ON THE CONVEX COMBINATIONS OF SLANTED HALF-PLANE HARMONIC MAPPINGS

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Abstract. The main result in this paper shows that convex combinations of slanted half-plane harmonic mappings are convex.

1. Introduction and Preliminaries

Let \( H \) denote the class of all complex-valued harmonic mappings \( f \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) normalized by \( f(0) = f_z(0) - 1 = 0 \). Let \( S_H \) be the subclass of \( H \) consisting of univalent and sense-preserving functions. Such functions can be written in the form

\[
 f(z) = h(z) + g(z)
\]

where

\[
 h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n
\]

are analytic in \( \mathbb{D} \) and the Jacobian \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \). Equivalently, \( h' \neq 0 \) in the unit disk and the analytic complex dilatation \( \omega \) of \( f \) defined by \( \omega = g'/h' \) satisfies \( |\omega(z)| < 1, z \in \mathbb{D} \). The classical family \( S \) of analytic, univalent and normalized functions in \( \mathbb{D} \) is a subclass of \( S_H \) with \( g(z) \equiv 0 \). The family of all functions \( f \in S_H \) with the additional property that \( f_z(0) = 0 \) is denoted by \( S_H^0 \).

Further information about planar harmonic mappings can be found in the book [5].

A function \( f \in S_H \) is said to be convex if it maps the unit disk onto a convex domain. If the intersection of \( f(\mathbb{D}) \) with every line parallel to the one passing through the origin and \( e^{i\theta} (0 \leq \theta < \pi) \) is connected, we say that \( f \) is convex in the direction \( \theta \). It is obvious that any convex function is convex in every direction.

The shear construction introduced by Clunie and Sheil-Small in [2] provides a way of producing univalent harmonic functions in the unit disk to regions which are convex in one direction. More specifically, the following result is an easy consequence of [2 Thm. 5.3].

**Theorem A.** A harmonic function \( f = h + \overline{g} \), locally univalent in \( \mathbb{D} \), is a univalent mapping of onto a domain convex in the direction \( \theta \) if and only if \( h - e^{2i\theta}g \) is an analytic univalent mapping in \( \mathbb{D} \) onto a domain convex in the direction \( \theta \).

As usual, we use \( C_H^0 \) to denote the set of all sense-preserving convex harmonic mappings \( f = h + \overline{g} \) in \( \mathbb{D} \) satisfying \( h(0) = g(0) = g'(0) = 1 - h'(0) = 0 \). Indeed, in
what follows, all the functions considered here will satisfy these normalizations so that we won’t mention this again.

Of particular importance among the functions in \( C_0^H \) (because of their relationship with certain extremal problems in this class, for instance) is the family of functions that map the unit disk onto the half-plane \( H = \{ w \in \mathbb{C} : \Re(w) > -1/2 \} \). It is known (see [3 Cor. 3.2]) that any sense-preserving univalent function \( f = h + g \) with \( f(D) = H \) satisfies

\[
h(z) + g(z) = \frac{z}{1-z}, \quad z \in \mathbb{D}. \tag{1.1}
\]

We say that a normalized sense-preserving harmonic mapping \( f = h + \overline{g} \) belongs to \( \mathcal{F}_0 \) if and only if \([1]\) holds. Note that since the function \( h(z) = z/(1-z) \) is convex in the unit disk, by Theorem A we see that any function \( f \in \mathcal{F}_0 \) will be univalent in the unit disk and also convex in the vertical direction (this is, convex in the direction \( \pi/2 \)). In fact, a direct application of [1 Lemma 1], which proves that any sense-preserving univalent harmonic mapping \( f = h + \overline{g} \) satisfying \([1.1]\) is convex, shows that any function in \( f \in \mathcal{F}_0 \) maps the unit disk onto a convex domain.

If a function \( f \in C_0^H \) maps \( \mathbb{D} \) onto the half-plane \( \{ \Re(e^{i\gamma}w) > -1/2 \} \) for some \( 0 \leq \gamma < 2\pi \), then \( f \) is said to be a slanted half-plane mapping with parameter \( \gamma \). Obviously, any slanted half-plane mapping with parameter \( \gamma = 0 \) maps the unit disk onto \( H \). Moreover, \( f \) is a slanted half-plane mapping with parameter \( \gamma \) if and only if \( \hat{f}(z) = e^{i\gamma} f(e^{-i\gamma}z) \) satisfies \( \hat{f}(\mathbb{D}) = H \). Therefore, any such slanted half-plane mapping \( f = h + \overline{g} \) satisfies

\[
h(z) + e^{-2i\gamma}g(z) = \frac{z}{1-e^{i\gamma}z}, \quad z \in \mathbb{D}. \tag{1.2}
\]

We will use \( \mathcal{F}_\gamma \) to denote the family of sense-preserving harmonic mappings \( f = h + \overline{g} \) in the unit disk satisfying \([1.2]\). Again, using Theorem A, we see that any function \( f \in \mathcal{F}_\gamma \) is necessarily univalent in \( \mathbb{D} \) and also convex in certain direction that depends on \( \gamma \). Note that using the same (invertible) correspondence \( f \to \hat{f} \) mentioned above between slanted half-plane mappings with parameter \( \gamma \) and functions that map the unit disk onto \( H \) one can prove that \( f \in \mathcal{F}_\gamma \) if and only if \( \hat{f} \in \mathcal{F}_0 \). We refer the reader to [1, 4, 6, 7] for recent investigations in the subclass \( \mathcal{F}_\gamma \).

A common way to try to construct new functions with a given property is to take convex combinations of functions satisfying the desired condition. More concretely, given the functions \( f_j = h_j + \overline{g_j} \in S_0^H \) and the real numbers \( t_j \) with \( 0 \leq t_j \leq 1 \), \( j = 1, 2, \ldots, n \), we construct a new harmonic mapping

\[
F = \sum_{j=1}^{n} t_j f_j = \sum_{j=1}^{n} t_j h_j + \sum_{j=1}^{n} t_j \overline{g_j} = H + \overline{G}, \quad \sum_{j=1}^{n} t_j = 1, \tag{1.3}
\]

with dilatation \( \omega = G'/H' \).

Wang et al. [8] proved a sufficient condition for the convex combination of the form \( F = t f_1 + (1-t) f_2 \) with \( h_j + g_j = z/(1-z) \), \( j = 1, 2 \), to be univalent and convex in the horizontal direction.

The aim of this paper is to show that as long as the functions \( f_j \) belong to \( \mathcal{F}_\gamma \), the corresponding function \( F \) as in \([1.3]\) will map the unit disk onto a convex
domain, thus generalizing the main results obtained in [8]. The following lemma will be fundamental to prove this result.

**Lemma 1.1.** Let \( k_j \ (j = 1, 2) \) be analytic in \( \mathbb{D} \) with \( |k_j(z)| < 1 \) and \( k_j(0) = 0 \). Then, in the unit disk,

\[
\Re \left( \frac{1 - k_1k_2}{(1 + e^{-2i\beta}k_1)(1 + e^{2i\beta}k_2)} \right) > 0
\]

for all \( \beta \in \mathbb{R} \).

**Proof.** Note that the functions \( z \rightarrow 1/(1+z) \) and \( z \rightarrow z/(1-z) \) map the unit disk \( \mathbb{D} \) onto \( \{ w \in \mathbb{C} : \Re(w) > 1/2 \} \) and \( \{ w \in \mathbb{C} : \Re(w) > -1/2 \} \), respectively. Hence, for any given \( \lambda = 1 \) and any function \( k \) analytic in \( \mathbb{D} \) with \( |k| < 1 \), we have

\[
\Re \left( \frac{1}{1 + \lambda k} \right) > \frac{1}{2} \quad \text{and} \quad \Re \left( \frac{-\lambda k}{1 + \lambda k} \right) > -\frac{1}{2}
\]

Since

\[
\frac{1 - k_1k_2}{(1 + e^{-2i\beta}k_1)(1 + e^{2i\beta}k_2)} = \frac{1}{1 + e^{-2i\beta}k_2} - \frac{e^{-2i\beta}k_1}{1 + e^{-2i\beta}k_1},
\]

the result easily follows. \( \Box \)

### 2. Main Result

We now state the main theorem.

**Theorem 2.1.** Any convex combination of functions in \( F_\gamma \) is convex. Therefore, any convex combination of slanted half-plane mappings with parameter \( \gamma \) is convex.

**Proof.** For \( j = 1, 2, \ldots, n \), let \( t_j \) be real numbers (with \( 0 \leq t_j \leq 1 \) and such that \( \sum_{j=1}^{n} t_j = 1 \)) and \( f_j = h_j + g_j \in F_\gamma \); this is,

\[
h_j + e^{-2i\gamma}g_j = \frac{z}{1 - e^{i\gamma}z}, \quad 0 \leq \gamma < 2\pi, \quad z \in \mathbb{D}.
\]

Then, the mappings \( \hat{f}_j = \hat{h}_j + \overline{g_j} \), defined by \( \hat{f}_j(z) = e^{i\gamma}f_j(e^{-i\gamma}z) \in F_0 \), hence

\[
\hat{h}_j(z) + \overline{g_j(z)} = \frac{z}{1 - z}.
\]

Moreover, the function

\[
\hat{F} = \sum_{j=1}^{n} t_j \hat{f}_j = \hat{H} + \overline{G}
\]

satisfies \( \hat{F}(z) = e^{i\gamma}F(e^{-i\gamma}z) \), where \( F \) is as in (1.3). Therefore, to show that \( F \) is convex, it suffices to check that \( \hat{F} \) is convex. Since

\[
\hat{H}(z) + \overline{G}(z) = \frac{z}{1 - z},
\]

we see that according to [1, Lemma 1], the result will follow once it is proved that \( \hat{F} \) is locally univalent.

To prove that \( \hat{F} \) is locally univalent, let us use \( \omega_j \) to denote the dilatation of \( \hat{f}_j \) and define

\[
\Phi = \left| \sum_{j=1}^{n} \frac{t_j}{1 + \omega_j} \right|^2 - \left| \sum_{j=1}^{n} \frac{t_j\omega_j}{1 + \omega_j} \right|^2.
\]
We find that
\[
\Phi = \left(\sum_{j=1}^{n} \frac{t_j}{1 + \omega_j}\right) \left(\sum_{j=1}^{n} \frac{t_j}{1 + \omega_j}\right) - \left(\sum_{j=1}^{n} t_j\omega_j\right) \left(\sum_{j=1}^{n} \frac{t_j\overline{\omega_j}}{1 + \overline{\omega_j}}\right)
\]
\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t_j t_k}{(1 + \omega_j)(1 + \overline{\omega_k})} - \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t_j t_k \omega_j \overline{\omega_k}}{(1 + \omega_j)(1 + \overline{\omega_k})}
\]
\[
= \frac{2 \sum_{j=1}^{n} \sum_{k<j} \Re \left( \frac{t_j t_k (1 - \omega_j \overline{\omega_k})}{(1 + \omega_j)(1 + \overline{\omega_k})} \right) + \sum_{j=1}^{n} \frac{t_j^2 (1 - |\omega_j|^2)}{1 + |\omega_j|^2}}{\sum_{j=1}^{n} \sum_{k<j} \Re \left( \frac{t_j t_k (1 - \omega_j \overline{\omega_k})}{(1 + \omega_j)(1 + \overline{\omega_k})} \right) + \sum_{j=1}^{n} \frac{t_j^2 (1 - |\omega_j|^2)}{1 + |\omega_j|^2}}
\]

Since \(\omega_j\) are analytic and \(|\omega_j| < 1\) \((j = 1, 2, \ldots, n)\), by Lemma 1.1 we see that \(\Phi > 0\) in the unit disk. Then, on the one hand,
\[
\left| \sum_{j=1}^{n} \frac{t_j}{1 + \omega_j} \right| > 0 \quad (2.2)
\]
and, on the other hand
\[
\frac{\sum_{j=1}^{n} \frac{t_j\omega_j}{1 + \omega_j}}{\sum_{j=1}^{n} \frac{t_j}{1 + \omega_j}} < 1. \quad (2.3)
\]

Now, using (2.1), we have
\[
\hat{h}'_j(z) = \frac{1}{(1 + \omega_j(z))(1 - z)^2},
\]
so that
\[
\hat{H}'(z) = \sum_{j=1}^{n} \frac{t_j}{(1 + \omega_j(z))(1 - z)^2}
\]
and by (2.2) we obtain that \(\hat{H}' \neq 0\) in the unit disk. Moreover, the modulus of the dilatation \(\omega\) of \(\hat{F}\) equals
\[
|\omega| = \left| \frac{\sum_{j=1}^{n} t_j \partial_j f_j}{\sum_{j=1}^{n} t_j h'_j} \right| = \left| \frac{\sum_{j=1}^{n} t_j \omega_j \hat{h}'_j}{\sum_{j=1}^{n} t_j h'_j} \right| = \left| \frac{\sum_{j=1}^{n} t_j \omega_j}{\sum_{j=1}^{n} t_j} \right|.
\]

Hence, by (2.3), we get that \(|\omega(z)| < 1\) for all \(z \in \mathbb{D}\). This shows that \(\hat{F}\) is locally univalent in \(\mathbb{D}\) and ends the proof of the theorem. \(\Box\)

To finish this paper, we state the following corollary of Theorem A and the proof of Theorem 2.1.

**Corollary 2.2.** Let \(f_j = h_j + \overline{g_j} \in \mathcal{H}^s_0\) satisfy
\[
h_j - e^{-2i\gamma} g_j = \frac{z}{1 - e^{\gamma} z}, \quad 0 \leq \gamma < 2\pi, \quad j = 1, 2, \ldots, n.
\]

Then, the function \(F\) defined by (1.3) is convex in the direction \(-\gamma\).
Proof. The function $\hat{F} = \hat{H} + \hat{G}$ defined as in the proof of Theorem 2.1 satisfies $\hat{H} - \hat{G} = z/(1 - z)$. A similar procedure as the one employed above shows that $\hat{F}$ is locally univalent in the unit disk. Hence, by Theorem A, $\hat{F}$ is convex in the horizontal direction. This is equivalent to the fact that the function $F$ itself is convex in the direction $-\gamma$. □

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