FIXED AND COMMON FIXED POINT FOR MAPPINGS SATISFYING SOME NONLINEAR CONTRACTIONS IN $b$-METRIC SPACES

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Abstract. In this paper, we introduce the concept of a $c$-comparison function with base $s$ and utilized our new notion to establish some common fixed point theorems for nonlinear contractions in a complete $b$-metric space with base $s$ in the sense of Czerwik [Czerwik S., Nonlinear set-valued contraction mappings in $b$-metric spaces, Atti Sem. Mat. Univ. Modena, 1998, 46, 263-276].

1. Introduction and preliminaries

Czerwik [1] introduced the notion of $b$-metric spaces as a generalization of metric spaces. Then after, many authors established many fixed point theorems in $b$-metric spaces (see [3]-[24]). Aydi et al. [5] established a nice fixed point theorem for set-valid quasi contraction in $b$-metric space. Also, Vahid Parvaneh et al. [17] studied some tripled coincidence point in ordered metric spaces. While Roshan et al. [19] studied some common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered $b$-metric spaces. Moreover, Shi et al. [21] studied some nice fixed point theorems in cone $b$-metric spaces and Mustafa et al. [14] studied some coupled coincidence point results in generalized $b$-metric spaces.

Now, we present the definition of $b$-metric spaces.

Definition 1.1. [1, 2] Let $X$ be an nonempty set. A real-valued function $d: X \times X \to [0, +\infty)$ is called a $b$-metric on $X$ with constant $s \geq 1$ if the next axioms are fulfilled:

1) $d(x, y) = 0$ if $x = y$,
2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
3) $d(x, y) \leq s(d(x, a) + d(a, y))$ for each $x, y, a \in X$.

The pair $(X, d)$ is called a $b$-metric space.

Note that every metric space is a $b$-metric space with base 1. Some authors established some $b$-metric spaces which is not metric spaces.

Following Boriceanu et al. [7], we have the following definitions and remarks:

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Definition 1.2. [7] Let \((X, d)\) be a \(b\)-metric space. Then a sequence \((x_n)\) in \(X\) is called:

1. convergent if and only if there exists \(x \in X\) such that \(d(x_n, x) \to 0\) as \(n \to +\infty\). In this case, we write \(\lim_{n \to +\infty} x_n = x\).
2. Cauchy if and only if \(d(x_n, x_m) \to 0\) as \(m, n \to +\infty\).

Definition 1.3. [7] The \(b\)-metric space \((X, d)\) is complete if every Cauchy sequence in \(X\) converges.

The following remark is crucial in our subsequent arguments.

Remark. [7] In a \(b\)-metric space \((X, d)\) the following assertions hold:

1. a convergent sequence has a unique limit;
2. each convergent sequence is Cauchy;
3. in general, a \(b\)-metric is not continuous.

Definition 1.4. [7] Let \((X, d)\) and \((X', d')\) be \(b\)-metric spaces with constant \(s\) and \(s'\) respectively. Then the mapping \(T: X \to X'\) is said to be continuous if whenever \((x_n)\) is a sequence in \(X\) converges to some \(x \in X\) with respect to \(d\), then \((Tx_n)\) converges to \(Tx\) with respect to \(d'\).

2. Main Results

We start our work by introducing the following definitions.

Definition 2.1. Let \(s\) be a constant with \(s \geq 1\). A map \(\phi: [0, +\infty) \to [0, +\infty)\) is called a \(c\)-comparison function with base \(s\) if \(\phi\) satisfies the following:

1. \(\phi\) is monotone nondecreasing,
2. \(\sum_{n=0}^{\infty} s^n \phi^n(st)\) converges for all \(t \geq 0\).

Remark. If \(\phi\) is a \(c\)-comparison function with base \(s\), then \(\phi(t) \leq t\) for all \(t > 0\).

Theorem 2.1. Let \((X, d)\) be a complete \(b\)-metric space with constant \(s \geq 1\) and \(T, S: X \to X\) be two mappings. Suppose that there exist a \(c\)-comparison function \(\phi\) with base \(s\) such that

\[
d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right)
\]

(2.1)

holds for all \(x, y \in X\). If \(\phi\) is a continuous function, then \(T\) and \(S\) have a unique common fixed point.

Proof. Since \(X\) is nonempty, we start with \(x_0 \in X\) and define a sequence \((x_n)\) in \(X\) inductively by putting \(x_{2n+1} = Tx_{2n}\) and \(x_{2n+2} = Sx_{2n+1}\). Suppose that \(x_n = x_{n+1}\) for some \(n \in \mathbb{N}\). Without loss of generality, we assume that \(n = 2k\) for
some $k \in \mathbb{N}$. By (2.1), we have
\[
\begin{align*}
sd(x_{2k+1}, x_{2k+2}) &= sd(Tx_{2k}, Sx_{2k+1}) \\
&\leq \phi \left( \max \left\{ sd(x_{2k}, x_{2k+1}), sd(x_{2k}, Tx_{2k}), sd(x_{2k+1}, Sx_{2k+1}), \right. \\
&\quad \left. \frac{1}{2}(d(x_{2k}, Sx_{2k+1}) + d(Tx_{2k}, x_{2k+1})) \right\} \right) \\
&= \phi \left( \max \left\{ sd(x_{2k+1}, x_{2k+2}), \frac{1}{2}d(x_{2k}, x_{2k+2}) \right\} \right) \\
&\leq \phi \left( \max \left\{ sd(x_{2k+1}, x_{2k+2}), \frac{8}{2}(d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})) \right\} \right) \\
&\leq \phi(sd(x_{2k+1}, x_{2k+2})).
\end{align*}
\]
Using the fact that $\phi(t) < t$ for all $t > 0$, we conclude that $sd(x_{2k+1}, x_{2k+2}) = 0$ and hence $x_{2k} = x_{2k+1} = x_{2k+2}$. Therefore $x_{2k} = Tx_{2k} = Sx_{2k}$. Therefore $x_k$ is a fixed point of $T$ and $S$. Thus, we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, we study the following cases.

**Case 1:** $n$ is even. Here $n = 2t$ for some $t \in \mathbb{N}$. By (2.1), we have
\[
\begin{align*}
sd(x_{2t+1}, x_{2t+2}) &= sd(x_{2t+1}, x_{2t}) \\
&= sd(Tx_{2t}, Sx_{2t-1}) \\
&\leq \phi(\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, Tx_{2t}), sd(x_{2t-1}, Sx_{2t-1})\)} \\
&\quad \left. \frac{1}{2}(d(x_{2t}, Sx_{2t-1}) + d(Tx_{2t}, x_{2t-1})) \right\}) \\
&= \phi \left( \max \left\{ sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1}), \frac{1}{2}d(x_{2t+1}, x_{2t-1}) \right\} \right) \\
&\leq \phi \left( \max \left\{ sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1}), \frac{8}{2}(d(x_{2t-1}, x_{2t}) + d(x_{2t}, x_{2t+1})) \right\} \right) \\
&= \phi(\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\})
\end{align*}
\]
If $\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\} = sd(x_{2t}, x_{2t+1})$, then
\[
\begin{align*}
sd(x_{2t}, x_{2t+1}) \leq \phi(sd(x_{2t}, x_{2t+1})) < sd(x_{2t}, x_{2t+1})
\end{align*}
\]
which is a contradiction. Thus
\[
\begin{align*}
\max\{sd(x_{2t}, x_{2t-1}), sd(x_{2t}, x_{2t+1})\} = sd(x_{2t}, x_{2t-1}).
\end{align*}
\]
Therefore
\[
\begin{align*}
\begin{cases}
\text{(2.2)} \\
sd(x_{2t}, x_{2t+1}) \leq \phi(sd(x_{2t}, x_{2t-1})).
\end{cases}
\end{align*}
\]

**Case 2:** $n$ is odd. Here $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. Using similar arguments as those given in Case 1, we can show that
\[
\begin{align*}
\begin{cases}
\text{(2.3)} \\
\text{sd}(x_{2t+1}, x_{2t+2}) \leq \phi(sd(x_{2t}, x_{2t+1})).
\end{cases}
\end{align*}
\]
Combining (2.2) and (2.3) together, we arrive to
\[
\begin{align*}
\begin{cases}
\text{(2.4)} \\
\text{sd}(x_n, x_{n+1}) \leq \phi(sd(x_{n-1}, x_n)).
\end{cases}
\end{align*}
\]
From (2.4), we have
\[ sd(x_n, x_{n+1}) \leq \phi(sd(x_{n-1}, x_n)) \]
\[ \leq \phi^2(sd(x_{n-2}, x_{n-1})) \]
\[ \leq \phi^3(sd(x_{n-3}, x_{n-2})) \]
\[ \vdots \]
\[ \leq \phi^n(sd(x_0, x_1)). \] (2.5)

Given \( n, m \in \mathbb{N} \) with \( m > n \). By the definition of \( b \)-metric and (2.5), we have
\[ d(x_n, x_m) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \]
\[ \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_m) \]
\[ \vdots \]
\[ \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) + \cdots + s^{m-n}d(x_{m-1}, x_m) \]
\[ \leq \phi^n(d(x_0, x_1)) + \phi^{n+1}(sd(x_0, x_1)) + \phi^{n+2}(sd(x_0, x_1)) + \cdots + \phi^{m-n-1}(sd(x_0, x_1)). \]

Using the fact that \( s \geq 1 \), we get
\[ d(x_n, x_m) \leq \sum_{j=n}^{+\infty} s^j \phi^j(sd(x_0, x_1)). \]

Since \( \phi \) is \((c)\)-comparison with base \( s \), we have
\[ \lim_{n,m \to +\infty} d(x_n, x_m) = 0. \] (2.6)

Thus \( (x_n) \) is Cauchy in the complete \( b \)-metric space \((X, d)\). So there exists \( u \in X \) such that \( x_n \to u \) as \( n \to +\infty \).

Since
\[ d(x_{2n+1}, x_{2n}) \leq s(d(x_{2n+1}, u) + d(u, x_{2n})) \]
we have \( \lim_{n \to +\infty} d(x_{2n+1}, x_{2n}) = 0 \). Now assume that \( d(u, Su) \neq 0 \). Using (2.1), we have
\[ sd(x_{2n+1}, Su) = sd(Tx_{2n}, Su) = \phi \left( \max \left\{ sd(x_{2n}, u), sd(x_{2n}, x_{2n+1}), sd(u, Su), \frac{1}{2}(d(x_{2n+1}, u) + d(x_{2n}, Su)) \right\} \right) \]
\[ \leq \phi \left( \max \left\{ sd(x_{2n}, u), sd(x_{2n}, x_{2n+1}), sd(u, x_{2n+1}) + sd(x_{2n+1}, Su), \right. \right. \]
\[ \left. \frac{1}{2}(d(x_{2n+1}, u) + sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, Su)) \right\} \).

Taking the limit sup and use the continuity of \( \phi \), we get the following
\[ \limsup_{n \to +\infty} sd(x_{2n+1}, Su) \leq \phi(\limsup_{n \to +\infty} sd(x_{2n+1}, Su)). \]

By the properties of \( \phi \), we get
\[ \lim_{n \to +\infty} d(x_{2n+1}, Su) = 0 \]
and hence
\[ \lim_{n \to +\infty} d(x_{2n+1}, Su) = 0. \]
On letting \( n \to +\infty \) in
\[ d(u, Su) \leq sd(u, x_{2n+1}) + sd(x_{2n+1}, Su), \]
we get \( d(u, Su) = 0 \) and hence \( Su = u \). From (2.1), we have
\[
\begin{align*}
sd(Tu, u) & = sd(Tu, Su) \\
& \leq \phi \left( \max \{ sd(u, Tu), sd(u, Su), \frac{1}{2}(d(Tu, u) + d(u, Su)) \} \right) \\
& = \phi(sd(u, Tu)).
\end{align*}
\]
By using the properties of \( \phi \), we get \( sd(u, Tu) = 0 \) and hence \( u = Tu \). Thus \( u \) is a common fixed point of \( T \) and \( S \).

By taking \( S = T \) in Theorem 2.1, we have the following results.

**Corollary 2.2.** Let \((X, d)\) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( T: X \to X \) be a mapping. Suppose that there exist a \( c \)-comparison function \( \phi \) with base \( s \) such that
\[
d(Tx, Ty) \leq \frac{1}{s} \phi \left( \max \{ sd(x, y), sd(x, Tx), sd(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y)) \} \right)
\]
holds for all \( x, y \in X \). If \( \phi \) is continuous, then \( T \) has a unique fixed point.

**Corollary 2.3.** Let \((X, d)\) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( T, S: X \to X \) be two mappings. Suppose the following inequality
\[
d(Tx, Sy) \leq \frac{1}{s + a} \max \{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y)) \}
\]
holds for all \( x, y \in X \). Then \( T \) and \( S \) have a unique common fixed point.

**Proof.** Define \( \phi : [0, +\infty) \to [0, +\infty) \) by \( \phi(t) = \frac{1}{s+a} t \), then \( \phi \) is a continuous \( c \)-comparison function with constant \( s \). Note that \( \phi \) satisfies the following inequality
\[
d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \} \right)
\]
for all \( x, y \in X \). So that \( T \) and \( S \) satisfy all the hypotheses of Theorem 2.1. Thus \( T \) and \( S \) have a unique common fixed point. \( \square \)

**Corollary 2.4.** Let \((X, d)\) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( T, S: X \to X \) be two mappings. Suppose the following inequality
\[
d(Tx, Sy) \leq \frac{1}{s^2} \max \{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2s}(d(x, Sy) + d(Tx, y)) \}
\]
holds for all \( x, y \in X \). Then \( T \) and \( S \) have a unique common fixed point.

**Proof.** Take \( a = s^2 - s \). Then \( a > 0 \) and \( s^2 = a + s \). Hence the result follows from Corollary 2.3. \( \square \)

If we take \( S = T \) in Corollary 2.3, we get the following result.
Corollary 2.5. Let \((X,d)\) be a complete \(b\)-metric space with constant \(s \geq 1\) and \(T,S: X \to X\) be two mappings. Suppose that there exist \(a > 0\) such that
\[
d(Tx,Ty) \leq \frac{1}{s + a} \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s} (d(x,Ty) + d(Tx,y)) \right\}
\]
holds for all \(x,y \in X\). Then \(T\) has a unique fixed point.

The continuity of \(\phi\) in Theorem 2.1 can be dropped. For this instance, we will assume that the mapping \(T\) or \(S\) is continuous.

Theorem 2.6. Let \((X,d)\) be a complete \(b\)-metric space with constant \(s \geq 1\) and \(T,S: X \to X\) be two mappings. Suppose that there exist a \(c\)-comparison function \(\phi\) with base \(s\) such that
\[
d(Tx,Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x,y), sd(x,Tx), d(y,Sy), \frac{1}{2} (d(x,Sy) + d(Tx,y)) \right\} \right)
\]
holds for all \(x,y \in X\). If \(T\) or \(S\) is continuous, then \(T\) and \(S\) have a unique common fixed point.

Proof. Using the same method in the proof of Theorem 2.1 we construct a Cauchy sequence \((x_n)\) in the \(b\)-metric space \((X,d)\). Since \(X\) is complete, then there is \(u \in X\) such that \(x_n \to u\) as \(n \to +\infty\). Now, we prove that \(Tu = Su = u\). It is an easy matter to realize that \(Tx_{2n} = x_{2n+1} \to u\) and \(Sx_{2n+1} = x_{2n+2} \to u\) as \(n \to +\infty\). Now, we prove that \(Su = u\) and \(Tu = u\). Without loss of generality, we may assume that \(S\) is continuous. Since \(x_{2n+1} \to u\) and \(S\) is continuous, we get that \(x_{2n+2} = Sx_{2n+1} \to Su\). By uniqueness of limit, we conclude that \(Su = u\). To prove that \(Tu = u\), suppose to the contrary. By (2.7), we have
\[
qd(Tu,u) = qd(Tu,Su) \leq \phi \left( \max \left\{ sd(u,Tu), sd(u,Su), \frac{1}{2} (d(Tu,u) + d(u,Su)) \right\} \right)
\]
Since \(\phi(sd(u,Tu)) < sd(u,Tu)\), we arrive to a contradiction. Thus \(u\) is a common fixed point of \(T\) and \(S\). To prove the uniqueness of a common fixed point of \(T\) and \(S\), we assume that \(T\) and \(S\) have two different common fixed point, say \(u\) and \(v\). By (2.7), we have
\[
d(u,v) = d(Tu,Sv) \leq \frac{1}{s} \phi \left( \max \left\{ sd(u,v), sd(u,Tu), sd(v,Sv), \frac{1}{2} (d(u,Sv) + d(Tu,v)) \right\} \right)
\]
Using the property of \(\phi\), we get \(d(u,v) = 0\) and hence \(u = v\).

By taking \(S = T\) in Theorem 2.6, we have the following results.
**Corollary 2.7.** Let $(X, d)$ be a complete $b$-metric space with constant $s$ and $T: X \to X$ be a mapping. Suppose that there exist a $c$-comparison function $\phi$ with base $s$ such that

$$d(Tx, Ty) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y)) \right\} \right)$$

holds for all $x, y \in X$. If $T$ is continuous, then $T$ has a unique fixed point.

**Corollary 2.8.** Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $T, S: X \to X$ be two mappings. Suppose that there exist a $c$-comparison function $\phi$ with base $s$ such that

$$d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right)$$

holds for all $x, y \in X$. If $\phi$ is a continuous function, then $T$ and $S$ have a unique common fixed point.

**Proof.** Follows from Theorem 2.1 and the fact that

$$\frac{1}{s} \phi \left( \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right).$$

**Corollary 2.9.** Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $T, S: X \to X$ be two mappings. Suppose that there exist a $c$-comparison function $\phi$ with base $s$ such that

$$d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right)$$

holds for all $x, y \in X$. If $T$ or $S$ is continuous, then $T$ and $S$ have a unique common fixed point.

**Proof.** Follows from Theorem 2.6 and the fact that

$$\frac{1}{s} \phi \left( \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right).$$

□

Taking $S = T$ in Corollary 2.8 and Corollary 2.9, we get the following results.

**Corollary 2.10.** Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $T, S: X \to X$ be two mappings. Suppose that there exist a $c$-comparison function $\phi$ with base $s$ such that

$$d(Tx, Ty) \leq \frac{1}{s} \phi \left( \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y)) \right\} \right)$$
holds for all \( x, y \in X \). If \( \phi \) is a continuous function, then \( T \) has a unique common fixed point.

**Corollary 2.11.** Let \( (X, d) \) be a complete \( b \)-metric space with constant \( s \geq 1 \) and \( T: X \to X \) be two mappings. Suppose that there exist a \( c \)-comparison function \( \phi \) with base \( s \) such that
\[
d(Tx, Ty) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Ty), \frac{1}{2}(d(x, Ty) + d(Tx, y)) \right\} \right)
\]
holds for all \( x, y \in X \). If \( T \) is continuous mapping, then \( T \) has a unique common fixed point.

### 3. Examples

In this section, we introduce some examples to show the validity of our main results.

**Example 3.1.** Let \( X = [0, +\infty) \). Consider the complete \( b \)-metric space \( d : X \times X \to [0, +\infty) \), \( d(x, y) = (x - y)^2 \) with constant \( s = 2 \). Define the mappings \( T, S : X \to X \) by \( Tx = \frac{1}{3}x \) and \( Sy = \frac{1}{6}x \). Also define \( \phi: [0, +\infty) \to [0, +\infty) \) by \( \phi(t) = \frac{1}{4}t \). Then

1. \( \phi \) is a continuous \( c \)-comparison function.
2. \( T, S \) and \( \phi \) satisfy the following inequality:
\[
d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right),
\]
   for all \( x, y \in X \).

**Proof.** It is clear that \( \phi \) is a nondecreasing continuous function. Now, given \( t \in [0, +\infty) \). Then, it is an easy matter to show that
\[
\phi^n(st) = \phi^n(2t) = \frac{1}{4^n}(2t).
\]
Thus
\[
\sum_{n=0}^{\infty} s^n \phi^n(st) = \sum_{n=0}^{\infty} 2^n \frac{1}{4^n}(2t) = 2t \sum_{n=0}^{\infty} \frac{1}{2^n} < +\infty.
\]
So \( \phi \) is a continuous \( c \)-comparison function. This prove (1).

To prove (2), let \( x, y \in X \). Then
\[
d(Tx, Sy) = d \left( \frac{1}{3}x, \frac{1}{6}y \right) = \left( \frac{1}{3}x - \frac{1}{6}y \right)^2 = \frac{1}{9} \left( x - \frac{1}{2}y \right)^2.
\]
Consider the following cases:

**Case I:** \( x = \frac{1}{2}y \). Here, we have
\[
d(Tx, Sy) = 0 \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2}(d(x, Sy) + d(Tx, y)) \right\} \right).
\]
Case II: \( x > \frac{1}{2}y \). Here, we have
\[
d(Tx, Sy) = \frac{1}{9} (x - \frac{1}{2}y)^2 \leq \frac{x^2}{6} \\
= \frac{1}{2} (\frac{2}{3} x)^2 (\frac{1}{4}) \\
= \frac{1}{2} \phi \left( 2(x - \frac{1}{3}x)^2 \right) \\
= \frac{1}{2} \phi \left( 2d \left( x, \frac{1}{3}x \right) \right) \\
= \frac{1}{s} \phi(sd(x, Tx)) \\
\leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right).
\]

Case III: \( x < \frac{1}{2}y \). Here, we have
\[
d(Tx, Sy) = \frac{1}{9} (x - \frac{1}{2}y)^2 \leq \frac{y^2}{36} \\
\leq \frac{25}{36} \left( \frac{y^2}{4} \right) \\
= \frac{1}{2} \phi \left( 2 \left( \frac{25}{36} \right) y^2 \right) \\
= \frac{1}{2} \phi \left( 2(y - \frac{1}{6}y)^2 \right) \\
= \frac{1}{2} \phi \left( 2d \left( y, \frac{1}{6}y \right) \right) \\
= \frac{1}{s} \phi(sd(y, Sx)) \\
\leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right).
\]

Note that \( T, S \) and \( \phi \) satisfy all the hypotheses of Theorem 2.1. So \( T \) and \( S \) have a unique common fixed point. \( \square \)

Example 3.2. Let \( X = \{1, 2, 3, \ldots, 100\} \). Define \( d : X \times X \to [0, +\infty) \) by
\[
d(x, y) = \begin{cases} 
0, & \text{if } x = y \\
\left| \frac{1}{2} - \frac{1}{y} \right|, & \text{if } x \text{ and } y \text{ are even with } x \neq y \\
5, & \text{if } x \text{ and } y \text{ are odd with } x \neq y \\
2, & \text{other wise.}
\end{cases}
\]
Also, define the two mappings \( T, S : X \to X \) by
\[
Tx = \begin{cases} 
2, & \text{if } x \text{ is odd;} \\
4, & \text{if } x \text{ is even}
\end{cases}
\]
and

$$Sx = \begin{cases} 6, & \text{if } x \text{ is odd;} \\ 4, & \text{if } x \text{ is even} \end{cases}$$

Also, define \( \phi : [0, +\infty) \to [0, +\infty) \) by

$$\phi(t) = \begin{cases} \frac{1}{2}t, & \text{if } t \geq 1 \\ 0, & \text{if } t < 1. \end{cases}$$

Then:

1. \((X, d)\) is a complete \(b\)-metric space with \(s = \frac{5}{4}\).
2. \(\phi\) is a noncontinuous \(c\)-comparison function.
3. \(T\) is continuous.
4. \(T, S\) and \(\phi\) satisfy the following inequality:

$$d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right),$$

for all \(x, y \in X\).

Proof. With a simple technique, one can easily prove (1). Note that \(\phi\) is nondecreasing. Now, given \(t \in [0, +\infty)\). Then

$$\phi^n(st) = \phi^n \left( \frac{5}{4} t \right) \leq \frac{5t}{(2^n)(4)}.$$

Hence

$$\sum_{n=0}^{\infty} s^n \phi^n(st) = \sum_{n=0}^{\infty} \frac{5^n}{4^n} \phi^n \left( \frac{5}{4} t \right) \leq \frac{5t}{4} \sum_{n=0}^{\infty} \left( \frac{5}{8} \right)^n < +\infty.$$

Thus \(\phi\) is a \(c\)-comparison function. Also, Since \(1 - \frac{1}{s} \to 1\), and \((\phi(1 - \frac{1}{s}))\) doesn’t converge to \(\phi(1)\), we conclude that \(\phi\) is not continuous. This prove (2). To prove (3), let \((x_n)\) be a sequence in \(X\) such that \(x_n \to x\) for some \(x \in X\). Then \(x_n = x\) for all but finitely many. So \(Tx_n = Tx\) for all but finitely many. So \(d(Tx_n, Tx) = 0\) for all but finitely many. Hence \(Tx_n \to Tx\). So \(T\) is continuous.

To prove (4), given \(x, y \in X\). We study the following cases:

**Case I:** \(x\) is odd and \(y\) is odd. Here, we have \(Tx = 2\) and \(Sy = 6\). Thus

$$d(Tx, Sy) = d(2, 6) = \frac{1}{3} \leq 2.5 = \left( \frac{25}{4} \right) \left( \frac{1}{2} \right)$$

Thus

$$d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right).$$

**Case II:** \(x\) is odd and \(y\) is even. Here, we have \(Tx = 2\) and \(Sy = 4\). Thus

$$d(Tx, Sy) = d(2, 4) = \frac{1}{4} \leq 1 = \left( \frac{10}{4} \right) \left( \frac{1}{2} \right)$$

Thus

$$d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ sd(x, y), sd(x, Tx), sd(y, Sy), \frac{1}{2} (d(x, Sy) + d(Tx, y)) \right\} \right).$$
Thus

\[d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ \, \frac{1}{2} (d(x, y) + d(Tx, y)) \right\} \right).\]

**Case III:** \(x\) is even and \(y\) is odd. Here, we have \(Tx = 4\) and \(Sy = 6\). Thus

\[d(Tx, Sy) = d(4, 6) = \frac{1}{12} \leq 1 = \left( \frac{4}{5} \right) \left( \frac{10}{4} \right) \left( \frac{1}{2} \right) \]

\[= \frac{4}{5} \phi \left( \frac{10}{4} \right) = \frac{4}{5} \phi \left( \frac{5}{4} d(x, y) \right) = \frac{1}{s} \phi (sd(x, y)).\]

Thus

\[d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ \, \frac{1}{2} (d(x, y) + d(Tx, y)) \right\} \right).\]

**Case IV:** \(x\) and \(y\) are even. Here \(Tx = Sy = 4\). Thus \(d(Tx, Sy) = 0\). Therefore

\[d(Tx, Sy) \leq \frac{1}{s} \phi \left( \max \left\{ \, \frac{1}{2} (d(x, y) + d(Tx, y)) \right\} \right).\]

So \(T, S\) and \(\phi\) satisfy all the hypotheses of Theorem 2.6. So \(T\) and \(S\) have a unique common fixed point. \(\square\)

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