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ON THE SECOND ORDER DIFFERENTIAL EQUATION SATISFIED BY PERTURBED CHEBYSHEV POLYNOMIALS

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Abstract. In some applications one is led to consider perturbations of orthogonal polynomials translated by a modification on the first coefficients of the second order recurrence relation satisfied by these polynomials. Moreover, the four Chebyshev families are among the most useful orthogonal sequences due to their exceptional features. Thus, it is important to clarify and explicit the properties of perturbed Chebyshev polynomials, in special the second order linear differential equation that assures their semi-classical character. This is the main goal of this work. By means of a symbolic algebraic algorithm based on Stieltjes equations, we are able to explicit new properties for the complete perturbation of order one and an special perturbation of order two for all four Chebyshev families.

1. Introduction

In [9], it was systematized a new general method, and the corresponding symbolic algorithm designated by $PSDF$, intended to explicit some semi–classical properties of perturbed second degree forms, namely: the functional equation, the class of the form, a closed formula for the Stieltjes function, the Stieltjes equation, a structure relation and the second order linear differential equation. First moments of perturbed forms are also computed by the algorithm as well as the generating functions for all perturbed Chebyshev. The Chebyshev form of second kind was taken as study case and new explicit results for two perturbations of order three were given therein. In fact, the four Chebyshev forms [15, 16, 35] are the most important cases of second degree forms [31] due to their remarkable properties and utility in applied mathematics, physic and other sciences [20], whereas the family of second kind is the simplest of the four forms of Chebyshev for the purposes of perturbation, because it is self-associated [33]. Thus, it was natural to start by studying this case that is often treated in the literature.

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In this work, by applying *PSDF* algorithm, we give new explicit results for the complete perturbation of order one and a perturbation of order two of the second kind Chebyshev form. Both perturbations allow to obtain easily the results for the other three forms of Chebyshev. For that purpose, we take advantage of the fact these last forms can be considered as perturbed of the form of second kind. The complete perturbation of order one includes as particular cases most of the perturbations treated in literature. The perturbation of order two includes the complete perturbation by dilatation of the same order that preserves the symmetry property of the original sequence. Furthermore, perturbations by dilatation are very common in applications [12, 24].

The so-called perturbation corresponds to a modification on the first coefficients of the recurrence relation of order two satisfied by orthogonal polynomial sequences. This transformation can promote a deep change of properties; nevertheless there is a large set of forms that are preserved by perturbation: the second degree forms [30]. Moreover, a second degree form is also a semi–classical one [27]. The aforementioned method is based on this crucial fact. It is worthy to mention that among the classical forms [7, 28, 29] only certain Jacobi forms are of second degree [3] and that the four Chebyshev forms are particular cases of Jacobi forms. From them other second degree forms can be generated by applying several transformations [3, 30, 31, 34]. Furthermore, all self-associated forms are also of second degree [33].

We notice that perturbed orthogonal polynomials have some applications [12, 21, 22, 24, 40], which motivate further their study. In fact, during the last years, several authors have worked on this subject considering perturbations of several orders with respect to classical [7, 28, 29], semi-classical [26, 27], Laguerre-Hahn [1, 2] and others families, studying several properties like Stieltjes functions, structure relations and differential equations, separation and distribution functions of zeros and integral representations among others. With respect to the co-recursive case (the perturbation of order 0), we would like to cite [6, 7, 19, 22, 38, 40], for the co-dilated situation refer to [12, 34], for the co-modified [10, 11, 37], for the generalized co-polynomials see [5, 13, 24]. Also, we call the attention to the general references [36, 41]. Moreover, there are some specific works about perturbed Chebyshev families namely [34] on the co-dilated case of the second kind form and [31, 39] concerning all the four forms.

Let us summarize the content of this article. In Section 2, we establish the theoretical framework, we recall briefly the mathematical background necessary to understand the subject of perturbed second degree forms following closely the references [1–3, 9, 26–28, 30, 31]. In particular, we have collected the formulas and procedures that allow to compute the results furnish herein. In the next section, we present the above mentioned properties leading, in particular, to the second order differential equation of the perturbed Chebyshev form of second kind and we show how we can obtain easily the corresponding results for the other three forms of Chebyshev. These results were derived by applying the *PSDF* algorithm in the algebraic manipulator *Mathematica*®. We notice that in applications often one is interested in specific values of the parameters of perturbation, in such a way the formulas given herein become much more simple. We finish this section with some graphical representations. At last, we propose some open problems concerning other important unknown properties of perturbed Chebyshev polynomials.

2. Theoretical framework

There are really four sequences of Chebyshev polynomials, not just two. They are called Chebyshev polynomials of first (T_n), second (U_n), third (V_n) and fourth (W_n) kinds. W. Gaustchi [15] named these last two sequences in this way, before they had been designated as airfoil polynomials (see, e. g., [14]).

Their trigonometric definitions are

$$T_n(x) = \frac{1}{2^{n-1}} \cos nt \quad , \quad U_n(x) = \frac{1}{2^n} \frac{\sin(n+1)t}{\sin t} \, , \tag{2.1}$$

$$V_n(x) = \frac{1}{2^n} \frac{\cos(n+\frac{1}{2})t}{\cos\frac{1}{2}t} \quad , \quad W_n(x) = \frac{1}{2^n} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \, , \tag{2.2}$$

where $x = \cos t$, $t \in [0,\pi]$, $n \geq 0$. Notice that, in this work, we always consider monic polynomials, i.e., with unit leading coefficient, thus some normalization constants must appear in the preceding definitions. From (2.1) and (2.2) is trivial to obtain explicit formulas for zeros. Also, using some trigonometric identities, it can be shown that these families satisfy the following initial conditions and *recurrence relation of order two* [35]

$$\begin{cases} P_0(x) = 1 \quad , \quad P_1(x) = x - \beta_0 \, , \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \quad , \quad \gamma_{n+1} \neq 0 \ , \ n \geq 0 \, , \end{cases} \tag{2.3}$$

with the *recurrence coefficients*

$$T_n : \qquad \beta_n = 0 \ , \ n \geq 0 \quad , \quad \gamma_1 = \frac{1}{2} \ , \ \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 1 \, , \tag{2.4}$$

$$U_n : \qquad \beta_n = 0 \ , \ n \geq 0 \quad , \quad \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 1 \, , \tag{2.5}$$

$$V_n : \quad \beta_0 = \tfrac{1}{2} \ , \ \beta_n = 0 \ , n \geq 1 \quad , \quad \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 0 \, , \tag{2.6}$$

$$W_n : \quad \beta_0 = -\tfrac{1}{2} \ , \ \beta_n = 0 \ , n \geq 1 \quad , \quad \gamma_{n+1} = \frac{1}{4} \ , \ n \geq 0 \, . \tag{2.7}$$

We remark that U_n has the most simple recurrence coefficients.

A polynomial sequence $\{P_n\}_{n\geq 0}$ is *symmetric* if and only if

$$P_n(-x) = (-1)^n P_n(x) \ , \ n \geq 0.$$

If (2.3) is verified, symmetry is equivalent to $\beta_n = 0$, $n \geq 0$ [7]. Thus, T_n and U_n are symmetric and V_n and W_n are not (see symmetries in graphical representations in Figure 1).

The so-called *r-associated* sequence of a sequence $\{P_n\}_{n\geq 0}$ satisfying (2.3) is a sequence $\{\widetilde{P}_n\}_{n\geq 0}$ defined by [7, 27]

$$\begin{cases} P_0^{(r)}(x) = 1 \quad , \quad P_1^{(r)}(x) = x - \beta_0^{(r)} \, , \\ P_{n+2}^{(r)}(x) = (x - \beta_{n+1}^{(r)}) P_{n+1}^{(r)}(x) - \gamma_{n+1}^{(r)} P_n^{(r)}(x) \quad , \quad n \ , \ r \geq 0 \, , \end{cases}$$

where

$$\beta_n^{(r)} = \beta_{n+r} \quad , \quad \gamma_{n+1}^{(r)} = \gamma_{n+1+r} \quad , \quad n \ , \ r \geq 0 \, .$$

Thus, $T_n^{(1)} \equiv V_n^{(1)} \equiv W_n^{(1)} \equiv U_n$. Moreover, $U_n^{(r)} \equiv U_n$, $\forall r \geq 0$, i.e., U_n is a *self-associated* sequence [33].

The so-called *r-perturbed* sequence of a sequence $\{P_n\}_{n\geq 0}$ satisfying (2.3) is a sequence $\{\widetilde{P}_n\}_{n\geq 0}$ defined by [27]

$$\begin{cases} \widetilde{P}_0(x) = 1 \quad , \quad \widetilde{P}_1(x) = x - \tilde{\beta}_0 \, , \\ \widetilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\widetilde{P}_n(x) \quad , \quad n \geq 0 \, , \end{cases}$$

where $r \geq 0$ and

$$\begin{aligned} \tilde{\beta}_0 &= \beta_0 + \mu_0 \, , \\ \tilde{\beta}_n &= \beta_n + \mu_n \; , \; \mu_n \in \mathbb{C} \quad ; \quad \tilde{\gamma}_n = \lambda_n \gamma_n \; , \; \lambda_n \in \mathbb{C} - \{0\} \; , \; 1 \leq n \leq r \, , \\ \tilde{\beta}_n &= \beta_n \quad ; \quad \tilde{\gamma}_n = \gamma_n \; , \; n \geq r+1 \, . \end{aligned}$$

Either $\mu_r \neq 0$ or $\lambda_r \neq 1$. With the notation $\mu := (\mu_1, \cdots, \mu_r)$, $\lambda := (\lambda_1, \cdots, \lambda_r)$, $r \geq 1$, we put

$$\widetilde{P}_n(x) \equiv P_n\left(\mu_0; \begin{matrix} \mu_1, \cdots, \mu_r \\ \lambda_1, \cdots, \lambda_r \end{matrix}; r\right)(x) \equiv P_n\left(\mu_0; \begin{matrix} \mu \\ \lambda \end{matrix}; r\right)(x) \, , \; n \geq 0 \, .$$

When $r = 0$, the perturbed sequence is usually designated as *co-recursive* sequence [6, 7] and it is noted by $\widetilde{P}_n \equiv P_n(\mu_0)$. In this manner, from (2.4)-(2.7), we can consider that T_n, V_n and W_n are *perturbed* of U_n as follows

$$T_n \equiv U_n\left(0; \begin{matrix} 0 \\ 2 \end{matrix}; 1\right) \quad , \quad V_n \equiv U_n\left(\frac{1}{2}\right) \quad , \quad W_n \equiv U_n\left(-\frac{1}{2}\right) .$$

Then, *perturbed* of T_n, V_n and W_n can be taken as *perturbed* of U_n with other values of the parameters of perturbation (the order of perturbation can eventually change).

A closed formula for the generating function $\tilde{f}(x,t) = \sum_{n\geq 0} \widetilde{P}_n(x)t^n$ of any perturbed Chebyshev sequence of order r, $r \geq 0$, is [9, Sec.4]

$$\tilde{f}(x,t) = \frac{\sum_{n=0}^{r+1} \widetilde{P}_k(x)t^k - xt\sum_{n=0}^{r} \widetilde{P}_k(x)t^k + \frac{1}{4}t^2 \sum_{n=0}^{r-1} \widetilde{P}_k(x)t^k}{1 + t(\frac{1}{4}t - x)} \; . \tag{2.8}$$

If $r = 0$, then $\sum_{n=0}^{r-1} \widetilde{P}_k(x)t^k = 0$.

By Shohat-Favard's theorem [7, 8, 25], a polynomial sequence $\{P_n\}_{n\geq 0}$ satisfies a recurrence relation of type (2.3) if and only if there is a form $u \in \mathcal{P}'$, the dual space of the vector space of polynomials with coefficients in $\mathbb{C}$, such that

$$< u, P_n P_m >= k_n \delta_{n,m} \quad , \quad n, m \geq 0 \quad ; \quad k_n \neq 0 \quad , \quad n \geq 0 \, ,$$

and we say that $\{P_n\}_{n\geq 0}$ is a *orthogonal polynomial sequence* with respect to u and that u is a *regular* form.

We note that the associated sequence $\{P_n^{(1)}\}_{n\geq 0}$ and the perturbed sequence $\{\widetilde{P}_n\}_{n\geq 0}$ are orthogonal with respect to the co-called associated form $u^{(1)}$ and the perturbed form $\tilde{u} = u\left(\mu_0; \begin{matrix} \mu \\ \lambda \end{matrix}; r\right)$, respectively. These forms are related to the original form u [27].

An orthogonal sequence $\{P_n\}_{n\geq 0}$ with respect to u is real if and only if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} \in \mathbb{R} - \{0\}$. These conditions are equivalent to $(u)_n =< u, x^n > \in \mathbb{R}$, $n \geq 0$, then u is real. If, in addition, we suppose that $\gamma_{n+1} > 0$, $n \geq 0$, we say that u is *positive definite*. All Chebyshev forms are positive definite. $\{P_n\}_{n\geq 0}$ is symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$ [7].

Let us represent by $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ the forms with respect to which T_n, U_n, V_n and W_n are orthogonal, normalized by the condition $(u)_0 = 1$. They are particular cases of *classical Jacobi* forms denoted by $\mathcal{J}(\alpha,\beta)$. A Jacobi form is regular if and only if $\alpha \neq -n$, $\beta \neq -n$, $\alpha+\beta \neq -(n+1)$, $n \geq 1$; it is positive definite if and only if $\alpha+1>0$ and $\beta+1>0$; it is symmetric for $\alpha=\beta$ and has the following integral representation for $Re(1+\alpha)>0$ and $Re(1+\beta)>0$ [7, 28, 29]

$$< \mathcal{J}(\alpha,\beta), f >= \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} (1+x)^{\alpha}(1-x)^{\beta} f(x)dx \ .$$

Following *Hahn property* [17, 18], a *classical sequence* is every orthogonal sequence, $\{P_n(x)\}_{n\geq 0}$, whose derivative sequence, $\left\{\frac{P'_{n+1}(x)}{n+1}\right\}_{n\geq 0}$, is also orthogonal. The Böchner [4] characterization of classical families states that an orthogonal polynomial sequence is classical if and only if there are two polynomials Φ and ψ and a sequence $\{\lambda_n\}_{n\geq 0}$, $\lambda_n \neq 0$, $n \geq 0$, such that

$$\Phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \lambda_n P_{n+1}(x), \ n \geq 0 \ , \tag{2.9}$$

with $\deg \Phi \leq 2$ and $\deg \psi = 1$. In the Jacobi case, [7, 28, 29]

$$\Phi(x) = x^2 - 1 \ , \ \psi(x) = -(\alpha+\beta+2)x + \alpha - \beta \ , \ \lambda_n = (n+1)(n - \psi'(0)) \neq 0 \ .$$

This characterization was generalized in [23, 32]. There are four classes of classical sequences [28, 29]: they are the families of Hermite, Laguerre, Bessel and Jacobi. Jacobi includes Gegenbauer ($\alpha=\beta$), Legendre ($\alpha=\beta=0$) and Chebyshev sequences of first ($\alpha=\beta=-\frac{1}{2}$), second ($\alpha=\beta=\frac{1}{2}$), third ($\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$) and fourth ($\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$) kinds.

Classical forms are *semi-classical* forms of class 0. A form u is called *semi-classical* (SC) [26, 27] if it is regular and satisfies the *functional equation*

$$(\Phi u)' + \psi u = 0 \ , \tag{2.10}$$

where (Φ,ψ) is a pair of polynomials, such that Φ is monic, $t = \deg \Phi$, and $p = \deg \psi \geq 1$. The *class* of u is the minimum integer that satisfies $s = \max(p-1, t-2)$. The class of a form is unique, but Φ and ψ are not [27]. The sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is also called *semi-classical of class s*. In (2.10) two operations on $\mathcal{P}'$ appear [26]: the *left product of a form by a polynomial* and the *derivative of a form* defined by transposition from the corresponding operations on $\mathcal{P}$ as $< fu, p >=< u, fp >$ and $< u', p >= - < u, p' >$, $f,\ p \in \mathcal{P}$.

Semi-classical forms are particular cases of *Laguerre-Hahn* forms [1, 2, 27]. A form u is a Laguerre-Hahn if it is regular and its *formal Stieltjes function* $S\big(u\big)(z) = -\sum_{n\geq 0} \frac{(u)_n}{z^{n+1}}$ satisfies the so-called *Stieltjes equation*

$$A(z)S'\big(u\big)(z) = B(z)S^2\big(u\big)(z) + C(z)S\big(u\big)(z) + D(z) \ , \tag{2.11}$$

where A, B, C and D are polynomials. The sequence $\{P_n\}_{n\geq 0}$ orthogonal with respect to u is also called a *Laguerre-Hahn sequence*. If $A=0$, then the form u is called a *second degree* form (SD) [27, 30]. If $A \neq 0$ and supposed monic, we let $A = \Phi$. Then if $B \neq 0$, the form u is called a *strict Laguerre-Hahn* form, whereas if $B = 0$, the form u is semi-classical. In this last case, (2.10) is equivalent to the following equation [27]

$$A(z)S'\big(u\big)(z) = C(z)S\big(u\big)(z) + D(z) \ , \tag{2.12}$$

with $A(z)=\Phi(z),\ C(z)=-\Phi'(z)-\psi(z),\ D(z)=-\big(u\theta_0\Phi\big)'(z)-\big(u\theta_0\psi\big)(z)$, where in $D(z)$ intervenes the *right product of a form by a polynomial* defined by [26]

$$\big(up\big)(x)=<u,\frac{xp(x)-\xi p(\xi)}{x-\xi}> \ \text{and}\ \big(\theta_c p\big)(x)=\frac{p(x)-p(c)}{x-c}\ ,\ p\in\mathcal{P}\ ,\ c\in\mathbb{C}\ .$$

Perturbation is a transformation that preserves the quadratic property of the Stieltjes equation (2.11) with $A=0$, in other words, if u is a second degree form, then $\tilde{u}$ is also of second degree, because it fullfils [30]

$$\widetilde{B}(z)S^2\big(\tilde{u}\big)(z)+\widetilde{C}(z)S\big(\tilde{u}\big)(z)+\widetilde{D}(z)=0, \tag{2.13}$$

with

$$k_r\widetilde{B}(z) = B(z)X_r^2(z)-C(z)X_r(z)Y_r(z)+D(z)Y_r^2(z), \tag{2.14}$$
$$k_r\widetilde{C}(z) = 2\big\{B(z)U_r(z)X_r(z)+D(z)V_r(z)Y_r(z)\big\} \tag{2.15}$$
$$-C(z)\big\{U_r(z)Y_r(z)+V_r(z)X_r(z)\big\}, \tag{2.16}$$
$$k_r\widetilde{D}(z) = B(z)U_r^2(z)-C(z)U_r(z)V_r(z)+D(z)V_r^2(z), \tag{2.17}$$

where k_r is a normalization constant chosen in order to make $\widetilde{B}$ monic and

$$U_r(z)=\gamma_r\Big\{\widetilde{P}^{(1)}_{r-1}(z)P^{(1)}_{r-2}(z)-\lambda_rP^{(1)}_{r-1}(z)\widetilde{P}^{(1)}_{r-2}(z)\Big\}-\mu_rP^{(1)}_{r-1}(z)\widetilde{P}^{(1)}_{r-1}, \tag{2.18}$$
$$V_r(z)=\gamma_r\Big\{\widetilde{P}^{(1)}_{r-1}(z)P_{r-1}(z)-\lambda_rP_r(z)\widetilde{P}^{(1)}_{r-2}(z)\Big\}-\mu_rP_r(z)\widetilde{P}^{(1)}_{r-1}(z), \tag{2.19}$$
$$X_r(z)=\gamma_r\Big\{\widetilde{P}_r(z)P^{(1)}_{r-2}(z)-\lambda_rP^{(1)}_{r-1}(z)\widetilde{P}_{r-1}(z)\Big\}-\mu_rP^{(1)}_{r-1}(z)\widetilde{P}_r(z), \tag{2.20}$$
$$Y_r(z)=\gamma_r\Big\{\widetilde{P}_r(z)P_{r-1}(z)-\lambda_rP_r(z)\widetilde{P}_{r-1}(z)\Big\}-\mu_rP_r(z)\widetilde{P}_r(z), \tag{2.21}$$

are the so-called *transfer polynomials* [27].

A closed formula of the Sieltjes function of the perturbed form $\tilde{u}$ is given by [27]

$$S\big(\tilde{u}\big)(z)=-\frac{U_r(z)+V_r(z)S\big(u\big)(z)}{X_r(z)+Y_r(z)S\big(u\big)(z)}\ . \tag{2.22}$$

Second degree forms are also semi-classical, because from (2.11) with $A=0$, one can deduce an equation of the type (2.12) with other coefficients $\widehat{A}(z)$, $\widehat{C}(z)$ and $\widehat{D}(z)$ given by [30]

$$\widehat{A}(z)=B(z)\big\{C^2(z)-4B(z)D(z)\big\}, \tag{2.23}$$
$$\widehat{C}(z)=2B(z)\big\{B'(z)D(z)-D'(z)B(z)\big\}+C(z)\big\{C'(z)B(z)-B'(z)C(z)\big\}, \tag{2.24}$$
$$\widehat{D}(z)=B(z)\big\{C'(z)D(z)-D'(z)C(z)\big\}+D(z)\big\{C'(z)B(z)-B'(z)C(z)\big\}. \tag{2.25}$$

Thus, in particular, the perturbed of a second degree form is semi-classical and satisfies an equation of type (2.12) with coefficients given by the preceding equalities taking $B=\widetilde{B}$, $C=\widetilde{C}$ and $D=\widetilde{D}$ furnished by (2.14)-(2.17).

Any polynomial P_{n+1} from a semi-classical sequence of class s satisfies the following *second-order linear differential equation* [27]

$$J(x;n)P''_{n+1}(x)+K(x;n)P'_{n+1}(x)+L(x;n)P_{n+1}(x)=0\quad,\quad n\geq 0\ , \tag{2.26}$$

with $\deg J(\cdot;n) \le 2s+2$, $\deg K(\cdot;n) \le 2s+1$, $\deg L(\cdot;n) \le 2s$, $n \ge 0$, and

$$J(x;n) = \Phi(x) D_{n+1}(x), \tag{2.27}$$

$$K(x;n) = C_0(x) D_{n+1}(x) - W\big(\Phi, D_{n+1}(x)\big)(x), \tag{2.28}$$

$$L(x;n) = W\left(\frac{1}{2}(C_{n+1}-C_0), D_{n+1}\right)(x) - D_{n+1}(x) \sum_{\nu=0}^{n} D_\nu(x), \tag{2.29}$$

for $n \ge 0$, where $W(f,g)$ denotes the wronskian of f and g, and C_n and D_n are given by [27]

$$A_0(z) = \Phi(z)\ ,\ B_0(z) = 0\ ,\ C_0(z) = C(z)\ ,\ D_0(z) = D(z)\ , \tag{2.30}$$

$$A_{r+1}(z) = \Phi(z)\ ,\ B_{r+1}(z) = \gamma_{r+1} D_r(z)\ , \tag{2.31}$$

$$C_{r+1}(z) = -C_r(z) + 2(z-\beta_r) D_r(z)\ , \tag{2.32}$$

$$\gamma_{r+1} D_{r+1}(z) = -\Phi(z) + B_r(z) - (z-\beta_r) C_r(z) + (z-\beta_r)^2 D_r(z)\ . \tag{2.33}$$

Reciprocally, when any polynomial P_{n+1} from an orthogonal sequence fullfils (2.26), then the sequence $\{P_n\}_{n\ge0}$ is semi-classical [27]. Thus, the differential equation (2.26) assures the semi-classical character of an orthogonal sequence. Furthermore, any semi-classical sequence $\{P_n\}_{n\ge0}$ satisfies the following *structure relation* [27]

$$\Phi(x) P'_{n+1}(x) = \frac{1}{2}\Big(C_{n+1}(x) - C_0(x)\Big) P_{n+1}(x) - \gamma_{n+1} D_{n+1}(x) P_n(x), \tag{2.34}$$

for $n \ge 0$, with $\deg C_n \le s+1$, $\deg D_n \le s$, $n \ge 0$. Notice that (2.26) corresponds to the Böchner differential equation (2.9), in the particular case of classical sequences when no perturbation occurs.

In order to obtain the differential equation (2.26) for perturbed second degree forms, one should take in (2.30) $\Phi(z) = \widehat{A}(z)$, $C(z) = \widehat{C}(z)$ and $D(z) = \widehat{D}(z)$ given by (2.23)-(2.25) with $B = \widetilde{B}$, $C = \widetilde{C}$ and $D = \widetilde{D}$ furnished by (2.14)-(2.17).

The coefficients C_n and D_n, for $n = 0, 1, \dots$ until a maximal order *nmax* can always be computed recursively from the relations (2.30)-(2.33), but in general it is not possible to solve analytically these recurrence relations in order to obtain closed formulas for C_n and D_n, for all $n \ge 0$. Otherwise, in the case we have a model for these formulas, we can use these relations to do a demonstration by induction. Considering that the coefficients of $C_n(z)$ and $D_n(z)$ in the canonical basis $< 1, z, z^2, \dots >$ are polynomials of limited and fixed degree in n, often of low degree, it is possible to find those closed formulas from the first few C_n and D_n, for $n = 0, 1, \dots, nmax$ by an interpolation procedure. The closed formulas for the coefficients of the differential equation (2.26) are obtained directly from the closed formulas of C_n and D_n, $n \ge 0$ by the identities (2.27)-(2.29), assuming that it is possible to find a closed formula for the finite summation that appears in (2.29). Finally, we notice that it is important to factorize the coefficients $J_n(x;n)$, $K_n(x;n)$ and $L_n(x;n)$, because often there are common factors between them that can be simplified in the differential equation. We can accomplish all these tasks in *Mathematica*® with the algorithm *PSDF*.

For Jacobi sequences, we have [28, 29]

$$C_n(x) = (2n+\alpha+\beta)x - \frac{\alpha^2-\beta^2}{2n+\alpha+\beta} \quad , \quad D_n(x) = 2n+\alpha+\beta+1\ \ n \ge 0\ .$$

Among the classical forms, only the Jacobi forms $\mathcal{J}(\alpha,\beta)$ corresponding to $\alpha = k - \frac{1}{2}$ and $\beta = l - \frac{1}{2}$, with $k + l \ge 0$ and $k,\ l \in \mathbb{Z}$ are of second degree [3, Theorem p.445]. The set of second degree forms is closed with respect to several transformations that preserve the quadratic property of the Stieltjes equation (2.11) for $A=0$ [3, 30, 31, 34]. Among these transformations is the perturbation.

3. New explicit results for the complete perturbation of order one and a perturbation of order two for the four Chebyshev families

In [9], was systematized a new general method intended to explicit some semi-classical properties of perturbed second degree forms leading to the second order linear differential equation. The symbolic algorithm *PSDF*, corresponding to the method, is prepared to work for any perturbation and any second degree form and was implemented in the algebraic manipulator *Mathematica*®. It is divided into four main steps. In step 1, we give some starting elements of the original second degree form u needed in the sequel of computations, they are: the recurrence coefficients (β_n, γ_{n+1}, $n\geq 0$), the closed formula of the Stieljtes function ($S(u)(z)$) and the coefficients of the Stieltjes equation as second degree form ($B^{SD}(x)$, $C^{SD}(x)$, $D^{SD}(x)$, $n\geq 0$). After that, in step 2, we do the perturbation, and we compute the coefficients of the Stieltjes equation of the perturbed form $\tilde{u}$ as second degree form ($\widetilde{B}^{SD}(x)$, $\widetilde{C}^{SD}(x)$, $\widetilde{D}^{SD}(x)$). For that we need to compute the transfer polynomials ($U_r(x)$, $V_r(x)$, $X_r(x)$, $Y_r(x)$) and from them we obtain also a closed formula for the Stieltjes function ($S(\tilde{u})(z)$). In step 3, we compute some elements of the perturbed form as semi-classical form, namely the coefficients of the functional equation ($\widetilde{\Phi}^{SC}(x)$, $\widetilde{\psi}^{SC}(x)$) and of the Stieltjes equation ($\widetilde{A}^{SC}(z)$, $\widetilde{C}^{SC}(z)$, $\widetilde{D}^{SC}(z)$), and the class s of $\tilde{u}$. For that, we need some first moments of $\tilde{u}$ ($(\tilde{u})_n$, $n=0,\dots,nmax$) obtained in step 2. In the last step, we begin by finding and demonstrating by induction the closed formulas of the coefficients of the structure equation ($\widetilde{C}_n^{SC}(z)$, $\widetilde{D}_n^{SC}(z)$, $n\geq 0$). From them, we obtain directly the closed formulas of the coefficients of the differential equation ($\widetilde{J}^{SC}(x;n)$, $\widetilde{K}^{SC}(x;n)$, $\widetilde{L}^{SC}(x;n)$, $n\geq 0$), our main goal in this work. In the particular case of Chebyshev sequences, we are able to compute closed formulas of the generating functions for any perturbation using (2.8).

In this section, we give new results, obtained with *PSDF*, for the complete perturbation of order 1 and the following perturbation of order two of the second kind Chebyshev form

$$\mathcal{U}\left(\mu_0;\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right),\mu_1\neq 0 \text{ or } \lambda_1\neq 1,\lambda_1\neq 0;\mathcal{U}\left(\mu_0;\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right),\lambda_1\neq 0,\lambda_2\neq 0,\lambda_2\neq 1,$$

from which we can easily obtain the results for the other three Chebyshev families as follows;

$$\mathcal{T}\left(\mu_0;\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right)=\mathcal{U}\left(\mu_0;\begin{matrix}\mu_1\\ \mathbf{2}\lambda_1\end{matrix};1\right),\mathcal{V}\left(\mu_0;\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right)=\mathcal{U}\left(\mu_0+\frac{\mathbf{1}}{\mathbf{2}};\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right),$$
$$\mathcal{W}\left(\mu_0;\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right)=\mathcal{U}\left(\mu_0-\frac{\mathbf{1}}{\mathbf{2}};\begin{matrix}\mu_1\\ \lambda_1\end{matrix};1\right);$$
$$\mathcal{T}\left(\mu_0;\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right)=\mathcal{U}\left(\mu_0;\begin{matrix}0\ ,\ 0\\ \mathbf{2}\lambda_1,\lambda_2\end{matrix};2\right),\mathcal{V}\left(\mu_0;\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right)=\mathcal{U}\left(\mu_0+\frac{\mathbf{1}}{\mathbf{2}};\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right),$$
$$\mathcal{W}\left(\mu_0;\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right)=\mathcal{U}\left(\mu_0-\frac{\mathbf{1}}{\mathbf{2}};\begin{matrix}0\ ,\ 0\\ \lambda_1,\lambda_2\end{matrix};2\right);$$

for that purpose the presence of the parameters μ_0 and λ_1 is essential. In the perturbation of order two, if we take $\mu_0 = 0$, we obtain the *c*omplete perturbation by dilatation of order two that preserves the type of symmetry of the original sequence.

PSDF algorithm starts from the following elements of $\mathcal{U}$ [9, 30, 31]

$$\beta_n = 0\ ,\ \gamma_{n+1} = \frac{1}{4}\ ,\ n \geq 0 \quad ; \quad S(\mathcal{U})(z) = -\frac{2}{z+\sqrt{z^2-1}}\ ;$$
$$B^{SD}(x) = 1\ ,\ C^{SD}(x) = 4x\ ,\ D^{SD}(x) = 4\ .$$

In next tables, we give all mentioned new explicit properties for the two perturbations.

We finish this section with an illustration and source of inspiration, we present in Figure 1 some graphical representations from which we can appreciate the fullness of interesting properties of perturbed Chebyshev polynomials. We call your attention to the symmetrical and unsymmetrical aspects, interception points, extremes and zeros.

FIGURE 1. Some perturbed of order 2 by dilatation, $\left(0; {}^{0,\ 0}_{1,\lambda_2}; 2\right)$, with positive parameter $\lambda_2 = 3(1)7$ (in red) and negative parameter $\lambda_2 = -5(1)-1$ (in black), of the four Chebyshev polynomials (in bleu) of degree $n = 8$.

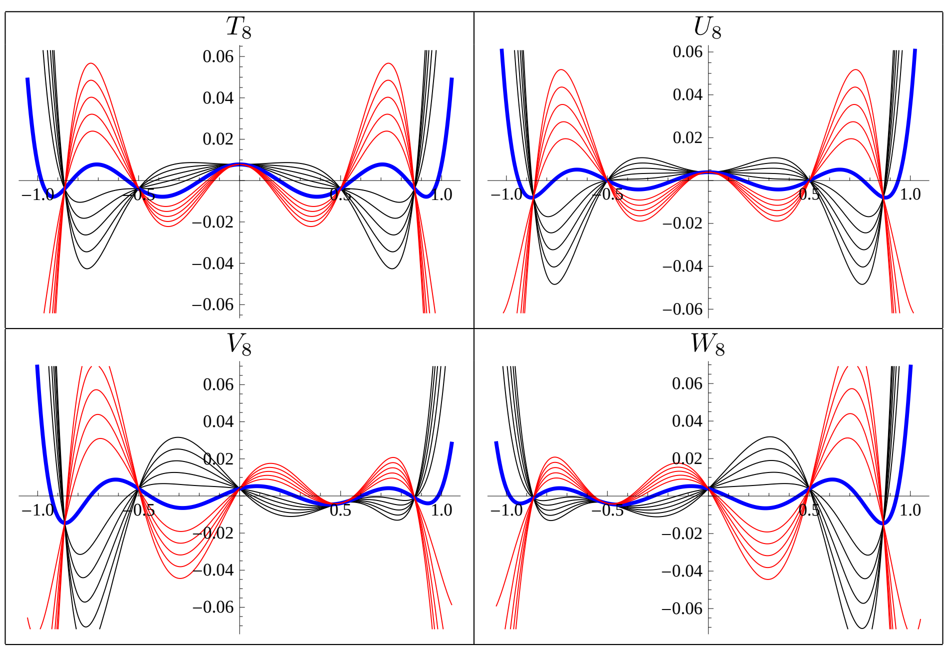

TABLE 1. Results for the **complete perturbation of order one**

The perturbed form $\tilde{u} = \mathcal{U}\left(\mu_0; {\mu_1 \atop \lambda_1}; 1\right)\ ,\ \mu_1 \neq 0\ ,\ \lambda_1 \neq 0\ .$	**Recurrence coefficients of $\tilde{u}$** $\tilde{\beta}_0 = \mu_0\ ,\ \tilde{\beta}_1 = \mu_1\ ,\ \tilde{\beta}_n = 0\ ,\ n \geq 2\ ;\ \tilde{\gamma}_1 = \frac{1}{4}\lambda_1\ ,\ \tilde{\gamma}_{n+1} = \frac{1}{4},\ n \geq 1.$
The generating function: $\tilde{f}(x,t) = \frac{1 - t\mu_0 + t^2\left(-x\mu_1 + \frac{1}{4}(1-\lambda_1+4\mu_0\mu_1)\right)}{1+t\left(\frac{t}{4}-x\right)}\ .$	
First moments: $(\tilde{u})_{n=0(1)5} = \Big\{1, \mu_0, \frac{1}{4}\left(\lambda_1 + 4\mu_0^2\right), \frac{1}{4}\left(2\lambda_1\mu_0 + 4\mu_0^3 + \lambda_1\mu_1\right), \frac{1}{16}\left(\lambda_1 + \lambda_1^2 + 12\lambda_1\mu_0^2 + 16\mu_0^4 + 8\lambda_1\mu_0\mu_1 + 4\lambda_1\mu_1^2\right),$ $\frac{1}{16}\left(2\lambda_1\mu_0 + 3\lambda_1^2\mu_0 + 16\lambda_1\mu_0^3 + 16\mu_0^5 + 2\lambda_1\mu_1 + 2\lambda_1^2\mu_1 + 12\lambda_1\mu_0^2\mu_1 + 8\lambda_1\mu_0\mu_1^2 + 4\lambda_1\mu_1^3\right)\Big\}\ .$	
Transfer polynomials $U_r(x) = -\mu_1\ ,\ V_r(x) = -\mu_1 x + \frac{1}{4}\ ,\ X_r(x) = -\mu_1 x + \frac{1}{4}\left(-\lambda_1 + 4\mu_0\mu_1\right)\ ,\ Y_r(x) = -\mu_1 x^2 + \frac{1}{4}\left(1 - \lambda_1 + 4\mu_0\mu_1\right)x - \frac{\mu_0}{4}\ .$	
The Stieltjes function of $\tilde{u}$: $S(\tilde{u})(z) = \frac{4\mu_1 z - 2 - 4\mu_1\sqrt{-1+z^2}}{-4\mu_1 z^2 + (2-\lambda_1+4\mu_0\mu_1)z - 2\mu_0 + \sqrt{-1+z^2}(4\mu_1 z - 4\mu_0\mu_1 + \lambda_1)}\ .$	
Coefficients of the Stieltjes equation of $\tilde{u}$ as second degree form $\widetilde{B}^{SD}(x) = x^3 - \frac{1}{4\mu_1}\Big(1 - \lambda_1 + 8\mu_0\mu_1 + 4\mu_1^2\Big)x^2 - \frac{1}{4\mu_1}\Big(-2\mu_0 + \lambda_1\mu_0 + 2\lambda_1\mu_1 - 4\mu_0^2\mu_1 - 8\mu_0\mu_1^2\Big)x$ $- \frac{1}{16\mu_1}\Big(\lambda_1^2 + 4\mu_0^2 - 8\lambda_1\mu_0\mu_1 + 16\mu_0^2\mu_1^2\Big),$ $\widetilde{C}^{SD}(x) = 2x^2 - \frac{1}{4\mu_1}\Big(2 - \lambda_1 + 8\mu_0\mu_1 + 8\mu_1^2\Big)x - \frac{1}{2\mu_1}\Big(-\mu_0 + \lambda_1\mu_1 - 4\mu_0\mu_1^2\Big)\quad ,\quad \widetilde{D}^{SD}(x) = x - \frac{1+4\mu_1^2}{4\mu_1}\ .$	
Coefficients of the functional equation of $\tilde{u}$ $\widetilde{\Phi}^{SC}(x) = (x^2-1)\widetilde{B}^{SD}(x)\ ,\ \tilde{\psi}^{SC}(x) = -3x\widetilde{B}^{SD}(x)\ .$	**$\tilde{u}$ is a semi–classical form of class 3.**

TABLE 2. Results for **the complete perturbation of order one** (continuation)

Coefficients of the Stieltjes equation of $\tilde{u}$ as semi–classical form

$$\widetilde{A}^{SC}(x)=\widetilde{\Phi}^{SC}(x)\ ,$$
$$\widetilde{C}^{SC}(x)=-2x^4+\frac{\left(1-\lambda_1+8\mu_0\mu_1+4\mu_1^2\right)}{4\mu_1}x^3+3x^2-\frac{\left(8-8\lambda_1+\lambda_1^2+4\mu_0^2+64\mu_0\mu_1-8\lambda_1\mu_0\mu_1+32\mu_1^2+16\mu_0^2\mu_1^2\right)}{16\mu_1}x-\frac{-2\mu_0+\lambda_1\mu_0+2\lambda_1\mu_1-4\mu_0^2\mu_1-8\mu_0\mu_1^2}{4\mu_1}\ ,$$
$$\widetilde{D}^{SC}(x)=-x^3-\frac{1}{4\mu_1}\Big(-\mu_0-8\mu_1+\lambda_1\mu_1-4\mu_0\mu_1^2\Big)x-\frac{1}{8\mu_1}\Big(2-\lambda_1+8\mu_0\mu_1+8\mu_1^2\Big)\ .$$

Coefficients of the Stieltjes equation of $\tilde{u}^{(n)}$ as Laguerre–Hahn form

$$\widetilde{A}_0^{SC}(x)=\widetilde{\Phi}^{SC}(x)\quad ;\quad \widetilde{B}_0^{SC}(x)=0\ ,\ \widetilde{B}_1^{SC}(x)=\frac{\lambda_1}{4}\widetilde{D}^{SC}(x)\ ,\ \widetilde{B}_n^{SC}(x)=\frac{1}{4}\widetilde{D}_{n-1}^{SC}(x)\ ,\ n\geq 2\ .$$

Coefficients of the structure relation of $\tilde{u}$ as semi–classical form

$$\widetilde{C}_0^{SC}(x)=\widetilde{C}^{SC}(x)\ ,$$
$$\begin{aligned}\widetilde{C}_1^{SC}(x)=&-\frac{1}{4\mu_1}\Big(1-\lambda_1+4\mu_1^2\Big)x^3+\frac{1}{2\mu_1}\Big(\mu_0+2\mu_1-\lambda_1\mu_1+4\mu_0\mu_1^2\Big)x^2-\frac{1}{16\mu_1}\Big(4\lambda_1-\lambda_1^2+4\mu_0^2+32\mu_0\mu_1+16\mu_0^2\mu_1^2\Big)x\\&+\frac{1}{2}\Big(\lambda_1+2\mu_0^2\Big)\ ,\end{aligned}$$
$$\begin{aligned}\widetilde{C}_n^{SC}(x)=&\,2(n-1)x^4-\frac{1}{4\mu_1}\Big\{(2n-1)-(2n-1)\lambda_1+16(n-1)\mu_0\mu_1+4(2n-3)\mu_1^2\Big\}x^3\\&+\frac{1}{2\mu_1}\Big\{(2n-1)\mu_0-(n-1)\lambda_1\mu_0-(2n-3)\lambda_1\mu_1+4(n-1)\mu_0^2\mu_1+4(2n-3)\mu_0\mu_1^2\Big\}x^2\\&-\frac{1}{16\mu_1}\Big\{4\lambda_1+(2n-3)\lambda_1^2+(8(n-1)+4)\mu_0^2-(16(n-1)-8)\lambda_1\mu_0\mu_1+(32(n-2)+16)\mu_0^2\mu_1^2\Big\}x\ ,\ n\geq 2\ ,\end{aligned}$$
$$\widetilde{D}_0^{SC}(x)=\widetilde{D}^{SC}(x)\ ,$$
$$\begin{aligned}\widetilde{D}_n^{SC}(x)=&\,(2n-1)x^3-\frac{1}{2\mu_1}\Big\{n-n\lambda_1+4(2n-1)\mu_0\mu_1+4(n-1)\mu_1^2\Big\}x^2\\&+\frac{1}{4\mu_1}\Big\{4n\mu_0-(2n-1)\lambda_1\mu_0-4(n-1)\lambda_1\mu_1+4(2n-1)\mu_0^2\mu_1+16(n-1)\mu_0\mu_1^2\Big\}x\\&\frac{1}{8\mu_1}\Big\{2\lambda_1+(n-1)\lambda_1^2+4n\mu_0^2-8(n-1)\lambda_1\mu_0\mu_1+16(n-1)\mu_0^2\mu_1^2\Big\}\ ,\ n\geq 1\ .\end{aligned}$$

TABLE 3. Results for **the complete perturbation of order one** (continuation)

Coefficients of the second order linear differential equation of $\tilde{u}$ as semi–classical form for $n \geq 0$

$$
\begin{aligned}
\widetilde{J}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{8\mu_1}(x^2-1)\Big\{ & 8(1+2n)\mu_1 x^3 + 4\Big(-1+\lambda_1-4\mu_0\mu_1+\left(-1+\lambda_1-8\mu_0\mu_1-4\mu_1^2\right)n\Big)x^2 \\
& +\tfrac{1}{4\mu_1}\Big(4n\mu_0-(2n-1)\lambda_1\mu_0-4(n-1)\lambda_1\mu_1+4(2n-1)\mu_0^2\mu_1+16(n-1)\mu_0\mu_1^2\Big)x \\
& -2\left(\lambda_1+2\mu_0^2\right)+\left(-\lambda_1^2-4\mu_0^2+8\lambda_1\mu_0\mu_1-16\mu_0^2\mu_1^2\right)n\Big\} \ ,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{K}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{8\mu_1}\Big\{ & 4\Big(-1+\lambda_1-4\mu_0\mu_1+\left(-1+\lambda_1-8\mu_0\mu_1-4\mu_1^2\right)n\Big)x^3 \\
& +4\Big\{4\mu_0-\lambda_1\mu_0+6\mu_1+4\mu_0^2\mu_1+2\left(2\mu_0-\lambda_1\mu_0+6\mu_1-2\lambda_1\mu_1+4\mu_0^2\mu_1+8\mu_0\mu_1^2\right)n\Big\}\,x^2 \\
& +\Big\{2\left(-4+\lambda_1-6\mu_0^2-16\mu_0\mu_1\right)+\left(-8+8\lambda_1-3\lambda_1^2-12\mu_0^2-64\mu_0\mu_1+24\lambda_1\mu_0\mu_1-32\mu_1^2-48\mu_0^2\mu_1^2\right)n\Big\}x \\
& +2\Big(\mu_0\left(4-\lambda_1+4\mu_0\mu_1\right)+2\left(2\mu_0-\lambda_1\mu_0-2\lambda_1\mu_1+4\mu_0^2\mu_1+8\mu_0\mu_1^2\right)n\Big)\Big\} \ ,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{L}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{8\mu_1}\Big\{ & -8n(1+n)(1+2n)\mu_1 x^3 \\
& -4\Big\{-1+\lambda_1-4\mu_0\mu_1+3\left(-1+\lambda_1-4\mu_0\mu_1\right)n+3\left(-1+\lambda_1-4\mu_0\mu_1\right)n^2+\left(-1+\lambda_1-8\mu_0\mu_1-4\mu_1^2\right)n^3\Big\}x^2 \\
& -2\Big\{6\left(\mu_0+2\mu_1\right)+\left(14\mu_0-\lambda_1\mu_0+24\mu_1-2\lambda_1\mu_1+4\mu_0^2\mu_1+8\mu_0\mu_1^2\right)n+3\mu_0\left(4-\lambda_1+4\mu_0\mu_1\right)n^2 \\
& +2\left(2\mu_0-\lambda_1\mu_0-2\lambda_1\mu_1+4\mu_0^2\mu_1+8\mu_0\mu_1^2\right)n^3\Big\}x+\left(\lambda_1^2+4\mu_0^2-8\lambda_1\mu_0\mu_1+16\mu_0^2\mu_1^2\right)n^3+6\left(\lambda_1+2\mu_0^2\right)n^2 \\
& +\left(8+4\lambda_1-\lambda_1^2+8\mu_0^2+16\mu_0\mu_1+8\lambda_1\mu_0\mu_1+32\mu_1^2-16\mu_0^2\mu_1^2\right)n-2\left(-4+\lambda_1-4\mu_0\mu_1\right)\Big\} \ .
\end{aligned}
$$

TABLE 4. Results for the perturbation of order 2

The perturbed form $\tilde{u} = \mathcal{U}\left(\mu_0; {}^{0,\ 0}_{\lambda_1, \lambda_2}; 2\right), \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_2 \neq 1.$	**Recurrence coefficients of** $\tilde{u}$ $\tilde{\beta}_0 = \mu_0, \tilde{\beta}_n = 0, n \geq 1$; $\tilde{\gamma}_1 = \frac{\lambda_1}{4}, \tilde{\gamma}_2 = \frac{\lambda_2}{4}, \tilde{\gamma}_{n+1} = \frac{1}{4}, n \geq 2.$
The generating function: $\tilde{f}(x,t) = \frac{1+\frac{1}{4}t^2(1-\lambda_1)-t\mu_0+t^3\left(\frac{1}{4}x(1-\lambda_2)+\frac{1}{4}(-1+\lambda_2)\mu_0\right)}{1+t\left(\frac{t}{4}-x\right)}.$	
First moments: $(\tilde{u})_{n=0(1)7} = \Big\{1, \mu_0, \frac{\lambda_1}{4} + \mu_0^2, \frac{\lambda_1\mu_0}{2} + \mu_0^3, \frac{1}{16}\left(\lambda_1^2 + 16\mu_0^4 + \lambda_1\left(\lambda_2 + 12\mu_0^2\right)\right),$ $\frac{3}{16}\lambda_1^2\mu_0 + \mu_0^5 + \lambda_1\left(\frac{\lambda_2\mu_0}{8} + \mu_0^3\right), \frac{1}{64}\left(\lambda_1^3 + 64\mu_0^6 + 2\lambda_1^2\left(\lambda_2 + 12\mu_0^2\right) + \lambda_1\left(\lambda_2^2 + 80\mu_0^4 + \lambda_2\left(1 + 12\mu_0^2\right)\right)\right),$ $\frac{1}{32}\mu_0\left(2\lambda_1^3 + 32\mu_0^6 + \lambda_1^2\left(3\lambda_2 + 20\mu_0^2\right) + \lambda_1\left(\lambda_2^2 + 48\mu_0^4 + \lambda_2\left(1 + 8\mu_0^2\right)\right)\right)\Big\}.$	
Transfer polynomials: $U_r(x) = \frac{1}{4}x\left(1 - \lambda_2\right)$, $V_r(x) = \frac{1}{4}x^2\left(1 - \lambda_2\right) + \frac{\lambda_2}{16}$, $X_r(x) = \frac{1}{4}x^2\left(1 - \lambda_2\right) + \frac{1}{4}x\left(-1 + \lambda_2\right)\mu_0 - \frac{\lambda_1}{16}$, $Y_r(x) = \frac{1}{4}x^3\left(1 - \lambda_2\right) + \frac{1}{4}x^2\left(-1 + \lambda_2\right)\mu_0 + \frac{1}{16}x\left(-\lambda_1 + \lambda_2\right) - \frac{\lambda_2\mu_0}{16}.$	
The Stieltjes function of $\tilde{u}$ $S(\tilde{u})(z) = \frac{-4z^2(-1+\lambda_2)+2\lambda_2+4z(-1+\lambda_2)\sqrt{-1+z^2}}{4z^3(-1+\lambda_2)-4z^2(-1+\lambda_2)\mu_0+z(\lambda_1-2\lambda_2)+2\lambda_2\mu_0+\sqrt{-1+z^2}(-4z^2(-1+\lambda_2)+4z(-1+\lambda_2)\mu_0-\lambda_1)}$.	
Coefficients of the Stieltjes equation of $\tilde{u}$ **as second degree form** $\widetilde{B}^{SD}(x) = x^4 - 2x^3\mu_0 - \frac{x^2\left(-2\lambda_1+\lambda_1\lambda_2+\lambda_2^2+4\mu_0^2-4\lambda_2\mu_0^2\right)}{4(-1+\lambda_2)} + \frac{x\left(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\right)\mu_0}{4(-1+\lambda_2)} - \frac{\lambda_1^2+4\lambda_2^2\mu_0^2}{16(-1+\lambda_2)}$, $\widetilde{C}^{SD}(x) = 2x^3 - 2x^2\mu_0 - \frac{x\left(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\right)}{4(-1+\lambda_2)} + \frac{\lambda_2^2\mu_0}{2(-1+\lambda_2)}$, $\widetilde{D}^{SD}(x) = x^2 - \frac{\lambda_2^2}{4(-1+\lambda_2)}$.	
Coefficients of the functional equation of $\tilde{u}$ $\widetilde{\Phi}^{SC}(x) = (x^2-1)\widetilde{B}^{SD}(x)$, $\widetilde{\psi}^{SC}(x) = -3x\widetilde{B}^{SD}(x)$.	$\tilde{u}$ **is a semi–classical form of class** 4.

TABLE 5. Results for the perturbation of order 2 (continuation)

Coefficients of the Stieltjes equation of $\tilde{u}$ as semi-classical form

$$\widetilde{A}^{SC}(x) = \widetilde{\Phi}^{SC}(x) \; ,$$

$$\widetilde{C}^{SC}(x) = -3x^5 + 4x^4\mu_0 + \frac{x^3\big(-16-2\lambda_1+16\lambda_2+\lambda_1\lambda_2+\lambda_2^2+4\mu_0^2-4\lambda_2\mu_0^2\big)}{4(-1+\lambda_2)} - 6x^2\mu_0 - \frac{x\big(-16\lambda_1+\lambda_1^2+8\lambda_1\lambda_2+8\lambda_2^2+32\mu_0^2-32\lambda_2\mu_0^2+4\lambda_2^2\mu_0^2\big)}{16(-1+\lambda_2)} + \frac{\big(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\big)\mu_0}{4(-1+\lambda_2)} \; ,$$

$$\widetilde{D}^{SC}(x) = -2x^4 + x^3\mu_0 + 3x^2 + \frac{x\big(8-8\lambda_2+\lambda_2^2\big)\mu_0}{4(-1+\lambda_2)} + \frac{2\lambda_1-\lambda_1\lambda_2-2\lambda_2^2}{8(-1+\lambda_2)} \; .$$

Coefficients of the Stieltjes equation of $\tilde{u}^{(n)}$ as Laguerre-Hahn form

$$\widetilde{A}_0^{SC}(x) = \widetilde{\Phi}^{SC}(x) \quad ; \quad \widetilde{B}_0^{SC}(x) = 0 \; , \; \widetilde{B}_1^{SC}(x) = \tfrac{\lambda_1}{4}\widetilde{D}_0^{SC}(x) \; , \; \widetilde{B}_2^{SC}(x) = \tfrac{\lambda_2}{4}\widetilde{D}_1^{SC}(x) \; , \; \widetilde{B}_n^{SC}(x) = \tfrac{1}{4}\widetilde{D}_{n-1}^{SC}(x) \; , \; n \geq 3 \; ,$$

Coefficients of the structure relation of $\tilde{u}$ as semi-classical form

$$\widetilde{C}_0^{SC}(x) = \widetilde{C}^{SC}(x) \; ,$$

$$\widetilde{C}_1^{SC}(x) = -x^5 + 2x^4\mu_0 - \frac{x^3\big(8-2\lambda_1-8\lambda_2+\lambda_1\lambda_2+\lambda_2^2-4\mu_0^2+4\lambda_2\mu_0^2\big)}{4(-1+\lambda_2)} + \frac{x^2\big(8-8\lambda_2+\lambda_2^2\big)\mu_0}{2(-1+\lambda_2)} - \frac{x\big(8\lambda_1-\lambda_1^2-4\lambda_1\lambda_2+32\mu_0^2-32\lambda_2\mu_0^2+4\lambda_2^2\mu_0^2\big)}{16(-1+\lambda_2)} \; ,$$

$$\widetilde{C}_n^{SC}(x) = (2n-3)x^5 - (4n-6)x^4\mu_0 - \frac{x^3\big(-(4n-6)\lambda_1+4\lambda_2+(2n-5)(\lambda_1+\lambda_2)\lambda_2+(8n-12)(1-\lambda_2)\mu_0^2\big)}{4(-1+\lambda_2)} + \frac{x^2\big(-(2n-3)\lambda_1+4\lambda_2+(n-2)\lambda_1\lambda_2+(2n-5)\lambda_2^2\big)\mu_0}{2(-1+\lambda_2)} - \frac{x\big((2n-3)\lambda_1^2+4\lambda_1\lambda_2+16\lambda_2\mu_0^2+(8n-20)\lambda_2^2\mu_0^2\big)}{16(-1+\lambda_2)} \; , \; n \geq 2 \; ;$$

$$\widetilde{D}_0^{SC}(x) = \widetilde{D}^{SC}(x) \; ,$$

$$\widetilde{D}_1^{SC}(x) = \frac{x^2(-2+\lambda_1+\lambda_2)}{2(-1+\lambda_2)} - \frac{x(-8+\lambda_1+4\lambda_2)\mu_0}{4(-1+\lambda_2)} - \frac{\lambda_1+4\mu_0^2-2\lambda_2\mu_0^2}{4(-1+\lambda_2)} \; ,$$

$$\widetilde{D}_n^{SC}(x) = 2(n-1)x^4 - 4(n-1)x^3\mu_0 - \frac{x^2\big(-2(n-1)\lambda_1+2\lambda_2+(n-2)(\lambda_1+\lambda_2)\lambda_2+4(n-1)\mu_0^2(1-\lambda_2)\big)}{2(-1+\lambda_2)} + \frac{x\big(-4(n-1)\lambda_1+8\lambda_2+(2n-3)\lambda_1\lambda_2+4(n-2)\lambda_2^2\big)\mu_0}{4(-1+\lambda_2)} - \frac{(n-1)\lambda_1^2+2\lambda_1\lambda_2+8\lambda_2\mu_0^2+4(n-2)\lambda_2^2\mu_0^2}{8(-1+\lambda_2)} \; , \; n \geq 2 \; .$$

TABLE 6. Results for the perturbation of order two (continuation)

Coefficients of the differential equation of $\tilde{u}$ as semi-classical form

$$\widetilde{J}^{SC}(0;x) = \frac{\widetilde{B}^{SD}(x)}{4(-1+\lambda_2)}(x^2-1)\Big\{2x^2\left(-2+\lambda_1+\lambda_2\right) - x\left(-8+\lambda_1+4\lambda_2\right)\mu_0 - 4\mu_0^2 + 2\lambda_2\mu_0^2 - \lambda_1\Big\}\ ,$$

$$\begin{aligned}\widetilde{J}^{SC}(n;x) = \frac{\widetilde{B}^{SD}(x)}{8(-1+\lambda_2)}(x^2-1)\Big\{&16nx^4\left(-1+\lambda_2\right) - 32nx^3\left(-1+\lambda_2\right)\mu_0 - \\ &4x^2\Big(-\lambda_2\left(-2+\lambda_1+\lambda_2\right) + n\left(-2\lambda_1+\lambda_1\lambda_2+\lambda_2^2+4\mu_0^2-4\lambda_2\mu_0^2\right)\Big) + \\ &2x\Big(-\lambda_2\left(-8+\lambda_1+4\lambda_2\right) + 2n\left(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\right)\Big)\mu_0 - 8\lambda_2\mu_0^2 + 4\lambda_2^2\mu_0^2 - 4n\lambda_2^2\mu_0^2 - n\lambda_1^2\Big\}\ ,\ n\geq 1\ .\end{aligned}$$

$$\widetilde{K}^{SC}(0;x) = \frac{\widetilde{B}^{SD}(x)}{4(-1+\lambda_2)}\left(x-\mu_0\right)\Big\{2x^2\left(-2+\lambda_1+\lambda_2\right) - 6x\left(-2+\lambda_2\right)\mu_0 - 8 + \lambda_1 + 4\lambda_2\Big\}\ ,$$

$$\begin{aligned}\widetilde{K}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{8(-1+\lambda_2)}\Big\{&16nx^5\left(-1+\lambda_2\right) + 4x^3\Big(-\lambda_2\left(-2+\lambda_1+\lambda_2\right) + n\left(16-2\lambda_1-16\lambda_2+\lambda_1\lambda_2+\lambda_2^2+4\mu_0^2-4\lambda_2\mu_0^2\right)\Big) - \\ &4x^2\Big(-\lambda_2\left(-8+\lambda_1+4\lambda_2\right) + 2n\left(12-2\lambda_1-12\lambda_2+\lambda_1\lambda_2+2\lambda_2^2\right)\Big) + \\ &x\Big(-2\lambda_2\left(-8+\lambda_1+4\lambda_2-12\mu_0^2+6\lambda_2\mu_0^2\right) + n\left(-16\lambda_1+3\lambda_1^2+8\lambda_1\lambda_2+8\lambda_2^2+32\mu_0^2-32\lambda_2\mu_0^2+12\lambda_2^2\mu_0^2\right)\Big) - \\ &2\Big(-\lambda_2\left(-8+\lambda_1+4\lambda_2\right) + 2n\left(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\right)\Big)\mu_0\Big\}\ ,\ n\geq 1\ .\end{aligned}$$

$$\widetilde{L}^{SC}(0;x) = -\frac{\widetilde{B}^{SD}(x)}{4(-1+\lambda_2)}\Big(2x^2\left(-2+\lambda_1+\lambda_2\right) - 6x\left(-2+\lambda_2\right)\mu_0 - 8 + \lambda_1 + 4\lambda_2\Big)\ ,$$

$$\begin{aligned}\widetilde{L}^{SC}(n;x) = -\frac{\widetilde{B}^{SD}(x)}{8(-1+\lambda_2)}\Big\{&16x^4(-1+n)n(1+n)\left(-1+\lambda_2\right) - \\ &4x^2\Big(-\lambda_2\left(-2+\lambda_1+\lambda_2\right) + 3n\left(8-2\lambda_1-8\lambda_2+\lambda_1\lambda_2+\lambda_2^2\right) - 3n^2\lambda_2\left(-2+\lambda_1+\lambda_2\right) + \\ &\quad n^3\left(-2\lambda_1+\lambda_1\lambda_2+\lambda_2^2+4\mu_0^2-4\lambda_2\mu_0^2\right)\Big) + \\ &2x\Big(2n^3\left(-2\lambda_1+\lambda_1\lambda_2+2\lambda_2^2\right) - 3n^2\lambda_2\left(-8+\lambda_1+4\lambda_2\right) + n\left(48-2\lambda_1-48\lambda_2+\lambda_1\lambda_2+14\lambda_2^2\right) - 6\left(-2+\lambda_2\right)\lambda_2\Big) + \\ &n^3\left(-\lambda_1^2-4\lambda_2^2\mu_0^2\right) - 6n^2\lambda_2\left(\lambda_1+4\mu_0^2-2\lambda_2\mu_0^2\right) + n\left(-8\lambda_1+\lambda_1^2+4\lambda_1\lambda_2-8\lambda_2^2-32\mu_0^2+32\lambda_2\mu_0^2-8\lambda_2^2\mu_0^2\right) + \\ &2\lambda_2\left(-8+\lambda_1+4\lambda_2\right)\Big\}\ ,\ n\geq 1\ .\end{aligned}$$

4. Open problems

There are some other important properties of perturbed Chebyshev polynomials that are not furnished by *PSDF* algorithm, namely the integral representation of perturbed forms and the properties of zeros that are crucial in quadrature formulas of numerical integration. An integral representation for a perturbation of order one by dilatation of the Chebyshev family of second kind was recently given in [34], but the corresponding result for the perturbations treated in this work, in particular for the dilatation of order two, remain unknown. Some of the results given herein are the starting point to obtain these properties, namely the closed formula of the Stieltjes function and the Stieltjes equations in order to deduce integral representations [34] and the differential equation in order to obtain results about the distribution function of zeros [24, 38].

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