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WEIGHTED COMPOSITION OPERATORS FROM ANALYTIC MORREY SPACES INTO BLOCH-TYPE SPACES

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Abstract. In this paper, the boundedness and compactness of weighted composition operators from analytic Morrey spaces into Bloch-type spaces and little Bloch-type spaces are studied.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $\alpha \in (0, \infty)$. The Bloch-type space (or α-Bloch space), denoted by $\mathcal{B}^\alpha$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Under the above norm, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space. Let $\mathcal{B}^\alpha_0$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ such that $(1 - |z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. $\mathcal{B}^\alpha_0$ is called the little Bloch-type space (or little α-Bloch space).

For an arc $I \subset \partial\mathbb{D}$ and f belonging to the Hardy space $H^p(\mathbb{D})$, we define $f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}$, where $|I| = \frac{1}{2\pi} \int_I |d\zeta|$ is the normalized length of I. Let $0 < \lambda \le 1$. The Morrey space, denoted by $\mathcal{L}^{2,\lambda}(\mathbb{D})$, is the set of all f belonging to the Hardy space $H^2(\mathbb{D})$ such that (see [17, 19])

$$\sup_{I \subset \partial\mathbb{D}} \Big(\frac{1}{|I|^\lambda} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} \Big)^{1/2} < \infty.$$

Clearly, $\mathcal{L}^{2,1}(\mathbb{D}) = BMOA$. If $0 < \lambda < 1$, from Theorem 3.1 of [19] we see that $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty,$$

where $S(I)$ is the Carleson box based on I with

$$S(I) = \{z \in \mathbb{D} : 1 - |I| \le |z| < 1 \text{ and } \frac{z}{|z|} \in I\}.$$

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Moreover, the norm of functions $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ can be defined as follows

$$\|f\|_{\mathcal{L}^{2,\lambda}} = |f(0)| + \sup_{I \subset \partial\mathbb{D}} \Big(\frac{1}{|I|^{\lambda}} \int_{S(I)} |f'(z)|^2 (1-|z|^2) dA(z) \Big)^{1/2}.$$

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by $(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D})$. Let $\psi \in H(\mathbb{D})$. The weighted composition operator induced by φ and ψ is defined by

$$(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

We can regard this operator as a generalization of a composition operator C_φ and a multiplication operator M_ψ, where $(M_\psi f)(z) = \psi(z) f(z)$. We refer [4, 15, 20] for the theory of the composition operator on function spaces.

Composition operators and weighted composition operators between Bloch-type spaces and some other spaces in one, as well as, in several complex variables were studied, for example, in [1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 13, 14, 18, 21].

In this paper we study the weighted composition operator from the Morrey space $\mathcal{L}^{2,\lambda}$ into the Bloch-type space $\mathcal{B}^\alpha$ and the little Bloch space $\mathcal{B}^\alpha_0$. Some sufficient and necessary conditions for the boundedness and compactness of the weighted composition operator are given.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In this section, we give some auxiliary results which will be used in proving the main results of the paper. They are incorporated in the lemmas which follow.

Lemma 2.1. [10] *Let $\lambda \in (0,1)$. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, then*

$$|f(z)| \preceq \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{1-\lambda}{2}}}, \quad z \in \mathbb{D}.$$

Arguing as the proof of Lemma 2.1, we can get the following result.

Lemma 2.2. *Let $\lambda \in (0,1)$. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, then*

$$|f'(z)| \preceq \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{\frac{3-\lambda}{2}}}, \quad z \in \mathbb{D}.$$

Lemma 2.3. [14] *Let $\alpha \in (0,\infty)$. A closed set Ω in $\mathcal{B}^\alpha_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in \Omega} (1-|z|^2)^\alpha |f'(z)| = 0.$$

The following lemma can be proved as Lemma 4.2 of [5].

Lemma 2.4. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\psi \in H(\mathbb{D})$, $\alpha \in (0,\infty)$ and $\lambda \in (0,1)$. Then the following statements hold.*

(i)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0$$

if and only if $\psi \in \mathcal{B}_0^\alpha$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0.$$

(ii)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0$$

if and only if $\lim_{|z|\to 1}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)| = 0$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

The following criterion for compactness follows by Lemma 3.7 of [16].

Lemma 2.5. *Suppose that* φ *is an analytic self-map of* $\mathbb{D}$*,* $\psi \in H(\mathbb{D})$*,* $\alpha \in (0,\infty)$ *and* $\lambda \in (0,1)$*. The operator* $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ *is compact if and only if for any bounded sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathcal{L}^{2,\lambda}$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$*, we have* $\|\psi C_\varphi f_n\|_{\mathcal{B}^\alpha} \to 0$ *as* $n \to \infty$*.*

3. Main results and proofs

In this section, we state and prove our main results.

Theorem 3.1. *Suppose that* φ *is an analytic self-map of* $\mathbb{D}$*,* $\psi \in H(\mathbb{D})$*,* $\alpha \in (0,\infty)$ *and* $\lambda \in (0,1)$*. Then* $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \infty \text{ and } \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty. \tag{3.1}$$

Proof. Suppose that (3.1) holds. For arbitrary z in $\mathbb{D}$ and $f \in \mathcal{L}^{2,\lambda}$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
& (1-|z|^2)^\alpha |(\psi C_\varphi f)'(z)| \\
\le\ & (1-|z|^2)^\alpha |\psi'(z)||f(\varphi(z))| + (1-|z|^2)^\alpha |f'(\varphi(z))||\psi(z)\varphi'(z)| \\
\preceq\ & (1-|z|^2)^\alpha |\psi'(z)| \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} + (1-|z|^2)^\alpha |\psi(z)\varphi'(z)| \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} \\
=\ & \Big(\frac{(1-|z|^2)^\alpha |\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} + \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} \Big) \|f\|_{\mathcal{L}^{2,\lambda}}.
\end{aligned} \tag{3.2}$$

Taking the supremum in (3.2) over $\mathbb{D}$ and then using (3.1) we get that $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is bounded.

Conversely, assume that $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is bounded. Taking the functions $f(z) = z$ and $f(z) = 1$, it follows that $\psi \in \mathcal{B}^\alpha$ and

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\psi(z)\varphi'(z) + \psi'(z)\varphi(z)| < \infty.$$

Thus by the boundedness of the function $\varphi(z)$, we obtain

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\psi(z)\varphi'(z)| < \infty. \tag{3.3}$$

For a fixed $a \in \mathbb{D}$, take

$$f_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}}. \tag{3.4}$$

Then $f_a \in \mathcal{L}^{2,\lambda}$ and $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{L}^{2,\lambda}} \preceq 1$(see [10]). Hence, we have

$$\begin{aligned} \|\psi C_\varphi\| &\succeq \|f_{\varphi(w)}\|_{\mathcal{L}^{2,\lambda}}\|\psi C_\varphi\| \geq \|\psi C_\varphi f_{\varphi(w)}\|_{\mathcal{B}^\alpha} \\ &\geq \Big|\frac{3-\lambda}{2}\frac{(1-|w|^2)|\psi(w)\overline{\varphi(w)}\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}} - \frac{(1-|w|^2)|\psi'(w)|}{(1-|\varphi(w)|^2)^{\frac{1-\lambda}{2}}}\Big| \end{aligned} \tag{3.5}$$

for every $w \in \mathbb{D}$, from which it follows that

$$\frac{(1-|w|^2)^\alpha|\psi'(w)|}{(1-|\varphi(w)|^2)^{\frac{1-\lambda}{2}}} \preceq \|\psi C_\varphi\| + \frac{3-\lambda}{2}\frac{(1-|w|^2)^\alpha|\psi(w)\overline{\varphi(w)}\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}}. \tag{3.6}$$

Further, for $a \in \mathbb{D}$, take

$$g_a(z) = \frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\frac{5-\lambda}{2}}} - \frac{1-|a|^2}{(1-\bar{a}z)^{\frac{3-\lambda}{2}}}. \tag{3.7}$$

Then, arguing as in the proof of Lemma 4 of [10] we get $\sup_{a\in\mathbb{D}} \|g_a\|_{\mathcal{L}^{2,\lambda}} \preceq 1$, $g_{\varphi(a)}(\varphi(a)) = 0, g'_{\varphi(a)}(\varphi(a)) = \frac{\overline{\varphi(a)}}{(1-|\varphi(a)|^2)^{\frac{3-\lambda}{2}}}$. Thus,

$$\|\psi C_\varphi\| \succeq \|\psi C_\varphi g_{\varphi(w)}\|_{\mathcal{B}^\alpha} \geq \frac{(1-|w|^2)^\alpha|\psi(w)\overline{\varphi(w)}\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}},$$

for every $w \in \mathbb{D}$, i.e., we have

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\alpha|\psi(w)\overline{\varphi(w)}\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}} < \infty. \tag{3.8}$$

Taking the supremum in (3.6) over $w \in \mathbb{D}$ and using (3.8), the first inequality in (3.1) follows. For a fixed $\delta \in (0,1)$, by (3.8),

$$\sup_{|\varphi(w)|>\delta} \frac{(1-|w|^2)^\alpha|\psi(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}} < \infty. \tag{3.9}$$

In addition, by (3.3) we obtain

$$\begin{aligned} \sup_{|\varphi(w)|\leq\delta} \frac{(1-|w|^2)^\alpha|\psi(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{3-\lambda}{2}}} &\leq \sup_{|\varphi(w)|\leq\delta} \frac{(1-|w|^2)^\alpha|\psi(w)\varphi'(w)|}{(1-\delta^2)^{\frac{3-\lambda}{2}}} \\ &\leq \frac{\sup_{w\in\mathbb{D}}(1-|w|^2)^\alpha|\psi(w)\varphi'(w)|}{(1-\delta^2)^{\frac{3-\lambda}{2}}} \\ &< \infty. \end{aligned} \tag{3.10}$$

The second inequality in (3.1) follows from (3.9) and (3.10). This completes the proof.

Theorem 3.2. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\psi \in H(\mathbb{D})$, $\alpha \in (0,\infty)$, $\lambda \in (0,1)$ and that $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is bounded. Then $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0 \textit{ and } \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0. \tag{3.11}$$

Proof. Suppose $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$ (If such a sequence does not exist conditions in (3.11) are automatically satisfied). Using the notation in (3.4), for $n \in \mathbb{N}$, let $f_n = f_{\varphi(z_n)}$. Then, $\sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{L}^{2,\lambda}} \le C$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since ψC_φ is compact, we have $\|\psi C_\varphi f_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. Thus, by (3.5) applied to $w = z_n$, we obtain

$$\lim_{n\to\infty} \frac{\frac{3-\lambda}{2}(1-|z_n|^2)^\alpha|\psi(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} = \lim_{n\to\infty} \frac{(1-|z_n|^2)^\alpha|\psi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{1-\lambda}{2}}}, \tag{3.12}$$

if one of these two limits exists.

For a sequence $(z_n)_{n\in\mathbb{N}}$ such that $|\varphi(z_n)| \to 1$, using the notation in the proof of Theorem 3.1, let $g_n = g_{\varphi(z_n)}$. Then $(g_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{L}^{2,\lambda}$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$, $g_n(\varphi(z_n)) = 0$ and $g_n'(\varphi(z_n)) = \frac{\overline{\varphi(z_n)}}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}$. Then

$$\frac{(1-|z_n|^2)^\alpha|\psi(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\frac{3-\lambda}{2}}} \le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(\psi C_\varphi g_n)'(z)| \to 0$$

as $n \to \infty$, which implies that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

Therefore by (3.12), we have

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0.$$

Conversely, we assume that (11) holds. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{L}^{2,\lambda}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. By (3.11) we have that for any $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} < \varepsilon \quad \text{and} \quad \frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \varepsilon.$$

Let $\Omega = \{w \in \mathbb{D} : |w| \le \delta\}$. From $\psi \in \mathcal{B}^\alpha$ and $M = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)| < \infty$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(\psi C_\varphi f_n)'(z)| \\
\le\ &\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi'(z)f_n(\varphi(z))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)f_n'(\varphi(z))\varphi'(z)| \\
\le\ &\sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\alpha|\psi'(z)f_n(\varphi(z))| + \sup_{\delta\le|\varphi(z)|<1}(1-|z|^2)^\alpha|\psi'(z)f_n(\varphi(z))| \\
+\ &\sup_{|\varphi(z)|\le\delta}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)||f_n'(\varphi(z))| \\
&+ \sup_{\delta\le|\varphi(z)|<1}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)||f_n'(\varphi(z))|
\end{aligned}$$

$$\begin{aligned}
&\preceq \quad \|\psi\|_{\mathcal{B}^\alpha} \sup_{w\in\Omega}|f_n(w)| + \sup_{\delta\le|\varphi(z)|<1}\frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}}\|f_n\|_{\mathcal{L}^{2,\lambda}} \\
&\quad\quad +M\sup_{w\in\Omega}|f_n'(w)| + \sup_{\delta\le|\varphi(z)|<1}\frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}}\|f_n\|_{\mathcal{L}^{2,\lambda}} \\
&\preceq \quad \|\psi\|_{\mathcal{B}^\alpha} \sup_{w\in\Omega}|f_n(w)| + \varepsilon + M\sup_{w\in\Omega}|f_n'(w)| + \varepsilon.
\end{aligned}$$

Since Ω is a compact subset of $\mathbb{D}$, it follows that $\lim_{n\to\infty}\sup_{w\in\Omega}|f_n(w)| = 0$. By Cauchy's estimate, the sequence f_n' also converges on compact subsets of $\mathbb{D}$ to zero. Moreover $\lim_{n\to\infty}|\psi(0)f_n(\varphi(0))| = 0$. So

$$\limsup_{n\to\infty}\|\psi C_\varphi f_n\|_{\mathcal{B}^\alpha} = \limsup_{n\to\infty}(\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(\psi C_\varphi f_n)'(z)| + |\psi(0)f_n(\varphi(0))|) \preceq \varepsilon.$$

Since ε is an arbitrary positive number it follows that $\limsup_{n\to\infty}\|\psi C_\varphi f_n\|_{\mathcal{B}^\alpha}\to 0$. Therefore, by Lemma 2.5 $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}^\alpha$ is compact.

From Theorems 3.1 and 3.2, we deduce the following corollary.

Corollary 3.3. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\alpha\in(0,\infty)$ and $\lambda\in(0,1)$. Then the following statements hold.*

(i) $C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} < \infty.$$

(ii) If $C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}^\alpha$ is bounded, then $C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

Next we characterize the boundedness and compactness of the weighted composition operators $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}_0^\alpha$. Arguing as the proof of Theorem 4.4 of [5], we get the following result.

Proposition 3.4. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\psi\in H(\mathbb{D})$, $\alpha\in(0,\infty)$ and $\lambda\in(0,1)$. Then $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}_0^\alpha$ is bounded if and only if $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}^\alpha$ is bounded, $\psi\in\mathcal{B}_0^\alpha$ and $\lim_{|z|\to 1}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)| = 0$.*

Theorem 3.5. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\psi\in H(\mathbb{D})$, $\alpha\in(0,\infty)$ and $\lambda\in(0,1)$. Then $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} = 0 \quad \text{and} \quad \lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0. \tag{3.13}$$

Proof. First, assume that $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}_0^\alpha$ is compact. Taking $f(z)\equiv 1$ we obtain that $\psi\in\mathcal{B}_0^\alpha$. Taking $f(z) = z$, and using the boundedness of $\psi C_\varphi : \mathcal{L}^{2,\lambda}\to\mathcal{B}_0^\alpha$ it follows that

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)| = 0. \tag{3.14}$$

Hence, if $\|\varphi\|_\infty < 1$, from $\psi\in\mathcal{B}_0^\alpha$ and (3.14), we obtain that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-|\varphi(z)|^2)^{\frac{1-\lambda}{2}}} \le \lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\psi'(z)|}{(1-\|\varphi\|_\infty^2)^{\frac{1-\lambda}{2}}} = 0$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} \leq \lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-\|\varphi\|_\infty^2)^{\frac{3-\lambda}{2}}} = 0,$$

proving the result in this case.

Next assume that $\|\varphi\|_\infty = 1$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. Let f_n and $g_n(z)$ be as in the proof of Theorem 2.3. Since $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is compact, by Theorem 3.2, (3.11) holds. Lemma 2.3 now yields the desired result.

Conversely, assume (3.13) holds. Taking the supremum in (3.2) over all $f \in \mathcal{L}^{2,\lambda}$ such that $\|f\|_{\mathcal{L}^{2,\lambda}} \leq 1$, and letting $|z| \to 1$, we obtain that

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}\leq 1} (1-|z|^2)^\alpha |(\psi C_\varphi(f))'(z)| = 0.$$

By Lemma 2.2 it follows that the operator $\psi C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}_0^\alpha$ is compact.

From Theorems 3.4 and 3.5, we obtain the following corollary:

Corollary 3.6. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $\alpha \in (0,\infty)$ and $\lambda \in (0,1)$. Then, the following two statements hold.*

(i) $C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}_0^\alpha$ is bounded if and only if $C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}^\alpha$ is bounded and $\varphi \in \mathcal{B}_0^\alpha$.

(ii) $C_\varphi : \mathcal{L}^{2,\lambda} \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{3-\lambda}{2}}} = 0.$$

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