alpha}y(q^{n_0-k-1})+q\frac{q^{\alpha}-1}{1-q}y(q^{n_0-k})\right] \\
\geq\ & q^{n_0-k+1}(q^{n_0-k}-q^{n_0-k+2})_q^{-\alpha}(\frac{1-q^{\alpha}}{1-q^2})(c_q(\alpha))^{k-1}y(q^{n_0}) \\
+\ & ..... \\
+\ & q^{n_0-1}(q^{n_0-k}-q^{n_0})_q^{-\alpha}(\frac{1-q^{\alpha}}{1-q^k})(c_q(\alpha))y(q^{n_0}) \\
+\ & q^{n_0}(q^{n_0-k}-q^{n_0+1})_q^{-\alpha}(\frac{1-q^{\alpha}}{1-q^{k+1}})y(q^{n_0})
\end{aligned}$$

Since $y(q^{n_0})\geq 0$, we conclude that $q^{\alpha}y(q^{n_0-k-1})+q\frac{q^{\alpha}-1}{1-q}y(q^{n_0-k})\geq 0$ which implies that $y(q^{n_0-k-1})\geq c_q(\alpha)y(q^{n_0-k})$, which completes the proof.

Using Theorem 1.5 and the following identity that relates (Riemann) $q-$fractional derivative ${}_q\nabla_a^{\alpha}$ and the Caputo $q-$fractional derivative ${}_qC_a^{\alpha}$ of order $0<\alpha<1$ [1]

$$({}_qC_a^{\alpha}f)(t)=({}_q\nabla_a^{\alpha}f)(t)-\frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)}y(a),$$

we can state the following Caputo monotonicity result:

Corollary 2.1. *Let $y:T_q\to\mathbb{R}$ be a function satisfying $y(a)\geq 0$, $a=q^{n_0}>0$. Suppose that*

$${}_qC_{aq}^{\alpha}y(t)\geq -\frac{(t-qa)_q^{-\alpha}}{\Gamma_q(1-\alpha)}y(qa)\ \textit{for each}\ \ t=q^n, n<n_0.$$

Then, y is $c_q(\alpha)-$increasing on $\{t\in T_q : t\geq a=q^{n_0}\}$.

Proof of Theorem 1.6: We want to prove that

$${}_q\nabla_{aq}^{\alpha}y(t)=\nabla_q\nabla_{aq}^{-(1-\alpha)}y(t)=\nabla_q\left[\frac{1}{\Gamma_q(1-\alpha)}\int_{aq}^{t}(t-qs)_q^{-\alpha}y(s)\nabla_q(s)\right]\geq 0$$

Let

$$s(t)=\left[\frac{1}{\Gamma_q(1-\alpha)}\int_{aq}^{t}(t-qs)_q^{-\alpha}y(s)\nabla_q(s)\right]=\frac{1-q}{\Gamma_q(1-\alpha)}\sum_{i=n_0}^{n}q^i(q^n-q^{i+1})_q^{-\alpha}y(q^i).$$

Since $\nabla_q s(t) \geq 0$. We need to show that $s(t)$ is increasing on T_q. i.e. we need to show that $s(q^{n_0-k-1}) - s(q^{n_0-k}) \geq 0$ for any natural number k with $k \geq 1$. In fact,

$$\begin{aligned}
& s(q^{n_0-k-1}) - s(q^{n_0-k}) \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)} \left[\sum_{i=n_0-k-1}^{n_0} q^i (q^{n_0-k-1} - q^{i+1})_q^{-\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i) \right] \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
+ \; & \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k-1} - q^{i+1})_q^{-\alpha} y(q^i) - \sum_{i=n_0-k}^{n_0} q^i (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i)] \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
+ \; & \sum_{i=n_0-k}^{n_0} q^i [\frac{q^\alpha - q^{i+1-n_0+k}}{1-q^{i+1-n_0+k}} - 1] (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^i)] \\
= \; & \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
+ \; & \sum_{i=n_0-k}^{n_0} q^i \frac{q^\alpha - 1}{1-q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1} - q^i) \\
- \; & \sum_{i=n_0-k}^{n_0} q^i \frac{q^\alpha - 1}{1-q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1})] \\
\geq \; & \frac{(1-q)}{\Gamma_q(1-\alpha)} [q^{n_0-k-1} (q^{n_0-k-1} - q^{n_0-k})_q^{-\alpha} y(q^{n_0-k-1}) \\
+ \; & \sum_{i=n_0-k}^{n_0} q^i \frac{1-q^\alpha}{1-q^{i+1-n_0+k}} (q^{n_0-k} - q^{i+1})_q^{-\alpha} y(q^{n_0-k-1})] \geq 0.
\end{aligned}$$

Which concludes the proof.

Theorem 2.2. *Let $y : T_q \to \mathbb{R}$ be a function satisfying $y(q^{n_0}) > 0$, $a = q^{n_0}$. Suppose that y is a strictly increasing function on T_q. Then,*

$$_q\nabla_{aq}^{\alpha} y(t) > 0 \text{ for each } t = q^n, n < n_0.$$

In a similar way, the above results can be obtained for the function which takes negative value at the initial point of its domain.

Theorem 2.3. *Let $y : T_q \to \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that*

$$_q\nabla_{aq}^{\alpha} y(t) \leq 0 \text{ for each } t = q^n, n < n_0.$$

Then, y is $c_q(\alpha)-$decreasing on T_q

Theorem 2.4. *Let $y : T_q \to \mathbb{R}$ be a function satisfying $y(q^{n_0}) \leq 0$. Suppose that y is a decreasing function on T_q. Then,*

$$_q\nabla_{aq}^{\alpha} y(t) \leq 0 \text{ for each } \; t = q^n, n < n_0.$$

3. An Application

In this section, we wish to prove a Mean-Value Theorem in $q-$fractional calculus. First, we need the following result:

Theorem 3.1. *Let f be defined on T_q and $a, b \in T_q$ with $a < b$. Then the following equality holds:*

$$ {}_q\nabla_a^{-\alpha}\ {}_q\nabla_{aq}^{\alpha} f(t)|_b = f(b) - \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)} f(a) \ \text{ where } \alpha \in (0,1). $$

Proof.

$$ \begin{aligned} & {}_q\nabla_a^{-\alpha}\ {}_q\nabla_{aq}^{\alpha} f(t)\,|_b = {}_q\nabla_a^{-\alpha}(\nabla_q\ {}_q\nabla_{aq}^{-(1-\alpha)} f(t))\,|_b \\ = \ & \nabla_q\ {}_q\nabla_a^{-\alpha}\ {}_q\nabla_{aq}^{-(1-\alpha)} f(t)\,|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)}\ {}_q\nabla_{aq}^{-(1-\alpha)} f(a)\,|_b \\ = \ & \nabla_q\ {}_q\nabla_a^{-\alpha}\left[{}_q\nabla_a^{-(1-\alpha)} f(t) + \frac{(1-q)a(t-qa)_q^{-\alpha} f(a)}{\Gamma_q(1-\alpha)}\right]|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)}\ {}_q\nabla_{aq}^{-(1-\alpha)} f(a)\,|_b \\ = \ & \nabla_q\ {}_q\nabla_a^{-1} f(t)\,|_b - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)}\ {}_q\nabla_{aq}^{-(1-\alpha)} f(a)\,|_b \\ = \ & f(b) - \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)} f(a). \end{aligned} $$

□

Theorem 3.2. *Assume f and g are defined on T_q, $a, b \in T_q$, $a > 0$ and g is strictly increasing on $\Delta = \{t \in T_q : t \geq a, t \leq b\}$ and satisfying $g(a) > 0$. Then, there exists $r_1, r_2 \in \Delta$ such that*

$$ \frac{{}_q\nabla_{aq}^{\alpha} f(r_1)}{{}_q\nabla_{aq}^{\alpha} g(r_1)} \leq \frac{f(b) - M_q(\alpha,a,b)f(a)}{g(b) - M_q(\alpha,a,b)g(a)} \leq \frac{{}_q\nabla_{aq}^{\alpha} f(r_2)}{{}_q\nabla_{aq}^{\alpha} g(r_2)}. $$

Where $M_q(\alpha,a,b) = \frac{(1-q)a^{1-\alpha}(b-a)_q^{\alpha-1}(1-q)_q^{-\alpha}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}$.

Proof. Suppose without loss of generality to the contrary that

$$ \frac{f(b) - M_q(\alpha,a,b)f(a)}{g(b) - M_q(\alpha,a,b)g(a)} > \frac{{}_q\nabla_{aq}^{\alpha} f(t)}{{}_q\nabla_{aq}^{\alpha} g(t)} $$

Since g is strictly increasing, Theorem (2.2) implies that ${}_q\nabla_{aq}^{\alpha} g(t) > 0$. Hence

$$ \frac{f(b) - M_q(\alpha,a,b)f(a)}{g(b) - M_q(\alpha,a,b)g(a)}\ {}_q\nabla_{aq}^{\alpha} g(t) > \ {}_q\nabla_{aq}^{\alpha} f(t). $$

We apply ${}_q\nabla_a^{-\alpha}$ at $t = b$ to both sides of the above inequality and make use of Theorem 3.1 to get

$$ \frac{f(b) - M_q(\alpha,a,b)f(a)}{g(b) - M_q(\alpha,a,b)g(a)}\,[g(b) - M_q(\alpha,a,b)g(a)] > f(b) - M_q(\alpha,a,b)f(a). $$

This leads to $g(b) > g(b)$, which is a contradiction. □

Remark.
- *Notice that if the edge point $a = 0$ then $M_q(\alpha,a,b) = 0$ and hence no Mean-Value Theorem.*
- *Notice that since $M_q(\alpha,a,b) < 1$ and $g(t)$ is strictly decreasing then $g(b) - M_q(\alpha,a,b)g(a)$ in the statement of Theorem 3.2 is not equal to zero.*

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