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DYNAMICAL EQUIVALENCE OF QUASILINEAR DYNAMIC EQUATIONS ON TIME SCALES

ANDREJS REINFELDS, DZINTRA ŠTEINBERGA

Abstract. Using a Green type map and an integral functional equation technique, we find sufficient conditions under which a quasilinear equation is dynamically equivalent to its corresponding linear equation. This result extends Hartman–Grobman theorem for dynamic equations without ordinary dichotomy on time scales calculus.

1. Introduction

The linearization problem in the theory of ordinary differential equations was explored by D.M. Grobman [8], P. Hartman [9], K.J. Palmer [13] and other mathematicians [15, 16]. Variants of the Hartman–Grobman theorem to impulsive differential equations can be found in [1, 5, 17, 18, 20, 23]. Grobman–Hartman–Palmer linearization theorems were extended also to systems of differential equations in $\mathbb{R}^n$ with generalized exponential and ordinary dichotomy [4, 6, 7, 11, 12, 19]. It is of interest to understand what is the most general class of systems for which the linearization problem can be solved. Recently, L. Barreira and C. Valls gave a version of the Grobman–Hartman theorem for nonuniformly hyperbolic dynamics [2]. There are several papers considering Hartman–Grobman theorem on time scales [10, 14, 21, 22, 24, 25].

In our research we generalize these results, even for $\mathbb{R}^n$, by relaxing conditions on linear part A and strengthening conditions on nonlinear part f. We use Green type map and integral functional equation technique [16] to substantially simplify the proof. Moreover, our method to prove the dynamical equivalence used in this paper is completely different from previous papers. Furthermore, for more general point of view we consider differential equations in arbitrary Banach space. To highlight our improvement comparing to previous results, we use an example where the linear part of the differential equation even does not possess an ordinary dichotomy.

2. Main result and proof

Let $\mathbb{T}$ be a unbounded above and below time scale. For basic terminology and more details, see the monograph [3]. Let $\mathbf{X}$ be a Banach space and let $\mathfrak{L}(\mathbf{X})$ be the

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Banach space of linear bounded endomorphisms. Consider the following dynamic equations

$$x^{\Delta} = A(t)x + f_1(t,x) \tag{2.1}$$

and

$$x^{\Delta} = A(t)x + f_2(t,x) \tag{2.2}$$

where:

(i) the map $A\colon \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is rd-continuous and the map $A(t)\colon \mathfrak{L}(\mathbf{X}) \to \mathfrak{L}(\mathbf{X})$ is regressive;

(ii) the maps $f_j\colon \mathbb{T} \times \mathbf{X} \to \mathbf{X}$, $j = 1,2$ are rd-continuous with respect to t for fixed x, and, in addition they satisfy the Lipschitz conditions

$$|f_j(t,x) - f_j(t,x')| \le \varepsilon(t)|x - x'|, \quad j = 1,2,$$

and the estimate

$$\sup_x |f_1(t,x) - f_2(t,x)| \le N(t) < +\infty,$$

where $N\colon \mathbb{T} \to \mathbb{R}_+$ and $\varepsilon\colon \mathbb{T} \to \mathbb{R}_+$ are integrable scalar functions;

(iii) the maps $I + \mu(t)A(t) + \mu(t)f_j(t,\cdot)\colon \mathbf{X} \to \mathbf{X}$, $j = 1,2$ are invertible, where I is the identity map.

Here $\sigma\colon \mathbb{T} \to \mathbb{T}$ is the *forward jump operator* defined by equality

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$$

and $\mu\colon \mathbb{T} \to [0,+\infty)$ is the *graininess function* defined by

$$\mu(t) = \sigma(t) - t.$$

[For details, see [3]].

Note that condition (iii) implies continuability of solutions (2.1) and (2.2) in the negative direction. Furthermore, this together with the Lipshitz property with respect to x of the right hand side ensures that there is a unique solution for initial value problem defined on $\mathbb{T}$.

Let $x_j(\cdot,s,x)\colon \mathbb{T} \to \mathbf{X}$, $j = 1,2$, be the solutions of dynamic equations (2.1), (2.2) respectively satisfying the initial conditions $x_j(s) = x$. So $x_j(s,s,x) = x$ and, because of uniqueness of solutions, for $t,\tau,s \in \mathbb{T}$ we have

$$x_j(t,s,x) = x_j(t,\tau,x_j(\tau,s,x)).$$

For short, we will use the notation $x_j(t) = x_j(t,s,x)$.

Local results, which hold under more realistic assumptions on the nonlinearity, can be deduced using standard bump function technique.

Definition 2.1. *The dynamic equations* (2.1) *and* (2.2) *are* globally dynamical equivalent *if there exists a map $H\colon \mathbb{T} \times \mathbf{X} \to \mathbf{X}$ such that*

(i) *for each fixed $t \in \mathbb{T}$ the map $H(t,\cdot)\colon \mathbf{X} \to \mathbf{X}$ is a homeomorphism;*

(ii) $\sup_{t,x} |H(t,x) - x| < +\infty$*;*

(iii) *for all $t \in \mathbb{T}$*

$$H(t,x_1(t,s,x)) = x_2(t,s,H(s,x)).$$

Green type map can be represented in the form

$$G(t,s) = \begin{cases} e_A(t,s)P(s), & \text{if } t > s \\ e_A(t,s)(P(s) - I), & \text{if } t < s \end{cases}$$

where $e_A(t,s)$ is the *exponential map* of linear dynamic equation

$$x^{\Delta} = A(t)x \tag{2.3}$$

and $P(s) \in \mathfrak{L}(\mathbf{X})$ is rd-continuous with respect to $s \in \mathbb{T}$. Note that the linear dynamic equation (2.3) has infinitely many Green type maps. But if $\mathbb{T} = \mathbb{R}$ and the linear dynamic equation (2.3) has an exponential dichotomy then moreover there exists a unique Green type map which satisfies the inequality

$$|G(t,s)| \le K\exp(-\lambda|t-s|), \quad K \ge 1, \lambda > 0.$$

Let us note that

$$e_A(t,\tau)e_A(\tau,s) = e_A(t,s).$$

The solutions of (2.1) and (2.2) can be represented in the form

$$x_j(t,s,x) = e_A(t,s)x + \int_s^t e_A(t,\sigma(\tau))f_j(\tau, x_j(\tau,s,x))\,\Delta\tau,\ j = 1,2.$$

Theorem 2.1. *Suppose that the linear dynamic equation* (2.3) *has a rd-continuous Green type map $G(s,\tau) \in \mathfrak{L}(\mathbf{X})$ such that*

$$\sup_s \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|N(\tau)\,\Delta\tau < +\infty$$

$$\sup_s \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)\,\Delta\tau = q < 1.$$

Then the dynamic equations (2.1) *and* (2.2) *are globally dynamical equivalent.*

Proof. Let $\mathbf{C}_{rd}(\mathbb{T}\times\mathbf{X},\mathbf{X})$ be a set of maps $h\colon \mathbb{T}\times\mathbf{X}\to\mathbf{X}$ that are rd-continuous with respect to t for fixed x and continuous with respect to x. The set

$$\mathfrak{M} = \left\{ h \in \mathbf{BC}_{rd}(\mathbb{T}\times\mathbf{X},\mathbf{X}) \,\middle|\, \sup_{s,x}|h(s,x)| < +\infty \right\}$$

is Banach space with the supremum norm

$$\|h\| = \sup_{s,x}|h(s,x)|.$$

We will seek the map establishing the equivalence of (2.1) and (2.2) in the form $H_1(s,x) = x + h_1(s,x)$. We examine the following integro-functional equation

$$h_1(s,x) = \int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau. \tag{2.4}$$

Let us consider the map $h_1 \mapsto \mathfrak{T}h_1$, $h_1 \in \mathfrak{M}$ defined by the equality

$$\mathfrak{T}h_1(s,x) = \int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau.$$

Because of Lipschitz condition and conditions of the Theorem 2.1, also $\mathfrak{T}h_1 \in \mathfrak{M}$. Next we get

$$\begin{aligned} &|\mathfrak{T}h_1(s,x) - \mathfrak{T}h_1'(s,x)| \\ &= \left|\int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_2(\tau,x_1(\tau)+h_1'(\tau,x_1(\tau))))\,\Delta\tau\right| \\ &\le \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)|h_1(\tau,x_1(\tau)) - h_1'(\tau,x_1(\tau))|\,\Delta\tau \end{aligned}$$

$$\leq \sup_s \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)\,\Delta\tau \|h_1 - h_1'\| = q\|h_1 - h_1'\|,$$

where $q < 1$. Thus the map $\mathfrak{T}$ is a contraction and consequently the integro-functional equation (2.4) has a unique solution in $\mathfrak{M}$.

We have

$$\begin{aligned}
&h_1(t,x_1(t)) \\
&= \int_{-\infty}^{+\infty} G(t,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&= \int_{-\infty}^{t} G(t,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&\quad + \int_{t}^{+\infty} G(t,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&= \int_{-\infty}^{s} e_A(t,s)G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&\quad + \int_{s}^{t} e_A(t,\sigma(\tau))P(\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&\quad + \int_{s}^{+\infty} e_A(t,s)G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&\quad + \int_{t}^{s} e_A(t,\sigma(\tau))(P(\sigma(\tau)) - I)(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&= e_A(t,s)\int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&\quad + \int_{s}^{t} e_A(t,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau \\
&= e_A(t,s)h_1(s,x) \\
&\quad + \int_{s}^{t} e_A(t,\sigma(\tau))(f_2(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))) - f_1(\tau,x_1(\tau)))\,\Delta\tau.
\end{aligned}$$

Consequently, we have

$$x_1(t,s,x) + h_1(t,x_1(t,s,x)) = x_2(t,s,x+h_1(s,x)).$$

Changing the roles of f_1 and f_2, we prove in the same way the existence of $h_2 \in \mathfrak{M}$ satisfying integro-functional equation

$$h_2(s,x) = \int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f_1(\tau,x_2(\tau)+h_2(\tau,x_2(\tau))) - f_2(\tau,x_2(\tau)))\,\Delta\tau. \quad (2.5)$$

that satisfies the equality

$$x_2(t,s,x) + h_2(t,x_2(t,s,x)) = x_1(t,s,x+h_2(s,x)).$$

Designing $H_2(s,x) = x + h_2(s,x)$, we get

$$H_2(t,H_1(t,x_1(t,s,x))) = x_1(t,s,H_2(s,H_1(s,x))),$$
$$H_1(t,H_2(t,x_2(t,s,x))) = x_2(t,s,H_1(s,H_2(s,x))).$$

Taking into account uniqueness of maps $H_2(t,H_1(t,\cdot)) - I$ and $H_1(t,H_2(t,\cdot)) - I$ in $\mathfrak{M}$ we have $H_2(t,H_1(t,\cdot)) = I$ and $H_1(t,H_2(t,\cdot)) = I$ and therefore $H_1(t,\cdot)$ is a homeomorphism establishing a dynamical equivalence of the (2.1) and (2.2). $\square$

Let $f_2(t,x) = 0$. Then Theorem 2.1 implies that the dynamic equations (2.1) and (2.3) are globally dynamical equivalent.

3. Example

Consider a dynamic equation in $\mathbb{R}^2$

$$x^\Delta = A(t)x + f(t,x) \tag{3.1}$$

where

$$A(t) = \begin{pmatrix} \ln 2 & 0 \\ 0 & -\frac{2t}{1+t^2} \end{pmatrix},$$

$$|f(t,x) - f(t,x')| \le \varepsilon(t)|x - x'|,$$

$$\sup_x |f(t,x)| \le N(t) < +\infty.$$

Then the exponential map of the dynamic equation takes the form

$$e_A(t,s) = \begin{pmatrix} 2^{t-s} & 0 \\ 0 & \frac{1+s^2}{1+t^2} \end{pmatrix}.$$

Corresponding Green type map can be represented in the form

$$G(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1+s^2}{1+t^2} \end{pmatrix}, \text{ if } t > s$$

and

$$G(t,s) = \begin{pmatrix} -2^{t-s} & 0 \\ 0 & 0 \end{pmatrix}, \text{ if } t < s.$$

If

$$\int_{-\infty}^{+\infty} (1+\tau^2)N(\tau)\,\Delta\tau < +\infty$$

and

$$\int_{-\infty}^{+\infty} (1+\tau^2)\varepsilon(\tau)\,\Delta\tau = q < 1,$$

then in accordance with Theorem 2.1 the dynamic equation (3.1) is globally dynamical equivalent to the linear one

$$x^\Delta = A(t)x. \tag{3.2}$$

Let us note that $|G(t,s)| = \frac{1+s^2}{1+t^2}$ for $t > s$. It means that $|G(t,s)|$ is not globally bounded. The dynamic equation (3.2) does not even have the uniform ordinary dichotomy.

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Andrejs Reinfelds
Institute of Mathematics and Computer Science, Raiņa bulv. 29, Rīga, LV-1459, Latvia
University of Latvia, Department of Mathematics, Zeļļu 25, Rīga, LV-1002, Latvia
E-mail address: `reinf@latnet.lv`

Dzintra Šteinberga
University of Latvia, Department of Mathematics, Zeļļu 25, Rīga, LV-1002, Latvia
E-mail address: `dzintra.steinberga@gmail.com`