Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://ilirias.com/jma
Volume 7 Issue 4(2016), Pages 74-82.

HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT OF HARMONICALLY CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

MEHMET KUNT, İMDAT İŞCAN AND NAZLI YAZICI

Abstract. In this paper, some Hermite-Hadamard type inequalities for products of two harmonically convex functions via Riemann-Liouville fractional integrals are established. Our results about harmonically convex functions are analogous generalizations for some other results proved by Pachpette, Chan and Noor for convex and harmonically h-convex functions.

1. Introduction

It is known that Hermite-hadamard integral inequality was built on a convex function. In [16], one can find general notations of convexity.

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities.. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [2, 7, 8, 10, 12, 16, 17, 18, 19, 20, 23, 26] and references therein.

Hermite-Hadamard type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [21]. In [22], Pachpette also established Hermite-hadamard type inequalities involving two log-convex functions. In [13], Kırmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and s-convex functions. In [19], Özdemir et al. established Hadamard-type inequalities for product of s-convex functions on co-ordinates. In [24], Sarıkaya et al. proved some Hermite-Hadamard type inequalities for products of two h-convex functions. In [1], Bakula et al. established Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [4, 6], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically s-convex functions. In [25], Varošanec, introduced concept of h-convex functions, in [17, 18], Noor et al. established some Hadamard's type inequalities for product of relative convex functions, relative h-convex functions and harmonically convex functions. In [27], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two

2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.
Key words and phrases. Hermite-Hadamard inequality, Harmonically convex functions, Riemann-Liouville fractional integral.

Submitted April 19, 2016. Published June 22, 2016.

convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to m-convex and (α, m)-convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two harmonically convex functions via Riemann-Liouville fractional integrals. Our results are analogous generalization for some results in [4, 5, 17, 21].

2. Preliminaries

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

is well known in the literature as Hermite-Hadamard's inequality [9].

In [21], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

Theorem 2.1. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{2.2}$$

and

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \quad &\leq \quad \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\ &\quad + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \end{aligned} \tag{2.3}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

In [5], Chan established following two Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville fractional integrals as follows:

Theorem 2.2. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b) + J_{b-}^{\alpha}f(a)\,g(a)\right] \\ \leq \quad &\left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)M(a,b) + \frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ \leq \quad &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)\,g(b) + J_{b-}^{\alpha}f(a)\,g(a)\right] \\ &+ \frac{\alpha}{(\alpha+1)(\alpha+2)}M(a,b) + \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)N(a,b) \end{aligned} \tag{2.5}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

In [12], İşcan gave definition of harmonically convex functions as follows:

Definition 1. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x) \tag{2.6}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2.6) is reversed, then f is said to be harmonically concave.

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 2. [15]. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

Some results about Hermite-Hadamard type inequality for some class of convex functions via Riemann-Liouville fractional integrals see [3, 5, 11, 14, 18, 20] and referances therein.

In [11], İşcan and Wu presented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral forms as follows:

Theorem 2.3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{1/a-}(f\circ h)(1/b) \\ +J^{\alpha}_{1/b+}(f\circ h)(1/a) \end{array}\right\} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{2.7}$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [14], Kunt et al. presented, Hermite-Hadamard inequality in fractional integral forms for harmonically convex functions as follows:

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a) \\ +J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ h)(1/b) \end{array}\right\} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{2.8}$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

3. General Results

Theorem 3.1. *Let f and $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be non-negative and harmonically convex functions with $a<b$ and $f\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}J^{\alpha}_{1/a-}(f\circ h)(1/b)(g\circ h)(1/b)\\+J^{\alpha}_{1/b+}(f\circ h)(1/a)(g\circ h)(1/a)\end{array}\right\}\\ \leq\ &\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)M(a,b)+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b)\end{aligned}\tag{3.1}$$

where $\alpha>0$, $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$, $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Theorem 3.1 is direct result of (2.4) in Theorem 2.1. In (2.4), if we replace the convex functions f and g with the convex functions F defined $F:=f\circ h$ and G defined $G:=g\circ h$, $h(x)=\frac{1}{x}$ and if we replace a and b with $\frac{1}{b}$ and $\frac{1}{a}$ respectively, we have (3.1). □

Corollary 3.2. *In Theorem 3.1, if we take $g:[a,b]\to\mathbb{R}$ as $g(x)=1$ for all $x\in[a,b]$, then we have*

$$\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c}J^{\alpha}_{1/a-}(f\circ h)(1/b)\\+J^{\alpha}_{1/b+}(f\circ h)(1/a)\end{array}\right\}\leq\frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.7).

Corollary 3.3. *In Theorem 3.1, if we take $\alpha=1$, then we recapture*

$$\frac{ab}{b-a}\int_a^b f(x)g(x)\frac{dx}{x^2}\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)$$

for harmonically convex functions which is given in [6, Remark 11].

Remark. *Corollary 3.3, about harmonically convex functions, we have an analogue result of (2.2) for convex functions by Pachpette in* [21]. *If one take $h(t)=t$ in* [17, Theorem 3.6.], *one has the same inequality.*

Theorem 3.4. *Let f and $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be non-negative and harmonically convex functions with $a<b$ and $f\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\leq\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\\&\times\left\{\begin{array}{c}J^{\alpha}_{1/a-}(f\circ h)(1/b)(g\circ h)(1/b)\\+J^{\alpha}_{1/b+}(f\circ h)(1/a)(g\circ h)(1/a)\end{array}\right\}\\&+\frac{\alpha}{(\alpha+1)(\alpha+2)}M(a,b)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(a,b)\end{aligned}\tag{3.2}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Theorem 3.4 is direct result of (2.5) in Theorem 2.1. In (2.5), if we replace the convex functions f and g with the convex functions F defined $F := f \circ h$ and G defined $G := g \circ h$, $h(x) = \frac{1}{x}$ and if we replace a and b with $\frac{1}{b}$ and $\frac{1}{a}$ respectively, we have (3.2). □

Corollary 3.5. *In Theorem* 3.4*, if we take* $g : [a,b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a,b]$*, then we have*

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{1/a-}(f\circ h)(1/b) + J^{\alpha}_{1/b+}(f\circ h)(1/a)\right\} \\ &\quad + \frac{f(a)+f(b)}{2}. \end{aligned}$$

Corollary 3.6. *In Theorem* 3.4*, if we take* $\alpha = 1$*, then we recapture*

$$2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b f(x)\,g(x)\,\frac{dx}{x^2} + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

for harmonically convex functions which is given in [6, Remark 14].

Remark. *Corollary* 3.6*, about harmonically convex functions, we have an analogue result of* (2.3) *for convex functions by Pachpette in* [21].

Theorem 3.7. *Let* f *and* g *be real-valued, non-negative and convex functions on* $[a,b]$ *with* $a < b$ *and* $f, g \in L[a,b]$*, then the following inequality for the fractional integrals holds:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left\{J^{\alpha}_{\frac{a+b}{2}+}f(b)\,g(b) + J^{\alpha}_{\frac{a+b}{2}-}f(a)\,g(a)\right\} \\ \leq\ &\left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right)M(a,b) \\ &+ \frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}N(a,b) \end{aligned} \tag{3.3}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ *and* $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

Proof. Since f and g are convex functions on $[a,b]$, then for all $t \in [0,1]$ we have

$$\begin{aligned} f(ta+(1-t)b) &\leq tf(a) + (1-t)f(b), \\ g(ta+(1-t)b) &\leq tg(a) + (1-t)g(b), \\ f(tb+(1-t)a) &\leq tf(b) + (1-t)f(a), \\ g(tb+(1-t)a) &\leq tg(b) + (1-t)g(a). \end{aligned}$$

A composition of this inequalities, we have

$$\begin{aligned} &f(ta+(1-t)b)\,g(ta+(1-t)b) + f(tb+(1-t)a)\,g(tb+(1-t)a) \\ \leq\ &\left(2t^2-2t+1\right)M(a,b) + 2t(1-t)N(a,b). \end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $t^{\alpha-1}\frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\begin{aligned}
& \frac{\alpha}{2^{1-\alpha}}\left\{\int_0^{\frac{1}{2}} t^{\alpha-1} f\left(ta+(1-t)b\right) g\left(ta+(1-t)b\right) dt\right. \\
& \left.+\int_0^{\frac{1}{2}} t^{\alpha-1} f\left(tb+(1-t)a\right) g\left(tb+(1-t)a\right) dt\right\} \\
= & \frac{\alpha}{2^{1-\alpha}(b-a)^{\alpha}}\left\{\int_{\frac{a+b}{2}}^{b}(b-u)^{\alpha-1} f(u) g(u) du\right. \\
& \left.+\int_a^{\frac{a+b}{2}}(u-a)^{\alpha-1} f(u) g(u) du\right\} \\
= & \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left\{J_{\frac{a+b}{2}+}^{\alpha} f(b) g(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a) g(a)\right\} \\
\leq & \frac{\alpha}{2^{1-\alpha}}\left[M(a,b)\int_0^{\frac{1}{2}} t^{\alpha-1}\left(2t^2-2t+1\right) dt+N(a,b)\int_0^{\frac{1}{2}} t^{\alpha-1} 2t(1-t) dt\right] \\
= & \left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right) M(a,b)+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)} N(a,b).
\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let f and $g:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ be non-negative and harmonically convex functions with $a<b$ and $f,g\in L[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J_{\frac{a+b}{2ab}+}^{\alpha}(f\circ h)(1/a)(g\circ h)(1/a) \\ +J_{\frac{a+b}{2ab}-}^{\alpha}(f\circ h)(1/b)(g\circ h)(1/b) \end{array}\right\} \\
\leq & \left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right) M(a,b) \\
& +\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)} N(a,b) \qquad (3.5)
\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Theorem 3.8 is direct result of (3.3) in Theorem 3.7. In (3.3), if we replace the convex functions f and g with the convex functions F defined $F:=f\circ h$ and G defined $G:=g\circ h$, $h(x)=\frac{1}{x}$ and if we replace a and b with $\frac{1}{b}$ and $\frac{1}{a}$ respectively, we have (3.5). □

Corollary 3.9. *In Theorem 3.8, if we take $g:[a,b]\to\mathbb{R}$ as $g(x)=1$ for all $x\in[a,b]$, then we have*

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{\frac{a+b}{2ab}+}^{\alpha}(f\circ h)(1/a)+J_{\frac{a+b}{2ab}-}^{\alpha}(f\circ h)(1/b)\right\} \\
\leq & \frac{f(a)+f(b)}{2}
\end{aligned}$$

which is the right hand side of (2.8).

Corollary 3.10. *In Theorem* 3.8,*if we take* $\alpha = 1$*, then we recapture*

$$\frac{ab}{b-a}\int_a^b f(x)\,g(x)\,\frac{dx}{x^2} \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

for harmonically convex functions which is given in [6, Remark 11].

Remark. *In Corollary* 3.10*, about harmonically convex functions, we have an analogue result of* (2.2) *for convex functions by Pachpette in* [21]*. If one take* $h(t) = t$ *in* [17, Theorem 3.6.]*, one has the same inequality.*

Theorem 3.11. *Let* f *and* g *be real-valued, non-negative and convex functions on* $[a,b]$ *with* $a < b$ *and* $f, g \in L[a,b]$*, then the following inequality for the fractional integrals holds:*

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq\ & \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left\{J^{\alpha}_{\frac{a+b}{2}+}f(b)\,g(b) + J^{\alpha}_{\frac{a+b}{2}-}f(a)\,g(a)\right\} \\ & + \frac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}M(a,b) \\ & + \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right)N(a,b) \end{aligned} \tag{3.6}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ *and* $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

Proof. Since f and g are convex functions on $[a,b]$, then for all $t \in [0,1]$ we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ =\ & f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ \leq\ & \frac{1}{4}\left[f(ta+(1-t)b) + f((1-t)a+tb)\right]\left[g(ta+(1-t)b) + g((1-t)a+tb)\right] \\ =\ & \frac{1}{4}\left[f(ta+(1-t)b)\,g(ta+(1-t)b) + f((1-t)a+tb)\,g((1-t)a+tb)\right] \\ & \frac{1}{4}\left[f(ta+(1-t)b)\,g((1-t)a+tb) + f((1-t)a+tb)\,g(ta+(1-t)b)\right] \\ \leq\ & \frac{1}{4}\left[f(ta+(1-t)b)\,g(ta+(1-t)b) + f((1-t)a+tb)\,g((1-t)a+tb)\right] \\ & \frac{1}{4}\left\{2t(1-t)M(a,b) + \left[(1-t)^2+t^2\right]N(a,b)\right\} \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $2^{1+\alpha}\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have the desired result. □

Theorem 3.12. *Let* f *and* $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ *be non-negative and harmonically convex functions with* $a < b$ *and* $f \in L[a,b]$*, then the following inequality for*

fractional integrals holds:

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha} \\ &\times\left\{\begin{array}{c} J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a)(g\circ h)(1/a) \\ +J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ h)(1/b)(g\circ h)(1/b) \end{array}\right\} \\ &+\frac{\alpha^2+3\alpha}{4(\alpha+1)(\alpha+2)}M(a,b) \\ &+\left(\frac{\alpha}{4(\alpha+2)}-\frac{\alpha}{2(\alpha+1)}+\frac{1}{2}\right)N(a,b) \end{aligned} \tag{3.8}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ *and* $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Theorem 3.12 is direct result of (3.6) in Theorem 3.11. In (3.6), if we replace the convex functions f and g with the convex functions F defined $F:=f\circ h$ and G defined $G:=g\circ h$, $h(x)=\frac{1}{x}$ and if we replace a and b with $\frac{1}{b}$ and $\frac{1}{a}$ respectively, we have (3.8). □

Corollary 3.13. *In Theorem* 3.12*, if we take* $g:[a,b]\to\mathbb{R}$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, then we have*

$$\begin{aligned} 2f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ h)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ h)(1/b)\right\} \\ &+\frac{f(a)+f(b)}{2}. \end{aligned}$$

Corollary 3.14. *In Theorem* 3.12*, if we take* $\alpha=1$*, then we recapture*

$$2f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b f(x)g(x)\frac{dx}{x^2}+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b)$$

for harmonically convex functions which is given in [6, Remark 14].

Remark. *In Corollary* 3.14*, about harmonically convex functions, we have an analogue result of* (2.3) *for convex functions by Pachpette in* [21].

4. Acknowledgments

The authors are very grateful to the referee for helpful comments and valuable suggestions.

References

[1] M. K. Bakula, M. E. Özdemir, J. Pečaric, *Hadamard type inequalities form m-convex and* (α,m)*-convex functions*, J. Inequal. Pure Appl. Math. **9 4** 96 (2008) 1–12.

[2] S-P Bai, S-H Wang, F. Qi, *Some Hermite–Hadamard type inequalities for n-time differentiable* (α,m)*-convex functions*, J. Inequal. Appl. **267** (2012) 1–11.

[3] F. Chen, *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Chinese J. Math. (2014), **173293**, 1–7.

[4] F. Chen, *A note on Hermite-Hadamard inequalities for Products of convex functions*, J. Appl. Math. **935020** (2013) 1–5.

[5] F. Chen, *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Italian J. Pure Appl. Math. **33** (2014) 299–306.

[6] F. Chen, S. Wu, *Some Hermite-Hadamard type inequalities for harmonically s-convex functions*, Sci. World J. **279158** (2014), 1–7.

[7] S. S. Dragomir, *Refinements of the Hermite-Hadamard integral inequality for log-convex functions*, Aust. Math. Soc. Gaz. **28 3** (2001) 129–134.

[8] S. S. Dragomir, J. Pecaric, L.-E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21 3** (1995) 335–341.

[9] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **45** (1893) 171–215.

[10] S. Hussain, S. Qaisar, *New integral inequalities of the type of Hermite-Hadamard trough quasi convexity*, Punjab Univ. J. Math. **45** (2013) 33–38.

[11] İ. İşcan, S. Wu, *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Math. Comput. **238** (2014) 327–244.

[12] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacet. J. Math. Stat. **43 6** (2014) 935–942.

[13] U. S. Kırmacı, M. K. Bakula, M. E. Özdemir, J. Pečari´c *Hadamard-type inequalities for s-convex functions*, Appl. Math. Comput. **193 1** (2007) 26-35.

[14] M. Kunt, İ. İşcan, N. Yazıcı and U. Gözütok, *On new inequalities of Hermite-Hadamard-Fejer type for harmonically convex functions via fractional integrals*. SpringerPlus, **5 635** (2016) 1–19.

[15] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam 2006.

[16] C. P. Niculescu, L.-E. Persson, *Convex functions and their application, A contemporary approach*, Springer, New York, 2006.

[17] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, *Some integral inequalities for harmonically h-convex functions*, U. P. B. Sci. Bull., Series A **77 1** (2015) 5–16.

[18] M. A. Noor, K. I. Noor and M. U. Awan, *Generalized convexity and integral inequalities*, Appl. Math. Infor. Sci. **9 1** (2015) 233–243.

[19] M. E. Özdemir, M. A. Latif, A. O. Akdemir, *On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates*, J.Inequal. Appl. **21** (2012) 1–13.

[20] M. E. Özdemir, Ç. Yıldız, *The Hadamard's inequality for quasi-convex functions via fractional integrals*, Annals Univ. Craiova, Math. Comput. Sci. Ser, **40 2** (2013) 167–173.

[21] B. G. Pachpatte, *On some inequalities for convex functions*, RGMIA Resear. Rep. Coll. E **6** (2003) 1–9.

[22] B. G. Pachpatte, *A note on integral inequalities involving two log-convex functions*, Math. Inequal. Appl. **7 4** (2004) 511–515.

[23] A. M. Rubinov, J. Dutta, *Hadamard inequality for quasi-convex functions in higher dimensions*, J. Math. Anal. Appl. **270** (2002) 80–91.

[24] M. Z. Sarıkaya, A. Sağlam, and H. Yıldırım, *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal. **2 3** (2008) 335–341.

[25] S. Varosanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007)303–311.

[26] G. S. Yang, *Refinements of Hadamard inequality for r-convex functions*, Indian J. Pure Appl. Math. **32 10** (2001) 1571–1579.

[27] H.-P. Yin, F. Qi, *Hermite-Hadamard type inequalities for the product of* (α, m)*-convex functions*, J. Nonlinear Sci. Appl. **8** (2015) 231–236.

Mehmet Kunt(Corresponding author), Nazli Yazici
Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey
E-mail address: mkunt@ktu.edu.tr, nazliyazici@ktu.edu.tr

İmdat İşcan
Department of Mathematics, Faculty of Sciences and Arts, Giresun University, 28200, Giresun, Turkey
E-mail address: imdat.iscan@giresun.edu.tr;imdati@yahoo.com