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EXISTENCE OF SOLUTION TO A NONLINEAR FIRST-ORDER DYNAMIC EQUATION ON TIME SCALES

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Abstract. We prove existence of solution to a nonlinear first-order nabla dynamic equation on an arbitrary bounded time scale with boundary conditions, where the right-hand side of the dynamic equation is a continuous function.

1. Introduction

In this work we prove existence of solution to the following system:

$$\begin{aligned} x^{\nabla}(t) &= f(t, x(t)), \quad t \in \mathbb{T}_k, \\ x(a) &= x(b). \end{aligned} \tag{1.1}$$

Here $\mathbb{T}$ is an arbitrary bounded time scale, where we denote $a := \min \mathbb{T}$, $b := \max \mathbb{T}$, $\mathbb{T}_\circ = \mathbb{T} \setminus \{a\}$, and $f : \mathbb{T}_\circ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Problem (1.1) unifies continuous and discrete problems. We use the notion of tube solution for system (1.1), in the spirit of the works of Gilbert and Frigon [7–9]. This notion is useful to get existence results for systems of differential equations of first order, as a generalization of lower and upper solutions [3, 6, 10, 11]. Our main result provides existence of solution to the nonlinear nabla boundary value problem (1.1).

The article is organized as follows. In Section 2 we review some basic definitions and theorems regarding ∇-differentiation and ∇-integration on time scales, and we prove some preliminary results. In Section 3 we introduce the notion of tube solution for system (1.1) and we prove our main result (Theorem 3.3). We end with Section 4, mentioning some directions for future work.

2. Preliminaries

A time scale $\mathbb{T}$ is defined to be any nonempty closed subset of $\mathbb{R}$. Then the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{t \in \mathbb{T} : s < t\}.$$

For $t \in \mathbb{T}$, we say that t is left-scattered (respectively right-scattered) if $\rho(t) < t$ (respectively $\sigma(t) > t$); that t is isolated if it is left-scattered and right-scattered. Similarly, if $t > \inf(\mathbb{T})$ and $\rho(t) = t$, then we say that t is left-dense; if $t < \sup(\mathbb{T})$

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and $\sigma(t) = t$, then we say that t is right-dense. Points that are simultaneously left- and right-dense are called dense. If $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, we set $\mathbb{T}_\kappa := \mathbb{T}$. The (backward) graininess $\nu : \mathbb{T}_\kappa \to [0, +\infty[$ is defined by $\nu(t) := t - \rho(t)$.

Definition 2.1 (See [1,9]). *For $f : \mathbb{T} \to \mathbb{R}^n$ and $t \in \mathbb{T}_\kappa$, the nabla derivative of f at t, denoted by $f^\nabla(t)$, is defined to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t such that*

$$\left\| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] \right\| \leq \epsilon |\rho(t) - s|$$

for all $s \in U$. If f is ∇-differentiable at t for every $t \in \mathbb{T}_\kappa$, then $f : \mathbb{T} \to \mathbb{R}^n$ is called the ∇-derivative of f on $\mathbb{T}_\kappa$.

Theorem 2.2 (See [5]). *Assume $f : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}_\kappa$. The following holds:*

(1) *If f is ∇-differentiable at t, then f is continuous at t.*

(2) *If f is continuous at the left-scattered point t, then f is ∇-differentiable at t with*
$$f^\nabla(t) = \frac{f(t) - f(\rho(t)}{\nu(t)}.$$

(3) *If t is left-dense, then f is nabla differentiable at t if and only if the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists in $\mathbb{R}^n$. In this case,
$$f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) *If f is nabla differentiable at t, then*
$$f^\rho(t) = f(t) - \nu(t) f^\nabla(t),$$
where $f^\rho(t) := f(\rho(t))$.

Theorem 2.3 (See [5]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then,*

(1) *$f + g$ is nabla differentiable at t and $(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t)$;*

(2) *αf is nabla differentiable at t for every $\alpha \in \mathbb{R}$ and $(\alpha f)^\nabla(t) = \alpha f^\nabla(t)$;*

(3) *fg is nabla differentiable at t and*
$$(fg)^\nabla(t) = f^\nabla(t) g(t) + f^\rho(t) g^\nabla(t) = f(t) g^\nabla(t) + f^\nabla(t) g^\rho(t);$$

(4) *if $g(t) g^\rho(t) \neq 0$, then $\frac{f}{g}$ is nabla differentiable at t and*
$$\left(\frac{f}{g}\right)^\nabla(t) = \frac{f^\nabla(t) g(t) - f(t) g^\nabla(t)}{g(t) g^\rho(t)};$$

(5) *if f and f^∇ are continuous, then*
$$\left(\int_a^t f(t, s) \nabla s\right)^\nabla = f(\rho(t), t) + \int_a^t f^\nabla(t, s) \nabla s.$$

Theorem 2.4. *Let W be an open set of $\mathbb{R}^n$ and $t \in \mathbb{T}$ be a left-dense point. If $g : \mathbb{T} \to \mathbb{R}^n$ is nabla-differentiable at t and $f : W \to \mathbb{R}$ is differentiable at $g(t) \in W$, then $f \circ g$ is nabla-differentiable at t with $(f \circ g)^\nabla(t) = \langle f'(g(t)), g^\nabla(t) \rangle$.*

Proof. Let $\epsilon > 0$. We need to show that there exists a neighborhood U of t such that $\left|f(g(t)) - f(g(s)) - \langle f'(g(t)), g^\nabla(t)\rangle(t-s)\right| \le \epsilon|t-s|$ for all $s \in U$. Let $k > 0$ be a constant and $\epsilon' = \frac{\epsilon}{k}$. By hypotheses, there exists a neighborhood U_1 of t where $\left\|g(t) - g(s) - g^\nabla(t)(t-s)\right\| \le \epsilon'|t-s|$ for all $s \in U_1$. In addition, there exists a neighborhood $V \subset W$ of $g(t)$ such that $|f(g(t)) - f(y) - \langle f'(g(t)), g(t)-y\rangle| \le \epsilon'|g(t)-y|$ for all $y \in V$. Since function g is ∇-differentiable at t, it is also continuous at this point, and there exists a neighborhood U_2 of t such that $g(s) \in V$ for all $s \in U_2$. Let $U := U_1 \cap U_2$. In this case U is a neighborhood of t and if $s \in U$, then

$$
\begin{aligned}
&|f(g(t)) - f(g(s)) - \langle f'(g(t)), g^\nabla(t)\rangle(t-s)| \\
&\quad \le |f(g(t)) - f(g(s)) - \langle f'(g(t)), g(t) - g(s)\rangle| \\
&\qquad + \left|\langle f'(g(t)), g(t)-g(s)\rangle - \langle f'(g(t)), g^\nabla(t)\rangle(t-s)\right| \\
&\quad \le \epsilon' \|g(t) - g(s)\| + \left|\langle f'(g(t)), g(t) - g(s) - g^\nabla(t)(t-s)\rangle\right| \\
&\quad \le \epsilon' \left(\epsilon'|t-s| + \|g^\nabla(t)(t-s)\|\right) + \|f'(g(t))\| \, \|g(t) - g(s) - g^\nabla(t)(t-s)\| \\
&\quad \le \epsilon'(1 + \|g^\nabla(t)\| + \|f'(g(t))\|)|t-s|.
\end{aligned}
$$

Put $k = 1 + \|g^\nabla(t)\| + \|f'(g(t))\|$ and the theorem is proved. □

Example 2.5. *Assume $x : \mathbb{T} \to \mathbb{R}^n$ is nabla differentiable at $t \in \mathbb{T}$. We know that $\|\cdot\| : \mathbb{R}^n \backslash \{0\} \to [0, +\infty[$ is differentiable if $t = \rho(t)$. It follows from Theorem 2.4 that*

$$\|x(t)\|^\nabla = \frac{\langle x(t), x^\nabla(t)\rangle}{\|x(t)\|}.$$

Definition 2.6. *A function $f : \mathbb{T} \to \mathbb{R}^n$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{ld}(\mathbb{T}, \mathbb{R}^n)$. The set of functions $f : \mathbb{T} \to \mathbb{R}^n$ that are nabla-differentiable and whose nabla-derivative is ld-continuous, is denoted by $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$. It is known that if f is ld-continuous, then there is a function F such that $F^\nabla = f$ [4]. In this case,*

$$\int_a^b f(t)\nabla t := F(b) - F(a).$$

Theorem 2.7 (See [2])**.** *Assume $a, b, c \in \mathbb{T}$. Then*

(1) $\int_a^b [f(t) + g(t)]\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t;$
(2) $\int_a^b kf(t)\nabla t = k\int_a^b f(t)\nabla t;$
(3) $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t;$
(4) $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t;$
(5) $\int_a^b f^\nabla(t)g(t)\nabla t = f(t)g(t)|_a^b - \int_a^b f^\rho(t)g^\nabla(t)\nabla t.$

Theorem 2.8 (See [2])**.** *The following inequalities hold:*

$$\left|\int_a^b f(t)g(t)\nabla t\right| \le \int_a^b |f(t)g(t)|\nabla t \le \left(\max_{\sigma(a)\le t\le b} |f(t)|\right)\int_a^b |g(t)|\nabla t.$$

Definition 2.9 (See [5])**.** *For $\epsilon > 0$, the (nabla) exponential function $\hat{e}_\epsilon(\cdot, t_0) : \mathbb{T} \to \mathbb{R}$ is defined as the unique solution to the initial value problem*

$$x^\nabla(t) = \epsilon x(t), \quad x(t_0) = 1.$$

More explicitly, the exponential function $\hat{e}_\epsilon(\cdot, t_0) : \mathbb{T} \to \mathbb{R}$ *is given by the formula*

$$\hat{e}_\epsilon(t, t_0) = exp\left(\int_{t_0}^{t} \hat{\xi}_\epsilon(\nu(s))\nabla s\right),$$

where for $h \geq 0$ *we define* $\hat{\xi}_\epsilon(h)$ *as*

$$\hat{\xi}_\epsilon(h) = \begin{cases} \epsilon & \text{if } h = 0, \\ -\frac{log(1-h\epsilon)}{h} & \text{otherwise.} \end{cases}$$

Proposition 2.10. *If* $g \in C^1(\mathbb{T}_\kappa, \mathbb{R}^n)$*, then function* $x : \mathbb{T} \to \mathbb{R}^n$ *defined by*

$$x(t) = \hat{e}_1(t, b)\left[\frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1}\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_1(\rho(s), b)}\nabla s - \int_{(t,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_1(\rho(s), b)}\nabla s\right]$$

is solution to the problem

$$\begin{aligned} x^\nabla(t) - x(t) = g(t), \quad t \in \mathbb{T}_\kappa, \\ x(a) = x(b). \end{aligned} \tag{2.1}$$

Proof. We check (2.1) for each pair (x_i, g_i), $i \in \{1, 2, \dots, n\}$, by direct calculation. To simplify notation, we omit the indices i and we write

$$k = \frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1}\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_1(\rho(s), b)}\nabla s.$$

From Theorem 2.3, we have that

$$\begin{aligned} x^\nabla(t) - x(t) &= \hat{e}_1(t, b)k - \hat{e}_1(t, b)\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_1(\rho(s), b)}\nabla s \\ &+ \hat{e}_1(\rho(t), b)\frac{g(t)}{\hat{e}_1(\rho(t), b)} - \hat{e}_1(t, b)k + \hat{e}_1(t, b)\int_{(a,b]\cap\mathbb{T}} \frac{g(s)}{\hat{e}_1(\rho(s), b)}\nabla s = g(t) \end{aligned}$$

for all $t \in \mathbb{T}_\kappa$. It is easy to verify that $x(a) = x(b)$. □

Lemma 2.11. *Let* $r \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ *be a function such that* $r^\nabla(t) < 0$ *for all* $t \in \{t \in \mathbb{T}_\kappa; r(t) > 0\}$*. If* $r(a) \geq r(b)$*, then* $r(t) \leq 0$ *for all* $t \in \mathbb{T}$*.*

Proof. Suppose that there exists a $t \in \mathbb{T}$ such that $r(t) > 0$. Then there exists a $t_0 \in \mathbb{T}$ such that $r(t_0) = \max_{t\in\mathbb{T}}(r(t) > 0)$. If $\rho(t_0) < t_0$, then

$$r^\nabla(t_0) = \frac{r(\rho(t_0)) - r(t_0}{\rho(t_0) - t_0} \geq 0,$$

which contradicts the hypothesis. If $t_0 > a$ and $t_0 = \rho(t_0)$, then there exists an interval $[t_1, t_0]$ such that $r(t) > 0$ for all $t \in [t_1, t_0]$. Thus

$$\int_{t_1}^{t_0} r^\nabla(s)\nabla s = r(t_0) - r(t_1) < 0,$$

which contradicts the maximality of $r(t_0)$. Finally, if $t_0 = a$, then by hypothesis $r(b) \geq r(a)$ gives $r(a) = r(b)$. Taking $t_0 = a$, one can check that $r(a) \leq 0$ by using previous steps of the proof. The lemma is proved. □

3. Main Result

In this section we prove existence of solution to problem (1.1). A solution of this problem is a function $x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ satisfying (1.1). Let us recall that $\mathbb{T}$ is bounded with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$. We introduce the notion of tube solution for problem (1.1) as follows.

Definition 3.1. *Let $(v, M) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \times C^1_{ld}(\mathbb{T}, [0, +\infty[)$. We say that (v, M) is a tube solution of (1.1) if*

(1) *$\langle x - v(t), f(t, x(t)) - v^{\nabla}(t)\rangle + M(t)\|x - v(t)\| \leq M(t)M^{\nabla}(t)$ for every $t \in \mathbb{T}_\kappa$ and for every $x \in \mathbb{R}^n$ such that $\|x - v(t)\| = M(t)$;*

(2) *$v^{\nabla}(t) = f(t, v(t))$ and $\|x - v(t)\| - M^{\nabla}(t) < 0$ for every $t \in \mathbb{T}_\kappa$ such that $M(t) = 0$;*

(3) *$\|v(a) - v(b)\| \leq M(a) - M(b)$.*

Let $\mathbf{T}(v, M) := \{x \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) : \|x(t) - v(t)\| \leq M(t)$ for every $t \in \mathbb{T}\}$. We consider the following problem:

$$\begin{aligned} x^{\nabla}(t) - x(t) &= f(t, \widehat{x}(t)) - \widehat{x}(t), \quad t \in \mathbb{T}_\kappa, \\ x(a) &= x(b), \end{aligned} \tag{3.1}$$

where

$$\hat{x}(t) = \begin{cases} \frac{M(t)}{\|x - v(t)\|}(x(t) - v(t)) + v(t) & \text{if } \|x - v(t)\| > M(t), \\ x(t) & \text{otherwise.} \end{cases}$$

Let us define the operator $\mathbf{T}_{\hat{p}} : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ by

$$\mathbf{T}_{\hat{p}}(x)(t) = \hat{e}_1(t, b)\left[\frac{\hat{e}_1(a, b)}{\hat{e}_1(a, b) - 1}\int_{(a,b]\cap\mathbb{T}} \frac{f(s, \hat{x}(s)) - \hat{x}(s)}{\hat{e}_1(\rho(s), b)}\nabla s - \int_{(t,b]\cap\mathbb{T}} \frac{f(s, \hat{x}(s)) - \hat{x}(s)}{\hat{e}_1(\rho(s), b)}\nabla s\right].$$

Proposition 3.2. *If $(v, M) \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n) \times C^1_{ld}(\mathbb{T}, [0, +\infty[)$ is a tube solution of (1.1), then $\mathbf{T}_{\hat{p}} : C(\mathbb{T}, \mathbb{R}^n) \to C(\mathbb{T}, \mathbb{R}^n)$ is compact.*

Proof. We first prove the continuity of the operator $\mathbf{T}_{\hat{p}}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T}, \mathbb{R}^n)$ converging to $x \in C(\mathbb{T}, \mathbb{R}^n)$. By Theorem 2.8,

$$\begin{aligned} &\|\mathbf{T}_{\hat{p}}(x_n)(t) - \mathbf{T}_{\hat{p}}(x)(t)\| \\ &\quad \leq (1 + c)\|\hat{e}_1(t, b)\| \left\|\int_{(a,b]\cap\mathbb{T}} \frac{f(s, \hat{x}_n(s)) - f(s, \hat{x}(s)) - (\hat{x}_n(s) - \hat{x}(s))}{\hat{e}_1(\rho(s), b)}\nabla s\right\| \\ &\quad \leq \frac{k(1 + c)}{M}\left(\int_{(a,b]\cap\mathbb{T}} \|f(s, \hat{x}_n(s)) - f(s, \hat{x}(s))\| + \|\hat{x}_n(s) - \hat{x}(s)\| \,\nabla s\right), \end{aligned}$$

where $k := \max_{t\in\mathbf{T}} |\hat{e}_1(t, b)|$, $M := \min_{t\in\mathbf{T}}(\hat{e}_1(t, b))$, and $c := \|\frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\|$. Since there is a constant $R > 0$ such that $\|\hat{x}\|_{C(\mathbb{T},\mathbb{R}^n)} < R$, there exists an index N such that $\|\hat{x}_n\|_{C(\mathbb{T},\mathbb{R}^n)} < R$ for all $n > N$. Thus f is uniformly continuous on $\mathbb{T}_\kappa \times B_R(0)$. Therefore, for $\epsilon > 0$ given, there is a $\delta > 0$ such that for all $x, y \in \mathbb{R}^n$, where

$$\|x - y\| < \delta < \frac{\epsilon M}{2k(1 + c)(b - a)},$$

one has

$$\|f(s,y)-f(s,x)\| < \frac{\epsilon M}{2k(1+c)(b-a)}.$$

By assumption, for all $s \in \mathbb{T}_\kappa$ it is possible to find an index $\hat{N} > N$ such that $\|\hat{x}_n - \hat{x}\|_{C(\mathbb{T},\mathbb{R}^n)} < \delta$ for $n > \hat{N}$. In this case,

$$\|\mathbf{T}_{\hat{p}}(x_n)(t) - \mathbf{T}_{\hat{p}}(x)(t)\| \leq \frac{2k(1+c)}{M}\int_{[a,b)\cap\mathbb{T}} \frac{\epsilon M}{2k(1+c)(b-a)}\nabla s \leq \epsilon.$$

This proves the continuity of $\mathbf{T}_{\hat{p}}$. We now show that the set $\mathbf{T}_{\hat{p}}(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $\mathbf{T}_{\hat{p}}(C(\mathbb{T},\mathbb{R}^n))$ for all $n \in \mathbb{N}$. It exists $x_n \in C(\mathbb{T},\mathbb{R}^n)$ such that $y_n = \mathbf{T}_{\hat{p}}(x_n)$. From Theorem 2.8 one has

$$\|\mathbf{T}_{\hat{p}}(x_n)(t)\| \leq \frac{k(1+c)}{M}\left(\int_{[a,b)\cap\mathbb{T}} \|f(s,\hat{x}_n(s))\|\nabla s + \int_{[a,b)\cap\mathbb{T}} \|\hat{x}_n(s)\|\nabla s\right).$$

By definition, there is an $R > 0$ such that $\|\hat{x}_n(s)\| \leq R$ for all $s \in \mathbb{T}$ and all $n \in \mathbb{N}$. Function f is compact on $\mathbb{T}_\kappa \times B_R(0)$ and we deduce the existence of a constant $A > 0$ such that $\|f(s,\hat{x}_n(s)\| \leq A$ for all $s \in \mathbb{T}_\kappa$ and all $n \in \mathbb{N}$. The sequence $\{y_n\}_n \in \mathbb{N}$ is uniformly bounded. Note also that

$$\|\mathbf{T}_{\hat{p}}(x_n)(t_2) - \mathbf{T}_{\hat{p}}(x_n)(t_1)\| \leq B\|\hat{e}_1(t_2,b) - \hat{e}_1(t_1,b)\|$$
$$+k\left\|\int_{(a,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right\| < B\|\hat{e}_1(t_2,b)-\hat{e}_1(t_1,b)\| + \frac{k(A+R)}{M}|t_2-t_1|$$

for $t_1, t_2 \in \mathbb{T}$, where B is a constant that can be chosen such that it is higher than

$$\sup_{n\in\mathbb{N}}\left\|\frac{\hat{e}_1(a,b)}{\hat{e}_1(a,b)-1}\int_{(a,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s + \int_{(t,b]\cap\mathbb{T}} \frac{f(s,\hat{x}_n(s)) - \hat{x}_n(s)}{\hat{e}_1(\rho(s),b)}\nabla s\right\|.$$

This proves that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is equicontinuous. It follows from the Arzelà–Ascoli theorem, adapted to our context, that $\mathbf{T}_{\hat{p}}(C(\mathbb{T},\mathbb{R}^n))$ is relatively compact. Hence $\mathbf{T}_{\hat{p}}$ is compact. □

Theorem 3.3. *If* $(v, M) \in C^1_{ld}(\mathbb{T},[0,+\infty[) \times C^1_{ld}(\mathbb{T},\mathbb{R}^n)$ *is a tube solution of* (1.1), *then problem* (1.1) *has a solution* $x \in C^1_{ld}(\mathbb{T},\mathbb{R}^n) \cap \mathbf{T}(v,M)$.

Proof. By Proposition 3.2, $\mathbf{T}_{\hat{p}}$ is compact. It has a fixed point by Schauder's fixed point theorem. Proposition 2.10 implies that this fixed point is a solution to problem (3.1). Then it suffices to show that for every solution x of (3.1) one has $x \in \mathbf{T}(v,M)$. Consider the set $A = \{t \in \mathbb{T}_\kappa : \|x(t) - v(t)\| > M(t)\}$. If $t \in A$ is left dense, then by virtue of Example 2.5 we have

$$(\| x(t) - v(t) \| - M(t))^\nabla = \frac{\langle x(t) - v(t), x^\nabla(t) - v^\nabla(t)\rangle}{\|x(t) - v(t)\|} - M^\nabla(t).$$

If $t \in A$ is left scattered, then

$$\begin{aligned}(\|x(t)-v(t)\| - M(t))^{\nabla} &= \|x(t)-v(t)\|^{\nabla} - M^{\nabla}(t)\\ &= \frac{\|x(t)-v(t)\|^2 - \|x(t)-v(t)\|\|x(\rho(t))-v(\rho(t))\|}{\nu(t)\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &\le \frac{\langle x(t)-v(t), x(t)-v(t)-x(\rho(t))+v(\rho(t))\rangle}{\nu(t)\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &= \frac{\langle x(t)-v(t), [f(t,\hat{x}(t)) - \hat{x}(t) + x(t)] - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t).\end{aligned}$$

We will show that if $t \in A$, then $(\|x(t)-v(t)\| - M(t))^{\nabla} < 0$. If $t \in A$ and $M(t) > 0$, then

$$\begin{aligned}(\|x(t)-v(t)\| - M(t))^{\nabla} &= \|x(t)-v(t)\|^{\nabla} - M^{\nabla}(t)\\ &= \frac{\|x(t)-v(t)\|^2 - \|x(t)-v(t)\|\|x(\rho(t))-v(\rho(t))\|}{\nu(t)\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &\le \frac{\langle x(t)-v(t), x(t)-v(t)-x(\rho(t))+v(\rho(t))\rangle}{\nu(t)\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &= \frac{\langle x(t)-v(t), x^{\nabla}(t)-v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &= \frac{\langle x(t)-v(t), f(t,\hat{x}(t))-v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} + \frac{\langle x(t)-v(t), -\hat{x}(t)+x(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &= \frac{\langle \hat{x}^{\nabla}(t)-v(t), f(t,\hat{x}(t))-v^{\nabla}(t)\rangle}{M(t)} - M(t) + \|x(t)-v(t)\| - M^{\nabla}(t)\\ &\le \frac{M(t)M^{\nabla}(t) - M(t)\|x(t)-v(t)\|}{M(t)} - M(t) + \|x(t)-v(t)\| - M^{\nabla}(t)\\ &= -M(t) < 0.\end{aligned}$$

In addition, if $M(t) = 0$, then

$$\begin{aligned}(\|x(t)-v(t)\| - M(t))^{\nabla} &= \frac{\langle x(t)-v(t), f(t,\hat{x}(t)) + [x(t)-\hat{x}(t)] - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} - M^{\nabla}(t)\\ &\le \frac{\langle x(t)-v(t), f(t,v(t)) - v^{\nabla}(t)\rangle}{\|x(t)-v(t)\|} + \|x(t)-v(t)\| - M^{\nabla}(t) < 0.\end{aligned}$$

If we set $r(t) := \|x(t)-v(t)\| - M(t)$, then $r^{\nabla}(t) < 0$ for every $t \in \{t \in \mathbb{T}_\kappa, r(t) \ge 0\}$. Moreover, since (v, M) is a tube solution of (1.1), one has

$$r(a) - r(b) \le \|v(a) - v(b)\| - (M(a) - M(b)) \le 0$$

and thus the hypotheses of Lemma 2.11 are satisfied, which proves the theorem. □

Example 3.4. *Consider the following boundary value problem on time scales:*

$$\begin{aligned} x^{\nabla}(t) = a_1\|x(t)\|^2 x(t) - a_2 x(t) + a_3\varphi(t), \quad t \in \mathbb{T}_\kappa,\\ x(a) = x(b), \end{aligned} \tag{3.2}$$

where a_1, a_2, $a_3 \ge 0$ are nonnegative real constants chosen such that $a_2 \ge a_1 + a_3 + 1$ and $\varphi : \mathbb{T}_\kappa \to \mathbb{R}^n$ is a continuous function satisfying $\|\varphi(t)\| = 1$ for every $t \in \mathbb{T}_\kappa$. It is easy to check that $(v, m) \equiv (0, 1)$ is a tube solution. By Theorem 3.3, problem (3.2) has a solution x such that $\|x(t)\| \le 1$ for every $t \in \mathbb{T}$.

4. Conclusion and Future Work

We proved existence of a solution to a nonlinear first-order nabla dynamic equation on time scales. For that the notion of tube solution is used, in the spirit of the works of Frigon and Gilbert [7–9]. Our results can be improved by using ∇-Caratheodory functions f on the right-hand side of equation (1.1), which are not necessarily continuous. For that one needs to define a proper Sobolev space and related nabla concepts. This is under investigation and will be addressed elsewhere.

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