Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 5 Issue 4 (2014), Pages 12-17.

QUASIHOMOGENEOUS TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES OF THE UNIT BALL

WALEED AL-RAWASHDEH

Abstract. In this paper, we study some algebraic properties of Toeplitz operatos on the weighted Bergman space $A^2_\gamma(\mathbb{B}_n)$ with quasihomogeneous and radial symbols. Then we give an example that answers the conjecture in [8] negatively.

1. Introduction

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$. For $\gamma > -1$, the weighted Lebesgue measure dv_γ is defined, for $z \in \mathbb{B}_n$, as:

$$dv_\gamma(z) = c_\gamma(1 - |z|^2)^\gamma dv(z)$$

where dv is a volume measure on $\mathbb{B}_n$ and

$$c_\gamma = \frac{\Gamma(n + \gamma + 1)}{n!\Gamma(\gamma + 1)}$$

is a normalizing constant so that $v_\gamma(\mathbb{B}_n) = 1$. The weighted Bergman space $A^2_\gamma(\mathbb{B}_n)$ is the space of all holomorphic functions in $L^2_\gamma(\mathbb{B}_n)$, where L^2_γ denotes the standard Lebesgue space on $\mathbb{B}_n$ with respect to the measure v_γ. It is known that $A^2_\gamma(\mathbb{B}_n)$ is a closed linear subspace of L^2_γ. Recall that the orthogonal projection P_γ from $L^2_\gamma(\mathbb{B}_n)$ onto $A^2_\gamma(\mathbb{B}_n)$ is given by the integral operator

$$P_\gamma(f)(z) = \langle f, K_z\rangle = \int_{\mathbb{B}_n} f(w)\overline{K_z(w)}dv_\gamma(w),$$

where $K_z(w) = \dfrac{1}{(1 - \langle w, z\rangle)^{n+\gamma+1}}$ is Bergman reproducing kernel. It is well-known that this operator is bounded from $L^2_\gamma(\mathbb{B}_n)$ to $A^2_\gamma(\mathbb{B}_n)$ when $\gamma > -1$. Let M_ϕ denote the operator of multiplication by the function ϕ, that is $M_\phi f = \phi f$. The Toeplitz operator with the symbol $\phi \in L^\infty_\gamma(\mathbb{B}_n)$ is defined by $T_\phi = P_\gamma M_\phi$. It is clear that T_ϕ is bounded on $A^2_\gamma(\mathbb{B}_n)$ with $\|T_\phi\| \leq \|\phi\|_\infty$.

The problem when two Toeplitz operators commute, first considered by Brown and Halmos [3] in 1964, they gave a complete answer to this question for Toeplitz

2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 47L80, 32A36.
Key words and phrases. Toeplitz operators, weighted Bergman spaces, radial symbol, quasi-homogeneous symbol.

Submitted August 31, 2014. Published November 25, 2014.

operators on the Hardy space of the unit circle $\mathbb{T}$ in $\mathbb{C}$. In fact, they proved that T_ϕ and T_ψ, for ϕ and ψ in $L^\infty(\mathbb{T})$, commute if and only if

(1) both ϕ and ψ are analytic, or
(2) both ϕ and ψ are coanalytic, or
(3) there exist $\alpha, \beta \in \mathbb{C}$ not both zero such that $\alpha\phi + \beta\psi$ is constant on $\mathbb{T}$.

On the unweighted Bergman space $A^2(\mathbb{D})$ of the unit disk, many authors have studied this problem, see for example ([1], [2], [5], [4], [7], and [8]). On the unweighted Bergman space $A^2(\mathbb{B}_n)$ of the unit ball, Zheng [9] studied commuting Toeplitz operators with pluriharmonic symbols, and the authors of [10] and [6] studied the commuting of Toeplitz operators with radial and quasihomogeneous symbols. The authors of these papers used Mellin transform to study the commutativity of quasihomogeneous Toeplitz operators on the spaces $A^2(\mathbb{D})$ and $A^2(\mathbb{B}_n)$.

This paper is motivated by work on the unit disk of Louhichi and Zakariasy [7], and by work on the unit ball of Dong and Zhou [10]. In this paper, we consider the weighted Bergman space $A^2_\gamma(\mathbb{B}_n)$ of the unit ball. We characterize when two Toeplitz operators with symbols $\zeta^p\overline{\zeta}^s\phi$ and $\zeta^q\overline{\zeta}^u\psi$ commute, and when radial and quasihomogeneous Toeplitz operators commute. Then, we give an example that answers Louhichi and Rao's conjecture in [8] negatively. Our approach here is different from those in the papers mentioned above, there is no need for us to use Mellin transform.

2. Commutating Toeplitz Operators

For a multi-index $p = (p_1, p_2, ..., p_n) \in \mathbb{N}^n$ and $z = (z_1, z_2, ..., z_n) \in \mathbb{B}_n$, we write

$$|p| = p_1 + p_2 + \cdots + p_n,$$
$$p! = p_1!p_2! \cdots p_n!, \text{ and}$$
$$z^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}.$$

For two multi-indexes $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ in $\mathbb{N}^n$, the notation $p \perp q$ means that

$$p_1q_1 + p_2q_2 + \cdots + p_nq_n = 0,$$

$p \succeq q$ means that

$$p_i \geq q_i, \quad \text{for every } i = 1, 2, ..., n,$$

and we say $p \not\succeq p$ if $p_i < q_i$ for some i. Moreover, we define $p - q = (p_1 - q_1, p_2 - q_2, \cdots, p_n - q_n)$ and if $p \succeq q$, then $|p - q| = |p| - |q|$.

Let $\mathbb{S}_n$ denote the unit sphere in $\mathbb{C}^n$ and let σ denote the rotation-invariant probability measure on $\mathbb{S}_n$. It is well-known that the monomials z^α are orthogonal in $L^2(\mathbb{S}_n, d\sigma)$, and by (Lemma 1.11, [11]) we have

$$\int_{\mathbb{S}_n} |\zeta^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(|\alpha| + n - 1)!} \tag{2.1}$$

Integrating in polar coordinates (Lemma 1.11, [11]) it follows that the monomials z^α are orthogonal in $A^2_\gamma(\mathbb{B}_n)$ and have square norm equal to

$$\int_{\mathbb{B}_n} |z^\alpha|^2 dv_\gamma(z) = \frac{\Gamma(n+\gamma+1)\alpha!}{\Gamma(|\alpha|+\gamma+n+1)} \tag{2.2}$$

Let $\phi \in L^1(\mathbb{B}_n, dv_\gamma)$, we say ϕ is a radial function if $\phi(z)$ depends only on $|z|$, that is $\phi(z) = \phi(|z|)$ for $z \in \mathbb{B}_n$. The definition of quasihomogeneous function on the unit disk has been given in many papers, for example see [7] and [8], and recently it has been given on the unit ball in the papers [6] and [10]. Similarly, we next define the quasihomogeneous function on $\mathbb{B}_n$. Let $k \in \mathbb{Z}^n$ and $f \in L^1(\mathbb{B}_n, dv_\gamma)$. We say f is a quasihomogeneous function of degree K if and only if $f(r\zeta) = \zeta^k \phi(r)$, where ϕ is a radial function, $\zeta \in \mathbb{S}^n$, and $r \in [0,1)$. By separating positive and negative integers in k, k can be uniquely written as $k = p - s$, where $p, s \in \mathbb{N}^n$ with $p \perp s$. So in what follows, for any $k \in \mathbb{Z}^n$ we write $\zeta^k = \zeta^p \overline{\zeta}^s$ for $\zeta \in \mathbb{S}^n$.

A simple calculation gives the following two lemmas, they tell us how the radial and quasihomogeneous Toeplitz operators act on the monomials z^α for $z \in \mathbb{B}_n$ and $\alpha \in \mathbb{N}^n$.

Lemma 2.1. *Let $\gamma > -1$, let p and s be two multi-indexes. If ϕ is a bounded radial function on $\mathbb{B}_n$, then*

$$T_{\zeta^p\overline{\zeta}^s\phi}(z^\alpha) = \begin{cases} 0, & \text{if } \alpha + p \not\succeq s \\ \lambda_\alpha z^{\alpha+p-s}, & \text{if } \alpha + p \succeq s, \end{cases}$$

where

$$\lambda_\alpha = \frac{2(\alpha+p)!\Gamma(n+|\alpha|+|p|-|s|+\gamma+1)}{(\alpha+p-s)!(n-1+|\alpha|+|p|)!\Gamma(\gamma+1)} \int_0^1 r^{2n+2|\alpha|+|p|-|s|-1}\phi(r)(1-r^2)^\gamma dr.$$

Proof. For all multi-indexes α and β, we have

$$\begin{aligned} \langle T_{\zeta^p\overline{\zeta}^s\phi}(z^\alpha), z^\beta\rangle &= \int_{\mathbb{B}_n} \phi(z)\zeta^p\overline{\zeta}^s z^\alpha \overline{z}^\beta (1-|z|^2) dv_\gamma(z) \\ &= 2nc_\gamma \int_0^1 r^{2n+|\alpha|+|\beta|-1}\phi(r)(1-r^2)^\alpha \int_{\mathbb{S}_n} \zeta^{p+\alpha}\overline{\zeta}^{s+\beta} d\sigma(\zeta) dr \\ &= \begin{cases} 0, & \text{if } \alpha+p \not\succeq s \\ 2nc_\gamma \left(\int_0^1 r^{2n+|\alpha|+|\beta|-1}\phi(r)(1-r^2)^\alpha dr\right) \int_{\mathbb{S}_n} \zeta^{p+\alpha}\overline{\zeta}^{s+\beta} d\sigma(\zeta), & \text{if } \alpha+p \succeq s. \end{cases} \end{aligned}$$

If $\alpha + p \succeq s$, then by Equation (2.1)

$$\int_{\mathbb{S}_n} \zeta^{p+\alpha}\overline{\zeta}^{s+\beta} d\sigma(\zeta) = \begin{cases} 0, & \text{if } \beta \neq \alpha+p-s \\ \dfrac{(n-1)!(\alpha+p)!}{(|\alpha|+|p|+n-1)!}, & \text{if } \beta = \alpha+p-s. \end{cases}$$

For $\beta = \alpha + p - s$, we get by Equation (2.2)

$$\langle z^{\alpha+p-s}, z^\beta\rangle = \frac{\Gamma(n+\gamma+1)(\alpha+p-s)!}{\Gamma(|\alpha|+|p|-|s|+n+\gamma+1)}.$$

Hence, for $\beta = \alpha + p - s$ and $\alpha + p \succeq s$ we get

$$T_{\zeta^p\overline{\zeta}^s\phi}(z^\alpha)$$
$$= 2(n!)c_\gamma\frac{(\alpha+p)!\Gamma(|\alpha|+|p|-|s|+n+\gamma+1)}{(\alpha+p-s)!(|\alpha|+|p|+n-1)!\Gamma(n+\gamma+1)}\left(\int_0^1 r^{2n+2|\alpha|+|p|-|s|-1}\phi(r)(1-r^2)^\gamma dr\right)z^{\alpha+p-s}.$$

This completes the proof. □

The following lemma is a consequence of the previous Lemma 2.1, so we omit the proof's details. In this lemma if $\gamma = 0$, we get Lemma 4.3 in [10].

Lemma 2.2. *Let $\gamma > -1$, let p and s be two multi-indexes. If ϕ is bounded radial function on $\mathbb{B}_n$, then*

$$T_\phi(z^\alpha) = \frac{2(\alpha!)\Gamma(n+|\alpha|+\gamma+1)}{(\alpha!)(n-1+|\alpha|)!\Gamma(\gamma+1)}\left(\int_0^1 r^{2n+2|\alpha|-1}\phi(r)(1-r^2)^\gamma dr\right)z^\alpha,$$

$$T_{\zeta^p\phi}(z^\alpha) = \frac{2(\alpha+p)!\Gamma(n+|\alpha|+|p|+\gamma+1)}{(\alpha+p)!(n-1+|\alpha|+|p|)!\Gamma(\gamma+1)}\left(\int_0^1 r^{2n+2|\alpha|+|p|-1}\phi(r)(1-r^2)^\gamma dr\right)z^{\alpha+p},$$

and $T_{\overline{\zeta}^s\phi}(z^\alpha)$

$$= \begin{cases} 0, & \text{if } \alpha \not\succeq s \\ \dfrac{2(\alpha!)\Gamma(n+|\alpha|-|s|+\gamma+1)}{(\alpha-s)!(n-1+|\alpha|)!\Gamma(\gamma+1)}\left(\displaystyle\int_0^1 r^{2n+2|\alpha|+|p|-|s|-1}\phi(r)(1-r^2)^\gamma dr\right)z^{\alpha-s}, & \text{if } \alpha \succeq s. \end{cases}$$

The following theorem characterizes when a radial Toeplitz operator commutes with a quasihomogenous Toeplitz operator. This theorem is a generalization of the theorems (Theorem 4.4, [10]) and (Theorem 2.2, [6]) in the unweighted Bergman space of the unit ball and in the unweighted Pluriharmonic Bergman space, respectively, the authors of these papers used Mellin transform to prove their results. In this paper we can not use Mellin transform, in fact if $\gamma = 0$ we get the results in [10] and [6].

Theorem 2.3. *Let $\gamma > -1$ and let $p, s \in \mathbb{N}^n$ be multi-indexes such that $p \perp s$. If ψ and ϕ are non-identically zero bounded radial functions on $\mathbb{B}_n$, then*

$$T_\psi T_{\zeta^p\overline{\zeta}^s\phi} = T_{\zeta^p\overline{\zeta}^s\phi}T_\psi$$

if and only if $|p| = |s|$.

Proof. It follows from the previous lemmas that for $\alpha + p \succeq s$ we have

$$T_{\zeta^p\overline{\zeta}^p\phi}T_\psi(z^\alpha) = \frac{4\Gamma(n+|\alpha|+|p|-|s|+\gamma+1)\Gamma(n+|\alpha|+\gamma+1)(\alpha+p)!}{(\Gamma(\alpha+1))^2(n-1+|\alpha|+|p|)!(n-1+|\alpha|)!(\alpha+p-s)!}$$

$$\times\left(\int_0^1 \phi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2|p|-2|s|+2n-1}\right)dr\right)$$

$$\times\left(\int_0^1 \psi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2n-1}\right)dr\right)z^{\alpha+p-s},$$

and

$$T_{\zeta^p\overline{\zeta}^p\phi}T_\psi(z^\alpha) = \frac{4\left(\Gamma(n+|\alpha|+|p|-|s|+\gamma+1)\right)^2(\alpha+p)!}{\left(\Gamma(\alpha+1)\right)^2(n-1+|\alpha|+|p|-|s|)!(n-1+|\alpha|+|p|)!(\alpha+p-s)!}$$

$$\times\left(\int_0^1 \psi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2|p|-2|s|+2n-1}\right)dr\right)$$

$$\times\left(\int_0^1 \phi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2|p|-2|s|+2n-1}\right)dr\right)z^{\alpha+p-s}$$

It is clear that T_ψ and $T_{\zeta^p\overline{\zeta}^s\phi}$ commute if and only if $|p|=|s|$. □

The following theorem is the main result of this paper, which characterizes when two different quasihomogenous Toeplitz operators commute in the weighted Bergman space of the unit ball.

Theorem 2.4. *Let $p,s,q,$ and $u\in\mathbb{N}^n$ be multi-indexes with $|p|=|s|$, $|q|=|u|$, $p\perp s$ and $q\perp u$. Let ψ and ϕ be two bounded radial functions on $\mathbb{B}_n$. Then $T_{\zeta^p\overline{\zeta}^s\phi}$ and $T_{\zeta^q\overline{\zeta}^u\psi}$ commute in $A^2_\gamma(\mathbb{B}_n)$ if and only if one of the following conditions is satisfied*

(1) $p=s=0$,
(2) $q=u=0$,
(3) $p=q=0$,
(4) $s=u=0$.

Proof. For $\alpha+p\succeq s$ and $\alpha+p+q\succeq s+u$, we have

$$T_{\zeta^q\overline{\zeta}^u\psi}T_{\zeta^p\overline{\zeta}^s\phi}(z^\alpha) = \frac{4\left(\Gamma(n+|\alpha|+\gamma+1)\right)^2(\alpha+p-s+q)!(\alpha+p)!}{\left(\Gamma(\gamma+1)\right)^2(n-1+|\alpha|+|q|)!(n-1+|\alpha|+|p|)!(\alpha+p-s+q-u)!(\alpha+p-s)!}$$

$$\times\left(\int_0^1 \psi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2n-1}\right)dr\right)$$

$$\times\left(\int_0^1 \phi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2n-1}\right)dr\right)z^{\alpha+p-s+q-u}$$

For $\alpha+q\succeq u$ and $\alpha+p+q\succeq s+u$, we have

$$T_{\zeta^p\overline{\zeta}^s\phi}T_{\zeta^q\overline{\zeta}^u\psi}(z^\alpha) = \frac{4\left(\Gamma(n+|\alpha|+\gamma+1)\right)^2(\alpha+q-u+p)!(\alpha+q)!}{\left(\Gamma(\gamma+1)\right)^2(n-1+|\alpha|+|q|)!(n-1+|\alpha|+|q|)!(\alpha+q-u+p-s)!(\alpha+q-u)!}$$

$$\times\left(\int_0^1 \phi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2n-1}\right)dr\right)$$

$$\times\left(\int_0^1 \psi(r)\left(1-r^2\right)^\gamma\left(r^{2|\alpha|+2n-1}\right)dr\right)z^{\alpha+q-u+p-s}$$

Hence, $T_{\zeta^p\overline{\zeta}^s\phi}$ and $T_{\zeta^q\overline{\zeta}^u\psi}$ commute if and only if

$$\frac{(\alpha+p-s+q)!(\alpha+p)!}{(\alpha+p-s)!} = \frac{(\alpha+q-u+p)!(\alpha+q)!}{(\alpha+q-u)!}$$

It is clear that last equality holds if and only if one of the following is satisfied $p = s = 0$, $q = u = 0$, $p = q = 0$, or $s = u = 0$. This completes the proof. □

Remark. *Louhichi and Rao in* [8] *conjectured that* "if two Toeplitz operators commute with a third one, none of them being the identity, then they commute with each other"*. In what follows we give an example that shows this conjecture is false on the space $A^2_\gamma(\mathbb{B}_n)$ with $\gamma > -1$ and $n > 1$. Let ϕ, ψ and φ be bounded radial functions on $\mathbb{B}_n$. Then using Theorem 2.3 we get T_ϕ and $T_{\zeta^{(1,0)}\overline{\zeta}^{(0,1)}\psi}$ commute, we get also that T_ϕ and $T_{\zeta^{(0,2)}\overline{\zeta}^{(2,0)}\varphi}$ commute on $A^2_\gamma(\mathbb{B}_n)$. On the other hand, using Theorem 2.4 we get $T_{\zeta^{(1,0)}\overline{\zeta}^{(0,1)}\psi}$ and $T_{\zeta^{(0,2)}\overline{\zeta}^{(2,0)}\varphi}$ do not commute on $A^2_\gamma(\mathbb{B}_n)$.*

Acknowledgments. The Author would like to express his sincere thanks to the referee for his/her comments which improved the presentation in this paper.

References

[1] S. Axler and Ž. Čučković, Commuting operators with harmonic symbols, *Integr. Equ. Oper. Theory* 14(1991), 1-12.
[2] S. Axler, Ž. Čučković, and N. V. Rao, Commuatants of analytic Toeplitz operatos on the Bergman space, *Proc. Amer. Math. Soc.* 128(2000), 1951-1953.
[3] A. Brown and P. Halmos, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* 213(1964), 89102.
[4] Ž. Čučković, Commutants of Toeplitz operators on the Bergman space, *Pacific, J. Math.* 162(1994), 277-285.
[5] Ž. Čučković and N. V. Rao, Mellin Transform, monomial symbols, and commuting Toeplitz operators *J. Funct. Anal* 154(1998), 195-214.
[6] H. Guan, L. Liu, and Y. Lu, Algebraic Properies of Quasihomogeneous and Separately Quasihomogeneous Toeplitz Operators on the Pluriharmonic Bergman Space, *Abs. Appl. Anal.*, Vol. 2013, 1-12.
[7] I. Louhichi and L. Zakariasy, On Toeplitz operators with quasihomogeneous symbols, *Arch. Math.* 85(2005), 248257.
[8] I. Louhichi and N. V. Rao, Bicommutants of Toeplitz operators, *Arch. Math* 91(2008), 256-264.
[9] D. Zheng, Commuting Toeplitz operators with pluriharmonic symbols, *Trans. Amer. Math. Soc.* 350(1998), 15951618.
[10] Z. H. Zhou and X. T. Dong, Algebraic properties of Toeplitz operators with separately quasihomogeneous symbols on the Bergman space of the unit ball, *Integr. Equ. Oper. Theory* 64(2009), 137-154.
[11] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.

Waleed Al-Rawashdeh
Department of Mathematical Sciences
Montana Tech of The University of Montana
Butte, Montana 59701, USA
E-mail address: walrawashdeh@mtech.edu