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# ON LACUNARY STATISTICAL QUASI-CAUCHY SEQUENCES IN $n$-NORMED SPACES

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Abstract. In this paper, we introduce double lacunary statistical ward continuity and double lacunary statistically quasi-Cauchy sequences in $n$-normed spaces. We make an effort to prove some theorems related to compactness, continuity, ward continuity and uniform continuity.

## 1. Introduction

The concept of continuity plays an important role not only in pure mathematics but also in other branches of sciences involving mathematics especially in computer science, information theory, biological science etc. Menger [19] introduced a notion called a generalized metric in 1928 and ten years later Vulich [32] defined a notion of a higher dimensional normed linear spaces. Unfortunately, these studies had been neglected by many analysts for a long time. The concept of 2-normed space was developed by Gähler in the middle of 1960's ([10], [11] [12]). Since then, Mashadi [17], Gurdal [13], Mazaheri and Kazemi [18] have studied this concept and obtained various results.

Using the main idea in the definition of sequential continuity many kinds of continuities were introduced by some mathematicians (see [1], [3], [2], [7], [31]). The concept of lacunary statistical convergence was introduced by Fridy and Orhan in [8] and further studied in [9]. Lacunary statistical ward continuity of real functions was given in [4].

Let $\mathbb{N}$ and $\mathbb{R}$ be the set of all positive integers and the set of all real numbers, respectively. The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists .}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0,$$

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In this case we write $S-\lim_k x_k = L$ or $x \to L(S)$. The set of all statistical convergent sequences are denoted by $S$.
By the convergence of a double sequence we mean the convergence in Pringsheim's sense [25]. A double sequence $x = (x_{k,l})$ has a Pringsheim limit $L$ (denoted by $P - \lim x = L$) provided that given an $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. For details on double sequence spaces (see [22], [24]) and references therein.
Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be two dimentional set of positive integers and let $K(n,m)$ be the numbers of $(i,j)$ in $K$ such that $i \leq n$ and $j \leq m$, (see [25]). Then the lower natural density of $K$ is defined as

$$P - \liminf_{n,m} \frac{|K(n,m)|}{nm} = \delta_2(K).$$

In the case when the sequence $(\frac{K(n,m)}{mn})$ has a limit in Pringsheim's sense, we say that K has a double natural density and it is defined by

$$P - \lim_{n,m} \frac{|K(n,m)|}{nm} = \delta_2(K).$$

Mursaleen and Edely [22] defined the statistical analogue for double sequences $x = (x_{k,l})$ as follows. A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to $L$ if for each $\epsilon > 0$,

$$P - \lim_{n,m} \frac{1}{nm} |\{(k,l) : k < n \text{ and } l < m, |x_{k,l} - L| \geq \epsilon\}| = 0.$$

In this case we write $S_2 - \lim_{k,l} x_{k,l} = L$ and denote the set of all statistical convergent double sequences by $S_2$.
By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_r$. Recently, lacunary sequence spaces were studied in ([23], [29], [30]).
The double sequence $\theta = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

We denote $k_{r,s} = k_r l_s$ and $h_{r,s} = h_r \bar{h}_s$. Also the following intervals are determined by $\theta$ are:
$I_r = \{k : k_{r-1} < k \leq k_r\}$, $I_s = \{l : l_{s-1} < l \leq l_s\}$, $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_r}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$ (see [24]).

**Definition 1.1.** *A double sequence $x = (x_{k,l})$ is said to be lacunary statistical convergent to a number $L$ if for every $\epsilon > 0$, the set $\{(k,l) : |x_{k,l} - L| \geq \epsilon\}$ has double lacunary density zero. In this case, we write $S_2^\theta - \lim x = L$. The space of all lacunary statistical convergent double sequences is denoted by $S_2^\theta$.*

The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of $n$-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan

([14], [15]) and Gunawan and Mashadi [16]. Let $n \in \mathbb{N}$ and $X$ be a linear space over the field of real numbers $\mathbb{R}$ of dimension $d$, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on $X^n$ satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in $X$,
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an $n$-norm on $X$, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a *$n$-normed space* over the field $\mathbb{R}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the $n$-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the $n$-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$.
Let $(X, ||\cdot, \cdots, \cdot||)$ be an $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in $X$. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on $X^{n-1}$ as defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

is called an $(n-1)$-norm on $X$ with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to *converges* to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

A sequence $(x_k)$ in a $n$-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be *Cauchy* if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \ \text{ for every } \ z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be *complete* with respect to the $n$-norm. Any complete $n$-normed space is said to be *$n$-Banach space*. For more details about sequence spaces and $n$-normed spaces (see [21], [26], [27], [28]) and references therein.

**Definition 1.2.** *A double sequence $(x_{k,l})$ in an $n$-normed space $(X, \|., ..., .\|)$ is said to be double statistical convergent to $L$ with respect to $n$-norm if for each $\epsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{rs} |\{k \leq r, l \leq s : \|x_{k,l} - L, z_1, ..., z_{n-1}\| \geq \epsilon\}| = 0,$$

*for every $z_1, ..., z_{n-1} \in X$. Let $S_2(X)$ denote the set of all double statistical convergent sequences in an $n$-normed space $X$.*

**Definition 1.3.** *A double sequence $(x_{k,l})$ in an $n$-normed space $(X, \|., ..., .\|)$ is said to be double statistical Cauchy with respect to $n$-norm for each $\epsilon > 0$ there exist positive integers $p, q$ such that the set $\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|x_{k,l} - x_{p,q}, z_1, ..., z_{n-1}\| \geq \epsilon\}$ has double natural density zero, for every $z_1, ..., z_{n-1} \in X$.*

The main objective of this paper is to introduce the concept of ward continuity in $n$-normed spaces. We make an effort to study lacunary statistical sequential

continuity, lacunary statistical ward continuity and lacunary statistically quasi-Cauchy sequences in $n$-normed spaces.

## 2. Main Results

**Definition 2.1.** *A double sequence $(x_{k,l})$ in an $n$-normed space $(X, \|., ..., .\|)$ is said to be double lacunary statistical convergent or $S_{\theta_{r,s}}$-convergent to $L$ with respect to $n$-norm, denoted by $S_{\theta_{r,s}} - \lim\limits_{k,l\to\infty} x_{k,l} = L$, if for each $\epsilon > 0$, we have*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k,l} - L, z_1, ..., z_{n-1}\| \geq \epsilon\}| = 0,$$

*where $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$.*

**Definition 2.2.** *A double sequence $(x_{k,l})$ in an $n$-normed space $(X, \|., ..., .\|)$ is said to be double lacunary statistical Cauchy or $S_{\theta_{r,s}}$-Cauchy with respect to $n$-norm if there is a subsequence $(x_{\bar{k}_r,\bar{l}_s})$ of $(x_{k,l})$ such that $(\bar{k_r}, \bar{l_s}) \in I_{r,s}$ for each $r, s$,*

$$P - \lim_{r,s} \|x_{\bar{k}_r,\bar{l}_s} - L, z_1, ..., z_{n-1}\| = 0$$

*and for each $\epsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k,l} - x_{\bar{k}_r,\bar{l}_s}, z_1, ..., z_{n-1}\| \geq \epsilon\}| = 0.$$

**Theorem 2.3.** *Let $X$ be an $n$-normed space. If $(x_{k,l})$ is a double sequence in $X$ such that $S_{\theta_{r,s}} - \lim\limits_{k,l\to\infty} x_{k,l} = L$ exists, then the limit is unique.*

*Proof.* Suppose that there exist elements $L_1, L_2 (L_1 \neq L_2)$ in $X$ such that

$$S_{\theta_{r,s}} - \lim_{k,l\to\infty} x_{k,l} = L_1; \quad S_{\theta_{r,s}} - \lim_{k,l\to\infty} x_{k,l} = L_2.$$

Since $L_1 - L_2 \neq 0$, there exist $z_1, ..., z_{n-1} \in X$ such that $L_1 - L_2$ and $z_1, ..., z_{n-1}$ are linearly independent. Then

$$\epsilon = \frac{1}{2} \|L_1 - L_2, z_1, ..., z_{n-1}\|$$

is positive. Note that

$$\begin{aligned} 2\epsilon &= \|(L_1 - x_{k,l} + x_{k,l} - L_2), z_1, ..., z_{n-1}\| \\ &\leq \|x_{k,l} - L_1, z_1, ..., z_{n-1}\| + \|x_{k,l} - L_2, z_1, ..., z_{n-1}\|. \end{aligned}$$

So

$$\begin{aligned} &\{(k,l) \in I_{r,s} : \|x_{k,l} - L_2, z_1, ..., z_{n-1}\| \geq \epsilon\} \\ &\qquad \subseteq \{(k,l) \in I_{r,s} : \|x_{k,l} - L_1, z_1, ..., z_{n-1}\| \geq \epsilon\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k,l} - L_2, z_1, ..., z_{n-1}\| \geq \epsilon\}| \\ &\qquad \leq = \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k,l} - L_1, z_1, ..., z_{n-1}\| \geq \epsilon\}|. \end{aligned}$$

Since $S_{\theta_{r,s}} - \lim\limits_{k,l\to\infty} x_{k,l} = L_1$, it follows that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|x_{k,l} - L_2, z_1, ..., z_{n-1}\| \geq \epsilon\}| \leq 0$$

and consequently,

$$P-\lim_{r,s}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}:\|x_{k,l}-L_2,z_1,...,z_{n-1}\|\geq\epsilon\}|=0$$

as it cannot be negative. This contradicts the fact that $S_{\theta_{r,s}}-\lim_{k,l\to\infty}x_{k,l}=L_2$ and the proof is complete. □

**Definition 2.4.** *A subset $E$ of $X$ is called $S_{\theta_{r,s}}$-sequentially compact if any double sequence of points in $E$ has an $S_{\theta_{r,s}}$-convergent subsequence with an $S_{\theta_{r,s}}$-limit in $E$.*

We note that the union of two $S_{\theta_{r,s}}$-sequentially compact subsets of $X$ is $S_{\theta_{r,s}}$-sequentially compact, the sum of two $S_{\theta_{r,s}}$-sequentially compact subsets of $X$ is $S_{\theta_{r,s}}$-sequentially compact, the intersection of any family of two $S_{\theta_{r,s}}$-sequentially compact subsets is $S_{\theta_{r,s}}$-sequentially compact, any compact subset of $X$ is $S_{\theta_{r,s}}$-sequentially compact.

**Definition 2.5.** *A function $f$ defined on a subset $E$ of $X$ is said to $S_{\theta_{r,s}}$-sequentially continuous on $E$ if it preserves $S_{\theta_{r,s}}$-convergent sequence whenever $(x_{k,l})$ is an $S_{\theta_{r,s}}$-convergent sequence.*

We see that if $(x_{k,l})$ is an $S_{\theta_{r,s}}$-convergent sequence with $S_{\theta_{r,s}}-\lim_{k,l\to\infty}\|x_{k,l},z_1,...,z_{n-1}\|$ $=\|x_{0,0},z_1,...,z_{n-1}\|$ for every $z_1,...,z_{n-1}\in X$, then $(f(x_{k,l}))$ is an $S_{\theta_{r,s}}$-convergent sequence with $S_{\theta_{r,s}}-\lim_{k,l\to\infty}\|f(x_{k,l}),z_1,...,z_{n-1}\|=\|f(x_{0,0}),z_1,...,z_{n-1}\|$ for every $z_1,...,z_{n-1}\in X$. We note that the sum of two $S_{\theta_{r,s}}$-sequentially continuous function at a point $x_{0,0}$ of $X$ is $S_{\theta_{r,s}}$-sequentially continuous at $x_{0,0}$ and the composition of two $S_{\theta_{r,s}}$-sequentially continuous function at a point $x_{0,0}$ of $X$ is $S_{\theta_{r,s}}$-sequentially continuous at $x_{0,0}$.

**Theorem 2.6.** *Let $(f_{n,m})$ be a sequence of double lacunary statistically sequentially continuous functions defined on a subset $E$ of $X$ into $X$ for each $n,m\in\mathbb{N}$ and $(f_{n,m})$ be uniformly convergent to a function $f$ and then $f$ is double lacunary statistically sequentially continuous.*

*Proof.* Suppose $(f_{n,m})$ be a uniformly convergent sequence with uniform limit $f$ and $(x_{k,l})$ be any $S_{\theta_{r,s}}$-convergent sequence of points in $E$ with $S_{\theta_{r,s}}-\lim_{k,l\to\infty}\|x_{k,l},z_1,...,z_{n-1}\|$ $=\|x_{0,0},z_1,...,z_{n-1}\|$ for every $z_1,...,z_{n-1}\in X$. Let any $\epsilon>0$, by uniform convergence of $(f_{n,m})$ there exist $N,M\in\mathbb{N}$ such that $\|f(x)-f_{n,m}(x),z_1,...,z_{n-1}\|<\frac{\epsilon}{3}$ for every $n\geq N,m\geq M$, $x\in E$ and $z_1,...,z_{n-1}\in X$. Since $f_{N,M}$ is double lacunary statistically sequentially continuous on $E$, we have

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{0,0})-f_{N,M}(x_{k,l}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\right\}\right|=0.$$

On the other hand, we have

$$\begin{aligned}&\{(k,l)\in I_{r,s}:\|f(x_{0,0})-f(x_{k,l}),z_1,...,z_{n-1}\|\geq\epsilon\}\\&\qquad\subset\left\{(k,l)\in I_{r,s}:\|v_{k,l},z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\right\}\\&\qquad\cup\left\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{0,0})-f_{N,M}(x_{k,l}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\right\}\\&\qquad\cup\left\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k,l})-f(x_{k,l}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\right\},\end{aligned}$$

where $v_{k,l} = f(x_{0,0}) - f_{N,M}(x_{0,0})$ for every $k, l \in \mathbb{N}$. Thus

$$\begin{aligned}
&\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|f(x_{0,0}) - f(x_{n,m}), z_1, ..., z_{n-1}\| \geq \epsilon\}| \\
&\quad \leq \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \|v_{k,l}, z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3} \right\} \right| \\
&\quad + \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \|f_{N,M}(x_{0,0}) - f_{N,M}(x_{n,m}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3} \right\} \right| \\
&\quad + \lim_{r,s} \frac{1}{h_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : \|f_{N,M}(x_{n,m}) - f(x_{n,m}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3} \right\} \right| = 0
\end{aligned}$$

for every $z_1, ..., z_{n-1} \in X$. So $f$ is double lacunary statistically sequentially continuous on $E$ and the proof is completed. □

**Theorem 2.7.** *$S_{\theta_{r,s}}$-sequentially continuous image of any $S_{\theta_{r,s}}$-sequentially compact subset of $X$ is $S_{\theta_{r,s}}$-sequentially compact.*

*Proof.* Suppose that $f$ is an $S_{\theta_{r,s}}$-sequentially continuous function on a subset $E$ of $X$ and $A$ is an $S_{\theta_{r,s}}$-sequentially compact subset of $E$. Let $(f(x_{n,m}))$ be any sequence of points in $f(A)$ where $x_{n,m} \in A$ for each positive integers $n, m$. $S_{\theta_{r,s}}$-sequentially compactness of $A$ implies that there is a subsequence $(\gamma_{k,l}) = (x_{n_k,m_l})$ of $(x_{n,m})$ with $S_{\theta_{r,s}} - \lim_{k,l\to\infty} \|\gamma_{k,l}, z_1, ..., z_{n-1}\| = \|l_1, z_1, ..., z_{n-1}\|$ for every $z_1, ..., z_{n-1} \in E$. Let $(t_{k,l}) = (f(\gamma_{k,l}))$. As $f$ is $S_{\theta_{r,s}}$-sequentially continuous, $(f(\gamma_{k,l}))$ is $S_{\theta_{r,s}}$-sequentially convergent which is a subsequence of the double sequence $(f(x_{n,m}))$ with $S_{\theta_{r,s}} - \lim_{k,l} \|t_{k,l}, z_1, ..., z_{n-1}\| = \|l_1, z_1, ..., z_{n-1}\|$ for all $z_1, ..., z_{n-1} \in E$. This completes the proof of the theorem. □

The concept of strongly lacunary quasi-Cauchy sequences in a 2-normed space was studied in [5] and $S_\theta$-quasi-Cauchy was studied in [6]. Now we give the following definition of an $S_{\theta_{r,s}}$-quasi-Cauchy sequence in $n$-normed space.

**Definition 2.8.** *A double sequence $(x_{k,l})$ in an $n$-normed space $(X, \|., .., \|)$ is called to be double lacunary statistically quasi-Cauchy if $S_{\theta_{r,s}} - \lim_{k,l\to\infty} \|\Delta x_{k,l}, z_1, ..., z_{n-1}\| = 0$ for every $z_1, ..., z_{n-1} \in X$ where $\Delta x_{k,l} = x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}$ for each $k, l \in \mathbb{N}$. The set of double lacunary statistically quasi-Cauchy sequences is denoted by $\Delta S_{\theta_{r,s}}$.*

**Definition 2.9.** *A subset $E$ of $X$ is called $S_{\theta_{r,s}}$-ward compact if any double sequence of points in $E$ has an $S_{\theta_{r,s}}$-quasi-Cauchy subsequence.*

The union of two $S_{\theta_{r,s}}$-ward compact subset of $X$ is $S_{\theta_{r,s}}$-ward compact. The intersection of any family of $S_{\theta_{r,s}}$-ward compact subsets is $S_{\theta_{r,s}}$-ward compact. Any finite subset of $E$ is $S_{\theta_{r,s}}$-ward compact.

**Definition 2.10.** *A real valued function $f$ defined on a subset $E$ of an $n$-normed space is called double lacunary statistically ward continuous or $S_{\theta_{r,s}}$-ward continuous on $E$ if it preserves double lacunary statistically quasi-Cauchy sequence of points in $E$ where a double sequence $(x_{k,l})$ of points in $X$ is double lacunary statistically quasi-Cauchy if*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|\Delta x_{k,l}, z_1, ..., z_{n-1}\| \geq \epsilon\}| = 0$$

*for every positive real number* $\epsilon$ *and* $z_1, ..., z_{n-1} \in X$ *and* $(k_r, l_s)$ *is an increasing sequence of positive integers such that* $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ *as* $r \to \infty$ *and* $l_0 = 0, \bar{h}_r = l_s - l_{s-1} \to \infty$ *as* $s \to \infty$.

The sum of two double lacunary statistically ward continuous functions is double lacunary statistically ward continuous functions and also the composition of double lacunary statistically ward continuous functions is double lacunary statistically ward continuous functions.

**Theorem 2.11.** *If a real valued function is double lacunary statistically ward continuous on a subset* $E$ *of* $X$*, then it is double lacunary statistically sequentially continuous on* $E$.

*Proof.* Assume that $f$ is a double lacunary statistically ward continuous function on a subset $E$ of $X$. Let $(x_{n,m})$ be a double lacunary statistically quasi-Cauchy sequence in $E$. Then the double sequence $(x_{n,m})$ defined by

$$(x_{n,m}) = \begin{cases} x_{k,l}, \text{ if } n = 2k-1, m = 2l-1 \text{ for positive integers k, l,} \\ x_{0,0}, \text{ if } n, m \text{ is even.} \end{cases}$$

is a double lacunary statistically quasi-Cauchy sequence. Since $f$ is double lacunary statistically ward continuous, the transformed sequence $(y_{n,m})$ obtained by

$$(y_{n,m}) = \begin{cases} f(x_{k,l}), \text{ if } n = 2k-1, m = 2l-1 \text{ for positive integers k, l,} \\ f(x_{0,0}), \text{ if } n, m \text{ is even.} \end{cases}$$

is also double lacunary statistically quasi-Cauchy sequence. It follows that the sequence $(f(x_{k,l}))$ is double lacunary statistically converges to $f(x_{0,0})$. This completes the proof of the theorem. □

**Theorem 2.12.** *If a real valued function* $f$ *is uniformly continuous on a subset* $E$ *of* $X$*, then* $(f(x_{n,m}))$ *is double lacunary statistically quasi-Cauchy whenever* $(x_{n,m})$ *is a double quasi-Cauchy sequence of points in* $E$.

*Proof.* Let $f$ be uniformly continuous on $E$. Let any double quasi-Cauchy sequence $(x_{n,m})$ of points in $E$. Let $\epsilon$ be any positive real number. Since $f$ is uniformly continuous, there exists a $\delta > 0$ such that $\|f(x) - f(y), w_1, ..., w_{n-1}\| < \epsilon$ for $w_1, ..., w_{n-1} \in X$ whenever $\|x - y, z_1, ..., z_{n-1}\| < \delta$ and $z_1, ..., z_{n-1} \in X$. As $(x_{n,m})$ is double quasi-Cauchy sequence, for this $\delta$ there exist $n_0, m_0 \in \mathbb{N}$ such that $\|\Delta x_{n,m}, z_1, ..., z_{n-1}\| < \delta$ for $n \geq n_0, m \geq m_0$ for $z_1, ..., z_{n-1} \in X$. Therefore, $\|\Delta f(x_{n,m}), z_1, ..., z_{n-1}\| < \epsilon$ for $n \geq n_0, m \geq m_0$ so the number of indices $k, l$ for which $\|f(x_{n,m}) - f(x_{n,m+1}) - f(x_{n+1,m}) + f(x_{n+1,m+1}), z_1, ..., z_{n-1}\| \geq \epsilon$ is less than $n_0, m_0$. Hence
$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|f(x_{n,m}) - f(x_{n,m+1}) - f(x_{n+1,m}) + f(x_{n+1,m+1}), z_1, ..., z_{n-1}\| \geq \epsilon\}| \leq \lim_{r,s \to \infty} \frac{n_0 m_0}{h_{r,s}} = 0$. This completes the proof of the theorem. □

**Theorem 2.13.** *If* $(f_{n,m})$ *is a sequence of double lacunary statistically ward continuous functions on a subset* $E$ *of* $X$ *and* $(f_{n,m})$ *is uniformly convergent to a function* $f$*, then* $f$ *is double lacunary statistically ward continuous on* $A$.

*Proof.* Let $\epsilon$ be a positive real number and $(x_{n,m})$ be any double lacunary statistically quasi-Cauchy sequence of points in $E$. By uniform convergence of $(f_{n,m})$ there exist positive integers $N, M$ such that $\|f_{n,m}(x) - f(x)\| < \frac{\epsilon}{3}$ for all $x, z_1, ..., z_{n-1} \in E$

whenever $n \geq N, m \geq M$. As $f_{N,M}$ is double lacunary statistically ward continuous on $E$, we have

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\Big|\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k,l})-f_{N,M}(x_{k,l+1})-f_{N,M}(x_{k+1,l})+ f_{N,M}(x_{k+1,l+1}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\|\Big\}\Big|.$$

On the other hand, we have

$$\{(k,l)\in I_{r,s}:\|f(x_{k,l})-f(x_{k,l+1})-f(x_{k+1,l})+f(x_{k+1,l+1}),z_1,...,z_{n-1}\|\geq\epsilon\}$$
$$\cup\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k,l})-f_{N,M}(x_{k,l+1})-f_{N,M}(x_{k+1,l})+f_{N,M}(x_{k+1,l+1}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\Big\}$$
$$\cup\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k+1,l})-f_{N,M}(x_{k+1,l+1})-f(x_{k+1,l})+f(x_{k+1,l+1}),z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\Big\}.$$

Now it follows from this inclusion that

$$\lim_{r,s}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}:\|f(x_{k,l})-f(x_{k,l+1})-f(x_{k+1,l})+f(x_{k+1,l+1}),z_1,...,z_{n-1}\|\geq \epsilon\}|$$

$$\leq\lim_{r,s}\frac{1}{h_{r,s}}\Big|\Big\{(k,l)\in I_{r,s}:\|f(x_{k,l})-f(x_{k,l+1})-f_{N,M}(x_{k,l})+f_{N,M}(x_{k,l+1}),z_1,...,z_{n-1}\| \geq\frac{\epsilon}{3}\Big\}\Big|$$
$$+\lim_{r,s}\frac{1}{h_{r,s}}\Big|\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k,l})-f_{N,M}(x_{k,l+1})-f_{N,M}(x_{k+1,l})+f_{N,M}(x_{k+1,l+1}), z_1,...,z_{n-1}\|\geq\frac{\epsilon}{3}\Big\}\Big|$$
$$+\lim_{r,s}\frac{1}{h_{r,s}}\Big|\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k+1,l})-f_{N,M}(x_{k+1,l+1})-f(x_{k+1,l})+f(x_{k+1,l+1}),z_1 ,...,z_{n-1}\|\geq\frac{\epsilon}{3}\Big\}\Big|=0$$

for every $z_1,...,z_{n-1}\in X$. So $f$ is double lacunary statistically ward continuous on $E$. This completes the proof. □

**Theorem 2.14.** *Let $(f_{n,m})$ be a sequence of functions defined on a subset $E$ of $X$ into $X$ that transforms convergent sequences to double lacunary statistically quasi-Cauchy sequences for each $n,m\in\mathbb{N}$ and $(f_{n,m})$ be uniformly convergent to a function $f$, then $f$ transforms convergent sequences to double lacunary statistically quasi-Cauchy sequences.*

*Proof.* Suppose $(f_{n,m})$ be a uniformly convergent sequence with uniform limit $f$ and $(x_{k,l})$ be convergent sequence of points in $E$ with $\lim\|x_{k,l},z_1,...,z_{n-1}\| = \|x_{0,0},z_1,...,z_{n-1}\|$ for every $z_1,...,z_{n-1}\in X$. Take any $\epsilon>0$. As $(f_{n,m})$ is uniformly convergent then there exist $N,M\in\mathbb{N}$ such that $\|f(x)-f_{n,m}(x),z_1,...,z_{n-1}\|<\frac{\epsilon}{3}$ for $n\geq N, m\geq M$ and every $x\in E$ and $z_1,...,z_{n-1}\in X$. Since $f_{N,M}$ transforms convergent sequences to double lacunary statistically quasi-Cauchy sequences on $E$, we have

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\Big|\Big\{(k,l)\in I_{r,s}:\|f_{N,M}(x_{k,l})-f_{N,M}(x_{k,l+1})-f_{N,M}(x_{k+1,l})+$$

$f_{N,M}(x_{k+1,l+1}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}\Big| = 0.$

On the other hand, we have

$$\{(k,l) \in I_{r,s} : \|f(x_{k,l}) - f(x_{k,l+1}) - f(x_{k+1,l}) + f(x_{k+1,l+1}), z_1, ..., z_{n-1}\| \geq \epsilon\}$$
$$\subset \Big\{(k,l) \in I_{r,s} : \|f(x_{k,l}) - f(x_{k,l+1}) - f_{N,M}(x_{k+1,l}) + f_{N,M}(x_{k+1,l+1}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}$$
$$\cup \Big\{(k,l) \in I_{r,s} : \|f_{N,M}(x_{k+1,l}) - f_{N,M}(x_{k+1,l+1}) - f_{N,M}(x_{k,l}) + f_{N,M}(x_{k,l+1}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}$$
$$\cup \Big\{(k,l) \in I_{r,s} : \|f_{N,M}(x_{k,l}) - f_{N,M}(x_{k,l+1}) - f(x_{k+1,l}) + f(x_{k+1,l+1}), z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}.$$

So it follows from this inclusion that

$$\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : \|f(x_{k,l}) - f(x_{k,l+1}) - f(x_{k+1,l}) + f(x_{k+1,l+1}), z_1, ..., z_{n-1}\| \geq \epsilon\}|$$
$$\leq \lim_{r,s} \frac{1}{h_{r,s}} \Big|\Big\{(k,l) \in I_{r,s} : \|f(x_{k,l}) - f(x_{k,l+1}) - f_{N,M}(x_{k,l}) + f_{N,M}(x_{k,l+1}), z_1, ..., z_{n-1}\|$$
$$\geq \frac{\epsilon}{3}\Big\}\Big|$$
$$+ \lim_{r,s} \frac{1}{h_{r,s}} \Big|\Big\{(k,l) \in I_{r,s} : \|f_{N,M}(x_{k,l}) - f_{N,M}(x_{k,l+1}) - f_{N,M}(x_{k+1,l}) + f_{N,M}(x_{k+1,l+1}),$$
$$z_1, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}\Big|$$
$$+ \lim_{r,s} \frac{1}{h_{r,s}} \Big|\Big\{(k,l) \in I_{r,s} : \|f_{N,M}(x_{k+1,l}) - f_{N,M}(x_{k+1,l+1}) - f(x_{k+1,l}) + f(x_{k+1,l+1}), z_1$$
$$, ..., z_{n-1}\| \geq \frac{\epsilon}{3}\Big\}\Big| = 0$$

for every $z_1, ..., z_{n-1} \in X$. Thus $f$ transforms convergent sequences to double lacunary statistically quasi-Cauchy sequences, so the proof of the theorem is completed. □

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