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DIFFERENCE OPERATORS ON WEIGHTED SEQUENCE SPACES

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Abstract. The bounded, normal and compact composite difference operators are characterized in this paper.

1. Introduction and Preliminaries

Let N be the set of natural numbers and let $\lambda = \{\lambda_n\}$ be a sequence of strictly positive real numbers. Then the set function λ defined on $P(N)$, the power set of N by $\lambda(E) = \sum_{n \in E} \lambda_n$ for $E \in P(N)$ and $\lambda(\phi) = 0$ is a measure. For $1 \leq p < \infty$, the space $\ell^p_\lambda(N) = \{f | f : N \to C \text{ and } \sum_{n=1}^{\infty} |f_n|^p \lambda_n < \infty\}$ is a Banach space under the norm

$$||f||_p = (\sum_{n=1}^{\infty} |f_n|^p \lambda_n)^{\frac{1}{p}}$$

The space $\ell^p_\lambda(N)$ is known as a weighted sequence space. It is well known that $\ell^2_\lambda(N)$ is a Hilbert space under the inner product defined as

$$\langle f, g \rangle = \sum_{n=1}^{\infty} f_n \bar{g}_n \lambda_n.$$

The operator $D : \ell^2_\lambda(N) \to \ell^2_\lambda(N)$ defined by

$$(Df)(n) = f_n - f_{n-1}, \ f_0 = 0$$

is called a difference operator. The difference operators on sequence spaces are studied by Akhmedev and Baser [1], Altay and Basar [2]. These operators find wide applications in numerical methods and statistical methods. The symbol $B(\ell^2_\lambda(N))$ denotes the Banach algebra of all bounded linear operators from $\ell^2_\lambda(N)$ into itself. The set of eigen values of D is denoted by $\Pi_0(D)$.
In this paper our main purpose is to study important properties of difference operators on weighted sequence spaces.

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2. The Bounded Difference Operators

In case of $\ell^2(N)$, the difference operator is always bounded. We shall show that this is not true in case of difference operators on $\ell^2_\lambda(N)$. We obtain a necessary and sufficient condition for a difference operator to be bounded.

Theorem 2.1. *The operator $D : \ell^2_\lambda(N) \to \ell^2_\lambda(N)$ is bounded if and only if there exists $M > 0$ such that*

$$\sqrt{\frac{\lambda_{n+1}}{\lambda_n}} \leq M \text{ for every } n \in N$$

Proof. Assume first that the condition is true. Then for every $f \in \ell^2_\lambda(N)$.

$$\begin{aligned}
||Df||^2 &= (\sum_{n=1}^{\infty} |(Df)(n)|^2 \lambda_n)^{\frac{1}{2}} \\
&= (\sum_{n=1}^{\infty} |f_n - f_{n-1}|^2 \lambda_n)^{\frac{1}{2}}, \text{ where } f_0 = 0 \\
&\leq (\sum_{n=1}^{\infty} |f_n|^2 \lambda_n)^{\frac{1}{2}} + (\sum_{n=1}^{\infty} |f_{n-1}|^2 \lambda_n)^{\frac{1}{2}} \\
&= ||f||^2 + (\sum_{n=1}^{\infty} |f_n|^2 \lambda_{n+1})^{\frac{1}{2}} \\
&= ||f|| + (\sum_{n=1}^{\infty} |f_n|^2 . \frac{\lambda_{n+1}}{\lambda_n} \lambda_n)^{\frac{1}{2}} \\
&\leq ||f|| + M||f||, \quad \text{where } M = \sqrt{\sup_{1 \leq n < \infty} \frac{\lambda_{n+1}}{\lambda_n}} \\
&= (1 + M)||f||
\end{aligned}$$

This proves that D is a bounded operator.
Conversely suppose that D is bounded operator. Then there exists $M > 0$ such that

$$||De_n||^2 \leq M^2 ||e_n||^2 \; \forall n \in N, \text{ where } e_n(m) = \delta_{nm}, \text{ the Kronecker delta}$$

or

$$||e_n - e_{n+1}||^2 \leq M^2 \lambda_n$$

or

$$\lambda_n + \lambda_{n+1} \leq M^2 \lambda_n$$

Hence $\sqrt{\frac{\lambda_{n+1}}{\lambda_n}} \leq M \; \forall n \in N$. □

Example: Let $\lambda_n = n$ if n is odd number and $\lambda_n = 1$, if n is even number
Then

$$\begin{aligned}||D\frac{e_{2m}}{\sqrt{\lambda_{2m}}}||^2 &= \sum_{n=1}^{\infty}|e_{2m}(n)-e_{2m}(n-1)|^2\frac{\lambda_n}{\lambda_{2m}}\\ &= (\lambda_{2m}+\lambda_{2m+1})/\lambda_{2m}\\ &= 1+2m+1\\ &\to \infty \text{ as } m\to\infty\\ \text{and } \ ||\frac{e_{2m}}{\sqrt{\lambda_{2m}}}|| &= 1\end{aligned}$$

Therefore D is an unbounded operator. **Example:** Let $\lambda_n = \frac{1}{n}$ for every $n \in N$. Then

$$\frac{\lambda_{n+1}}{\lambda_n} = \frac{n}{n+1} \leq 1$$

Hence D is a bounded operator on $\ell^2_\lambda(N)$.
In the following theorem we obtain the adjoint of the difference operator. For $f \in \ell^2_\lambda(N)$, define $(Af)(n) = \frac{f_n\lambda_n - f_{n+1}\lambda_{n+1}}{\lambda_n}$ for every $n \in N$. We show that A is the adjoint of D.

Theorem 2.2. *Let $D \in B(\ell^2_\lambda(N))$. Then $D^* = A$.*

Proof. For $f, g \in \ell^2_\lambda(N)$, consider

$$\begin{aligned}\langle f, Dg\rangle &= \sum f(n)\overline{(Dg)}(n)\lambda_n\\ &= \sum_{n=1}^{\infty} f(n)\overline{[g(n)-g(n-1)]}\lambda_n\\ &= \sum_{n=1}^{\infty} f(n)\overline{g(n)}\lambda_n - \sum_{n=1}^{\infty} f(n)\overline{g(n-1)}\lambda_n\\ &= \sum_{n=1}^{\infty} f(n)\overline{g(n)}\lambda_n - \sum_{n=1}^{\infty} f(n+1)\overline{g(n)}\lambda_{n+1}\\ &= \sum_{n=1}^{\infty}[f(n)\lambda_n - f(n+1)\lambda_{n+1}]\overline{g(n)}\\ &= \sum_{n=1}^{\infty}\frac{f(n)\lambda_n - f(n+1)\lambda_{n+1}}{\lambda_n}\overline{g(n)}\lambda_n\\ &= \langle Af, g\rangle\end{aligned}$$

This shows that $D^* = A$.
It is easy to see that the difference operator on $\ell^2(N)$ is not normal. But if we replace N by Z, the set of integers, then the difference operator can be normal. In the following theoem a necessary and sufficient condition is obtained for difference operator on $\ell^2_\lambda(Z)$ to be normal. □

Theorem 2.3. *Let $D \in B(\ell^2_\lambda(Z))$. Then D is normal if and only if*

$$\frac{\lambda_{n+1}}{\lambda_n} = c \ \forall \ n \in Z,$$

where c is a constant.

Proof. Assume first that D is normal. A simple computation shows that

$$D^*De_n = (1+\frac{\lambda_{n+1}}{\lambda_n})e_n - (\frac{\lambda_n}{\lambda_{n-1}}e_{n-1} + e_{n+1})$$

and

$$DD^*e_n = (1+\frac{\lambda_{n-1}}{\lambda_n})e_n - (\frac{\lambda_n e_{n-1}}{\lambda_{n-1}} - e_{n+1}$$

From these two equations we obtain that

$$\frac{\lambda_n}{\lambda_{n-1}} = \frac{\lambda_{n+1}}{\lambda_n} \text{ for every } n \in N.$$

Hence $\frac{\lambda_{n+1}}{\lambda_n} = c$ for some constant c, for every $n \in Z$.
The converse is easy to prove by reversing the arguments. □

3. Spectrum and Numerical Range of Difference operator

In this section we show that point spectrum of difference operator on weighted sequence spaces is empty set. The numerical range of an operator A is defined by $W(A) = \{\langle Af, f\rangle : ||f|| = 1\}$

Theorem 3.1. *Let $D \in B(\ell^2_\lambda(N))$. Then*

$$W(D) \subseteq \{\alpha \in C : |\alpha - 1| < \sup_i \sqrt{\frac{\lambda_{i+1}}{\lambda_i}}\}$$

Proof. Suppose $\alpha \in W(D)$. Then

$$\begin{aligned} \alpha &= \langle Df, f\rangle \text{ for some unit vector } f \\ &= \sum_{i=1}^{\infty}(f_i - f_{i-1})\overline{f_i\lambda_i} \\ &= \sum_{i=1}^{\infty} f_i\bar{f_i}\lambda_i - \sum_{i=1}^{\infty} f_{i-1}\bar{f_i}\lambda_i \\ &= 1 - \sum_{i=1}^{\infty} f_{i-1}\bar{f_i}\lambda_i \end{aligned}$$

Therefore,

$$\begin{aligned} |1-\alpha| &\le \sum_{i=1}^{\infty} |f_{i-1}|\ |f_i|\lambda_i \\ &\le (\sum_{i=1}^{\infty} |f_{i-1}|^2\lambda_i)^{1/2}(\sum_{i=1}^{\infty} |f_i|^2\lambda_i)^{1/2} \\ &= (\sum_{i=1}^{\infty} |f_i|^2\lambda_{i+1})^{1/2} \\ &\le \sup_i \sqrt{\frac{\lambda_{i+1}}{\lambda_i}}(\sum_{i=1}^{\infty} |f_i|^2\lambda_i)^{1/2} \\ &\le \sup_i \sqrt{\frac{\lambda_{i+1}}{\lambda_i}} \end{aligned}$$

Hence $W(D) \subseteq \{\alpha \in C : |\alpha - 1| < \sup_{1\leq i<\infty} \sqrt{\frac{\lambda_{i+1}}{\lambda_i}}\}$. □

Theorem 3.2. *Let* $D \in B(\ell^2_\lambda(N))$. *Then* $\Pi_0(D) = \phi$ *and* $\Pi_0(D^*) \subset \{\alpha \in C : |\alpha - 1| < 1\}$. *Equality holds if* $\frac{\lambda_n}{\lambda_{n+1}} = 1 \ \forall \ n \in N$.

Proof. We first prove that $\Pi_0(D) = \phi$. Let $\lambda \in \Pi_0(D)$. Then there exists a non zero vector $f \in \ell^2_\lambda(N)$ such that $Df = \lambda f$. Since $f \neq 0$, let n_0 be the first natural number such that $f_{n_0} \neq 0$. Now $(Df)n_0) = \lambda f(n_0)$ implies that

$$f_{n_0} - f_{n_{0-1}} = \lambda f(n_0) \text{ or } f_{n_0} = \lambda f_{n_0}$$

as $f_{n_{0-1}} = 0$. Hence $\lambda = 1$.
Further $(Df)(n_0 + 1) = \lambda f_{n_{0+1}}$ implies that

$$f_{n_{0+1}} - f_{n_0} = f_{n_{0+1}}$$

or $f_{n_0} = 0$ which is a contradiction. Hence $\Pi_0(D) = \phi$.
Next we show that

$$\Pi_0(D^*) \subset \{\alpha \in C : |\alpha - 1| < 1\}$$

Let $\alpha \in \Pi_0(D^*)$. Then for some non-zero vector $f \in \ell^2_\lambda(N)$, we have

$$D^* f = \alpha f$$

Without loss of generality we can assume that $f_1 \neq 0$. From the equation

$$(D^* f)(1) = \alpha f(1)$$

or

$$\frac{f_1\lambda_1 - f_2\lambda_2}{\lambda_1} = \alpha f_1$$

we get

$$f_2 = (1 - \alpha) f_1 \frac{\lambda_1}{\lambda_2}$$

Similarly $(D^* f)(2) = \alpha f(2)$ implies that

$$f_3 = (1 - \alpha)^2 f_1 \frac{\lambda_1}{\lambda_3}$$

and so on

$$f_{n+1} = (1 - \alpha)^n f_1 \frac{\lambda_1}{\lambda_{n+1}}.$$

Since $f \in \ell^2_\lambda(N)$, so

$$||f||^2 \leq \sum_{n=1}^{\infty} |(1 - \alpha)|^{2n} |f_1|^2 \frac{\lambda_1^2}{\lambda_{n+1}^2} \lambda_{n+1} < \infty$$

This is possible only if $|1 - \alpha| < 1$ because $\frac{\lambda_n}{\lambda_{n+1}} = 1 \forall n \in N$
Hence $\Pi_0(D^*) \subset \{\alpha \in C : |1 - \alpha| < 1\}$ (1)
For the reverse inclusion, suppose $\frac{\lambda_n}{\lambda_{n+1}} = 1 \ \forall \ n \in N$
If $|1 - \alpha| < 1$, taking

$$f_n = \frac{(1 - \alpha)^{n-1}\alpha}{\lambda_n},$$

so that $f \in \ell^2_\lambda(N)$
and

$$\begin{aligned}
\lambda_{n+1} f_{n+1} &= (1-\alpha)^n \alpha \\
\lambda_{n+1} f_{n+1} &= (1-\alpha)^n \alpha \\
&= (1-\alpha)^{n-1}(1-\alpha)\alpha \\
&= \lambda_n f_n (1-\alpha) \\
\text{or} \quad \frac{\lambda_{n+1} - \lambda_{n+1} f_{n+1}}{\lambda_n} &= \alpha f_n
\end{aligned}$$

i.e. $(D^* f)(n) = \alpha f(n)\ \forall\ n \in N$ (2)
This proves that $\alpha \in \Pi_0(D^*)$. Combining (1) and (2), we get the equality. This completes the proof. □

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