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SOLVABILITY IN GEVREY CLASSES OF SOME NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we prove, under some growth conditions on the datas, the solvability in a Gevrey class $G_k([-1,1])$ of a nonlinear functional differential equations of the form

$$y'(x) = a(x)\Phi(y(\psi(x)) + b(x)$$

under the initial condition

$$y(d) = c$$

where a, b and ψ are holomorphic functions on some neighborhood in $\mathbb{C}$ of $[-1,1]$, Φ is an entire function and $d \in [-1,1]$ and $c \in \mathbb{R}$ are given numbers.

1. Introduction

Functional differential equations (FDE) are of great interest because they give a framework to the study of physical, biological and economical processes where the rate of change of the state of the systems depends not only on the present time but on other different times which are functions of the present time.

The systematic theory of FDE began in the twentieth century with A.D.Myshkis, E.M.Wright, R.Bellman and others. FDE have many applications in the theory of automatic control, the problems of rocket motion, the problems of economical planning, the theory of population dynamics, the study of cell cycle, the study of blood cell dynamics, the study of infectious disease dynamics...etc.

There is an extensive literature ([1] - [16], [18] - [20], [22]) dealing with FDE reflecting the increasingly rapid development of this branch of mathematical analysis. The lines of research on FDE are also various: solvability of FDE in Hölder functions, analytic functions or entire functions, existence of periodic solutions to FDE, existence of almost-periodic solutions to FDE, boundary value problems for FDE or sytems of FDE, stability for FDE, theory of functional differential inclusions, theory of iterative FDE, theory of stochastic FDE, FDE with fractional derivatives, numerical approximation of solutions to FDE, oscillatory properties of FDE, bifurcation theory of FDE, etc.

Our aim in this paper is to prove, under some growth conditions on the datas, the solvability in a Gevrey class $G_k([-1,1])$ of a nonlinear FDE of the form

$$y'(x) = a(x)\Phi(y(\psi(x)) + b(x)$$

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under the initial condition

$$y(d) = c$$

where a, b and ψ are holomorphic functions on some neighborhood in $\mathbb{C}$ of $[-1,1]$, Φ is an entire function and $d \in [-1,1]$ and $c \in \mathbb{R}$ are given numbers. The reason of our interest in Gevrey classes is firstly their extrem usefulness in the study of problems of mathematical physics, then the few number of published works on the solvability of FDE in these classes of functions. Our approach relies upon some classical results of functional analysis and some basic results of differential calculus in Banach spaces and of complex analysis.

2. A motivating example

Consider the following nonlinear FDE

$$y'(x) = -\frac{1}{3}t\ e^{y\ (\sin t\ )} + \frac{9}{100}\cosh t \tag{2.1}$$

with the initial condition

$$y(0) = 0 \tag{2.2}$$

The problem (2.1) - (2.2) is equivalent to the following integral equation

$$y(t) = -\frac{1}{3}\int_0^t s e^{y\ (\sin s)} ds + \frac{9}{100}\sinh t$$

where the unknown function is considered in the set $C^0([-1,1])$ of complex valued continuous functions on $[-1,1]$. It is clear that a solution of the problem (2.1) - (2.2) is a function of class C^∞on $[-1,1]$.

Let T_0 be the nonlinear operator defined on the Banach space $C^0([-1,1])$ of real valued continuous functions on $[-1,1]$ endowed with the uniform norm, by the following formula

$$T_0(\varphi)(t) := -\frac{1}{3}\int_0^t s e^{y\ (\sin s)} ds + \frac{9}{100}\sinh t\ , t \in [-1,1], \varphi \in C^0([-1,1])$$

Let be $r \geq 0$ and $\overline{\Delta}_\infty(0, r\ )$ the closed ball in $C^0([-1,1])$, of radius r and center the null function.

Direct computations show that

$$T_0(\overline{\Delta}_\infty(0,r)) \subset \overline{\Delta}_\infty(0, \frac{1}{6}e^r + \frac{9}{100}\sinh 1)$$

The study of variations of the function $r \to \frac{1}{6}e^r + \frac{9}{100}\sinh 1 - r$ shows that this function has exactly two zeros R_1 and R_2 such that

$$0 < R_1 < \ln 6 < R_2$$

Thence we have

$$T_0(\overline{\Delta}_\infty(0,R_1)) \subset \overline{\Delta}_\infty(0,R_1)$$

On the other hand direct computations show that for all $\varphi \in \overline{\Delta}_\infty(0,R_1)$, the function $T_0(\varphi)$ is lipshitzian with Lipshitz constant $\frac{1}{3}e^{R_1} + \frac{9}{100}\cosh 1$. It follows that the restriction of T_0 to $\overline{\Delta}_\infty(0,R_1)$ is uniformly bounded and equicontinuous. Thence according to the well known Arzelà theorem ([21], pages $440 - 441$), the restriction of the operator T_0 to $\overline{\Delta}_\infty(0,R_1)$ is compact. Futhermore since $\overline{\Delta}_\infty(0,R_1)$

is stable by T_0 it follows thanks to the Schauder principle ([21], pages 223 − 225) that the operators T_0 has a fixed point $u \in \overline{\Delta}_\infty(0, R_1)$. Thence u is a solution of the problem (2.1)-(2.2). The function u is then of class C^∞on $[-1, 1]$.

The natural question which arise is if we can prove that the function u is real-analytic on $[-1, 1]$.To answer this question we can adopt the following approach.

(i) Consider the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ defined on $[-1, 1]$ by the following relations

$$\begin{aligned} f_1 &= Q \\ f_{n+1} &= T_0(f_n), n \in \mathbb{N}^* \end{aligned}$$

where Q is a polynomial.

(ii) Choose Q sufficiently close to u for the uniform norm to ensure that the following properties hold:

(ii.1) the uniform convergence of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $[-1, 1]$ to the function u

(ii.2) the existence of a neighborhood U of $[-1, 1]$ such that the function f_n has for every $n \in \mathbb{N}$ an analytic continuation F_n on the neighborhood U

(ii.3) the uniform convergence of the sequence of functions $(F_n)_{n\in\mathbb{N}}$ on every compact subset of U to a function F which is therefore analytic on U.

As a conclusion to this approach we would have $u = F|_{[-1,1]}$ and the function u would be real-analytic on $[-1, 1]$.

Unfortunately for the achievement of this approach we need that $\sin(U) \subset U$ but this inclusion is never true since

$$\lim_{n\to+\infty} |\sin^{<n>}(it)| = +\infty \ , \ t \in \mathbb{R}^*$$

where $\sin^{<n>}$ denotes the iterate of order n of the function sin for the composition of mappings.

Thence this approach is not relevant to our situation.

The function sin do not stabilize any neighborhood of $[-1, 1]$ but it has the following property (for the proof, see section 5)

$$\sin([-1, 1]_{\frac{A}{p+1}}) \subset [-1, 1]_{\frac{A}{p}},\ p \in \mathbb{N}^*,\ A \in]0, 1[$$

where we set for every $s > 0$

$$[-1, 1]_s = \{x + re^{i\theta} : x \in [-1, 1],\ r \in [0, s],\ \theta \in [0, 2\pi]\}$$

This property of the function sin allows a correction of the previous approach which leads to a regularity result for the function u.

Indeed if the polynomial Q is sufficiently close to the function u for the uniform norm we can show the following properties:

(i') the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ is uniformly convergent on $[-1, 1]$ to the function u

(ii') there exists $s > 0$ such that every function f_n has an extension to a function F_n analytic on $[-1, 1]_{\frac{s}{n}}$ and the function $g_n : z \to F_{n+1}(z) - F_n(z)$ satisfies an estimate of the form

$$\sup_{z\in[-1,1]_{\frac{s}{n+1}}} |g_n(z)| \le C_0\sigma^n, n \in \mathbb{N}^*$$

where $C_0 > 0$ and $\sigma \in]0, 1[$ are real constants independent of the integer n.

Then in view of the Cauchy's inequalities we have for every $n \in \mathbb{N}^*$ and $p \in \mathbb{N}$

$$\begin{aligned} ||g_n^{(p)}||_{\infty,[-1,1]} &\leq C_0(\frac{2(n+1)}{s})^p \sigma^n p! \\ &\leq \frac{C_0}{\sigma}(\frac{2}{s})^p \sup_{x\geq 0}(x^p \sqrt{\sigma}^x)\sqrt{\sigma}^n p^p \end{aligned}$$

Direct computations show that

$$\sup_{x\geq 0}(x^p \sqrt{\sigma}^x) \leq (\frac{2}{e\ln(\frac{1}{\sigma})})^p p^p$$

It follows that

$$||g_n^{(p)}||_{\infty,[-1,1]} \leq \frac{C_0}{\sqrt{\sigma}}(\frac{4}{es\ln(\frac{1}{\sigma})})^p \sqrt{\sigma}^n p^{2p}$$

Thence the function we have

$$||(u - Q|_{[-1,1]})^{(p)}||_{\infty,[-1,1]} \leq \frac{C_0\sqrt{\sigma}}{1-\sqrt{\sigma}}(\frac{4}{es\ln(\frac{1}{\sigma})})^p p^{2p},\ p \in \mathbb{N}$$

Then the function $u - Q|_{[-1,1]}$ belongs to the Gevrey class $G_1([-1,1])$ (cf.definition below). It follows that u belongs also to the Gevrey class $G_1([-1,1])$.

In this paper we will study functional differential equations which are the generalisation of this example. We will follow in the proof of the general case the same strategy as in this example.

3. Notations, definitions and main result

Let S be a nonempty subset of $\mathbb{C}$ and $f : S \to \mathbb{C}$ a bounded function. $\|f\|_{\infty,S}$ denotes the quantity

$$\|f\|_{\infty,S} := \sup_{u\in S}|f(u)|$$

For every $z \in \mathbb{C}$, we denote by $\varrho(z,S)$ the quantity

$$\varrho(z,S) := \inf_{\zeta\in S}|z-\zeta|$$

Let be $a, b \in \mathbb{R}$, we set

$$\begin{aligned} a \vee b &: = \max(a,b),\ a \wedge b := \min(a,b) \\ sgn(0) &: = 0,\ sgn(a) := \frac{a}{|a|} \text{ if } a \neq 0 \end{aligned}$$

Let be p and q two integers, we denote by $\overline{p,q}$ the set

$$\overline{p,q} := \{j \in \mathbb{Z} : p \leq j \leq q\}$$

For every function g whose domain of definition contains $[-1,1]$ and whose restriction to $[-1,1]$ is continuous on $[-1,1]$, we set

$$||g||_1 := \int_{-1}^{1}|g(t)|dt$$

For $z \in \mathbb{C}$ and $h > 0$, $B(z,h)$ is the open ball in $\mathbb{C}$ with center z and radius h.

For $z_1, z_2 \in \mathbb{C}$, we denote by $\overrightarrow{z_1, z_2}$ the linear path joining z_1 to z_2.

For $r, k, A \in]0, +\infty[$, and $n \in \mathbb{N}^*$, we set for every nonempty interval I of $\mathbb{R}$

$$I_r := I + B(0,r),\ I_{k,A,n} := I + B(0, An^{\frac{-1}{k}})$$

$C^0([-1,1])$(resp.$C^\infty([-1,1])$) denotes the set of real-valued continuous functions on $[-1,1]$ (resp.the set of real-valued functions of class C^∞ on $[-1,1]$).

For $r \geq 0$ and $f \in C^0([-1,1])$ we denote by $\overline{\Delta}_\infty(f,r)$ the closed ball in the Banach space $(C^0([-1,1]), ||.||_{\infty,[-1,1]})$ of center f and of radius r.

We denote by $L(C^0([-1,1])$ the vector space of continuous linear mappings from the Banach space $C^0([-1,1]$ to itself. $L(C^0([-1,1])$ is a Banach space when endowed by the norm $|||.|||$ defined by

$$|||U||| := \sup_{f \neq 0} \frac{||U(f)||_{\infty,[-1,1]}}{||f||_{\infty,[-1,1]}}$$

Let E be a nonempty subset of $\mathbb{C}$. By $O(E)$ we denote the set of holomorphic functions on some neighborhood of E.

Along this paper $k > 0$ is a given real number. The Gevrey class $G_k([-1,1])$ is the set of all functions f of class C^∞on $[-1,1]$ such that there exists a constant $B > 0$ such that

$$||f^{(n)}||_{\infty,[-1,1]} \leq B^{n+1} n^{n(1+\frac{1}{k})}, n \in \mathbb{N}$$

with the convention that $0^0 = 1$.

A holomorphic function φ on a neighborhood $[-1,1]_r$ of $[-1,1]$ is said to verify the $E(k)$ property if the following conditions hold

(i) $\varphi([-1,1]) \subset [-1,1]$

(ii) there exists a constant $\tau_\varphi \in]0,r]$ depending only on φ such that for all A in $]0,\tau_\varphi]$ there exists an integer $N(A) \geq 1$ depending only on A such that for every positive integer $p \geq N(A)$, we have

$$\varphi([-1,1]_{k,A,p+1}) \subset [-1,1]_{k,A,p}$$

The real number τ_φ will be called a $k-$threshold of φ.

Remark 1. *Let be φ a holomorphic function on a neighborhood of $[-1,1]$ such that ψ verifies the $E(k)$ property. Thence we have*

$$\varphi([-1,1]_{k,A(2N(A))^{-\frac{1}{k}},p+1}) \subset [-1,1]_{k,A(2N(A))^{-\frac{1}{k}},p} \ , \ p \in \mathbb{N}^*$$

It follows that for every $A \in]0,\tau_\varphi[$ there exists $B \in]0,A[$ such that

$$\varphi([-1,1]_{k,B,p+1}) \subset [-1,1]_{k,B,p} \ , \ p \in \mathbb{N}^*$$

Our main result is the following.

Theorem 1. *Let Φ be an entire function such that Φ is bounded on some interval of the form $]-\infty,\eta]$ $(\eta \in \mathbb{R})$ and*

$$|\Phi(z)| \leq \alpha e^{\beta|z|}, z \in \mathbb{C}$$

where $\alpha > 0$ and $\beta > 0$ are two constants.Let a, b, ψ be holomorphic functions on some neighborhood $[-1,1]_\mu$ of $[-1,1]$ such that the function ψ verifies the $E(k)$ property, the function a is not identically vanishing on $[-1,1]$ and the functions a and b are real-valued on $[-1,1]$. Let be $c \in \mathbb{R}$ and $d \in [-1,1]$.We assume that

$$\Phi(\mathbb{R}) \subset \mathbb{R}^+ \tag{3.1}$$

$$sgn\ (t-d)a(t) \in \mathbb{R}^- \ , t \in [-1,1] \tag{3.2}$$

$$||\Phi'||_{\infty,[c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]},c+||b||_1]} < \frac{1}{||a||_1} \tag{3.3}$$

$$||b||_1 + |c| \; < \frac{\ln(\frac{1}{\alpha\beta e\|a\|_1})}{\beta} \tag{3.4}$$

$$||a||_{\infty,[-1,1]}||\Phi||_{\infty,[c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c\;+||b||_1]},c+||b||_1]} + ||b||_{\infty,[-1,1]} < 1 \tag{3.5}$$

Then the FDE

$$y'(x) = a(x)\Phi(y(\psi(x)) + b(x) \tag{3.6}$$

under the initial condition

$$y(d) = c \tag{3.7}$$

has a solution $u_{d,c}$ which belongs to the Gevrey class $G_k([-1,1])$.

4. Proof of the main result

We subdivide the proof of the main result in three steps.

4.1. The localisation of the solutions of the equation ($\Im$) $\mathbf{r} = \boldsymbol{\alpha}\|\mathbf{a}\|_1\mathbf{e}^{\;\beta r}$ $+\|\mathbf{b}\;\|_1+|\mathbf{c}|$.

The study of the variations of the function

$$H : t \to \alpha\|a\|_1 e^{\beta t} + \|b\|_1 + |c\;| - t$$

shows that H is strictly decreasing on $]-\infty, \frac{\ln\;(\frac{1}{\alpha\beta\|a\|_1})}{\beta}]$ and strictly increasing on $[\frac{\ln\;(\frac{1}{\alpha\beta\|a\|_1})}{\beta}, +\infty[$.

On the other hand it follows from the condition (3.4) that $\frac{\ln\;(\frac{1}{\alpha\beta\|a\|_1})}{\beta} > 0$ and $H(\frac{\ln\;(\frac{1}{\alpha\beta\|a\|_1})}{\beta}) < 0$.

We have $H(0) > 0$, thence the equation ($\Im$) has exactly two solutions $r_0 < r_1$ which satisfy the inequalities

$$0 < r_0 < \frac{\ln\;(\frac{1}{\alpha\beta\|a\|_1})}{\beta} < r_1$$

4.2. Proof of the existence in $C^\infty([-1,1])$ of a solution $u_{d,c}$ of the problem (3.6)-(3.7).

Consider the operator T defined in $C^0([-1,1])$ by the formula

$$T\;(f\;)(x) := \int_d^x a(t\;)\Phi(f\;(\psi(t\;)))dt + \int_d^x b\;(t\;)dt + c\;,\; x \in [-1,1]$$

We have for all $f \in \overline{\Delta}_\infty(0, r_0), T\;(f\;) \in C^0([-1,1])$ and

$$\begin{aligned}\|T(f\;)\|_{\infty,[-1,1]} &\leq \|a\|_1\|\Phi \circ f\; \circ \psi\|_{\infty,[-1,1]} + \|b\|_1 + |c\;| \\ &\leq \alpha\|a\|_1 e^{\;\beta\|f\;\|_{\infty,[-1,1]}}. + \|b\|_1 + |c\;| \\ &\leq \alpha\|a\|_1 e^{\;\beta r_0} + \|b\|_1 + |c\;| = r_0\end{aligned}$$

So the closed ball $\overline{\Delta}_\infty(0, r_0)$ is stable by the operator T.

On the other hand, we have for all $u \in \overline{\Delta}_\infty(0, r_0)$ and x_1, $x_2 \in [-1, 1]$

$$\begin{aligned}
|T(u)(x_1) - T(u)(x_2)| &\leq \left|\int_{x_2}^{x_1} a(t)\Phi(u(\psi(t)))dt\right| + \left|\int_{x_2}^{x_1} b(t)dt\right| \\
&\leq (||a||_{\infty,[-1,1]}||\Phi \circ u \circ \psi||_{\infty,[-1,1]} + ||b||_{\infty,[-1,1]})|x_1 - x_2| \\
&\leq (||a||_{\infty,[-1,1]}||\Phi||_{\infty,u(\psi([-1,1]))} + ||b||_{\infty,[-1,1]})|x_1 - x_2| \\
&\leq (||a||_{\infty,[-1,1]}||\Phi||_{\infty,u([-1,1])} + ||b||_{\infty,[-1,1]})|x_1 - x_2| \\
&\leq (||a||_{\infty,[-1,1]}||\Phi||_{\infty,[-r_0,r_0])} + ||b||_{\infty,[-1,1]})|x_1 - x_2|
\end{aligned}$$

Thence $T(\overline{\Delta}_\infty(0, r_0))$ is uniformly bounded and equicontinuous. Then it follows according to the Arzelà theorem that the restriction of the nonlinear operator T to the closed ball $\overline{\Delta}_\infty(0, r_0)$ is compact. Since $\overline{\Delta}_\infty(0, r_0)$ is stable by the operator T it follows thanks to the Schauder principle that T has a fixed point $u_{d,c} \in \overline{\Delta}_\infty(0, r_0)$. Thence $u_{d,c}$ is a solution in $C^\infty([-1, 1])$ of the problem (3.6)-(3.7).

4.3. Proof that $u_{d,c}$ belongs to the Gevrey class $G_k([-1, 1])$.

Proposition 1. *Under the conditions* ***(3.1), (3.2), (3.3)*** *and* ***(3.5)*** *there exists* $\varepsilon > 0$ *such that*

$$u_{d,c}([-1,1]) \subset]c - ||b||_1 - ||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]} - \varepsilon, c + ||b||_1 + \varepsilon[\tag{4.1}$$

$$||a||_{\infty,[-1,1]}||\Phi||_{\infty,]c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}-\varepsilon,c+||b||_1+\varepsilon[} + ||b||_{\infty,[-1,1]} < 1 \tag{4.2}$$

$$||a||_1||\Phi'||_{\infty,]c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}-\varepsilon,c+||b||_1+\varepsilon[} < 1 \tag{4.3}$$

Proof

By virtue of (3.1) and (3.2) we have for all $x \in [-1, 1]$

$$\begin{aligned}
u_{d,c}(x) &= c + \int_d^x a(t)\Phi(u_{d,c}(\psi(t))dt + \int_d^x b(t)dt \\
&= c + \int_{d\wedge x}^{d\vee x} sgn(t-d)a(t)\Phi(u_{d,c}(\psi(t))dt + \int_d^x b(t)dt \\
&= c - \int_{d\wedge x}^{d\vee x} |a(t)|\Phi(u_{d,c}(\psi(t))dt + \int_d^x b(t)dt \\
&\leq c + ||b||_1
\end{aligned}$$

We have also

$$u_{d,c}(x) \geq c - ||b||_1 - ||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}$$

Thence we have

$$u_{d,c}([-1,1]) \subset [c - ||b||_1 - ||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}, c + ||b||_1]$$

By the continuity of Φ' it follows from (3.3) and (3.5) that there exists $\varepsilon > 0$ such that

$$\begin{aligned}
||a||_{\infty,[-1,1]}||\Phi||_{\infty,]c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}-\varepsilon,c+||b||_1+\varepsilon[} + ||b||_{\infty,[-1,1]} &< 1 \\
||a||_1||\Phi'||_{\infty,]c-||b||_1-||a||_1||\Phi||_{\infty,]-\infty,c+||b||_1]}-\varepsilon,c+||b||_1+\varepsilon[} &< 1
\end{aligned}$$

Furthermore the following inclusion holds

$$u_{d,c}([-1,1]) \subset]c - ||b||_1 - ||a||_1||\Phi||_{\infty,]-\infty,c\ +||b||_1]} - \varepsilon, c + ||b||_1 + \varepsilon[$$

□

Direct computations show that the nonlinear operator T is of class C^1 on the Banach space $(C^0([-1,1]), ||.||_{\infty,[-1,1]})$ and that its differential at the point $u_{d,c}$ is the continuous linear mapping $dT(u_{d,c}) \in L(C^0([-1,1]))$ defined by

$$dT\ (u_{d,c})(h)(x) := \int_d^x a\ (t\ )\ \Phi'(u_{d,c}(\psi(t\ )))h\ (\psi(t))dt\ ,\ x \in [-1,1],\ h \in C^0([-1,1])$$

Furthermore it follows from 4.3 and the fact that ψ verifies the $E(k)$ property that the norm of $dT\ (u_{d,c})$ satisfies the inequality

$$\begin{aligned} |||dT\ (u_{d,c})||| &\leq ||a\ ||_1||\ \Phi' \circ u_{d,c}\ ||_{\infty,[-1,1]} \\ &\leq ||a\ ||_1||\ \Phi'||_{\infty,u_{d,c}([-1,1])} \\ &\leq ||a\ ||_1||\ \Phi'||_{\infty,J} < 1 \end{aligned}$$

where

$$J :=]c - ||b||_1 - ||a||_1||\Phi||_{\infty,]-\infty,c\ +||b||_1]} - \varepsilon, c + ||b||_1 + \varepsilon[$$

Thence there exists $r_2 \in]0,\mu[$ such that the operator T is lipshitzian on $\overline{\Delta}_\infty(u_{d,c}, r_2)$ with Lipshitz constant

$$\frac{1 + ||a||_1||\Phi'||_{\infty,J}}{2} \in [0,1[$$

It follows that $u_{d,c}$ is the unique solution of the problem (3.6) - (3.7) which belongs to $\overline{\Delta}_\infty(u_{d,c}, r_2)$.

On the other hand, since the function Λ defined on $[0,\mu[$ by

$$\Lambda(0) := [\sup_{x\in[-1,1]} (\int_{d\wedge x}^{d\vee x} |a(t)|dt)||\Phi'||_{\infty,J})] \vee (||a||_{\infty,[-1,1]}||\Phi||_{\infty,J} + ||b||_{\infty,[-1,1]})$$

and for s belonging to $]0,\mu[$, by

$$\Lambda(s) := [\sup_{z\in[-1,1]_s} (\int_{\overrightarrow{d,z}} |a(\zeta)||d\zeta|)||\Phi'||_{\infty,J_s})] \vee (||a||_{\infty,[-1,1]_s}||\Phi||_{\infty,J_s} + ||b||_{\infty,[-1,1]_s})$$

is continuous on $[0,\mu[$ and verifies by virtue of 4.2 and 4.3 the inequality

$$\Lambda(0) < 1$$

it follows that there exists $r_3 \in]0, r_2 \wedge \tau[$ such that

$$\Lambda([0,r_3]) \subset]0,1[$$

where τ is a k-threshold of ψ.

Furthermore thanks to the remark **1** there exists $s_1 \in]0,r_3[$ such that

$$\psi([-1,1]_{k,s_1,n+1)}) \subset [-1,1]_{k,,s_1,n}\ ,\ n \in \mathbb{N}^* \tag{4.4}$$

On the other hand according to the well known Weierstrass theorem (cf. [17], pages 87, $255-257, 266$) and in view of 4.1, there exists a polynomial Q with real coefficients such that

$$\begin{aligned} Q_{|[-1,1]} &\in \overline{\Delta}_\infty(u_{d,c}, s_1) \\ Q([-1,1]) &\subset J \end{aligned}$$

Consider then the sequence $(F_n)_{n\geq 1}$of functions $F_n : [-1,1]_{k,s_1,n} \to \mathbb{C}$ defined by the following relations

$$\begin{aligned} F_1 &= Q \\ F_{n+1}(z) &= c + \int\limits_{\overrightarrow{d,z}} a(\zeta)\Phi(F_n(\psi(\zeta))d\zeta + \int\limits_{\overrightarrow{d,z}} b(\zeta)d\zeta, z \in \ [-1,1]_{k,s_1,n+1}, n \in \mathbb{N}^* \end{aligned}$$

Then direct computations, based on 4.4, show that the function F_n is for every $n \in \mathbb{N}^*$ holomorphic on $[-1,1]_{k,s_1,n}$.

Proposition 2. *We have for every* $n \in \mathbb{N}^*$

$$F_n([-1,1]_{k,\frac{1}{\lambda}s_1,n}) \subset J_{k,s_1,n}$$

where

$$\lambda := ||Q'||_{\infty,[-1,1]_{k,s_1,1}} + 1$$

Proof

We denote the last inclusion by $A(n)$.

We denote for every $z \in \mathbb{C}$ by $\widehat{z}$ the closest point of $[-1,1]$ to z. Thence if $z \in [-1,1]_{k,\frac{1}{\lambda}s_1,1}$ we will have

$$\begin{aligned} \varrho(F_1(z), J) &\leq |Q(z) - Q(\widehat{z})| \\ &\leq ||Q'||_{\infty,[-1,1]_{k,s_1,1}} \varrho(z, [-1,1]) \\ &\leq s_1 \end{aligned}$$

It follows that $A(1)$ is true.

Assume for a certain $n \in \mathbb{N}^*$ that $A(p)$ is true for every $p \in \overline{1,n}$.

Let be $z \in [-1,1]_{k,\frac{1}{\lambda}s_1,n+1}$. We have then

$$\begin{aligned} \varrho(F_{n+1}(z), J) &\leq |F_{n+1}(z) - F_{n+1}(\widehat{z})| \\ &\leq |\int\limits_{\overrightarrow{\widehat{z},z}} a(\zeta)\Phi(F_n(\psi(\zeta))d\zeta + \int\limits_{\overrightarrow{\widehat{z},z}} b(\zeta)d\zeta| \\ &\leq [||a||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n}} ||\Phi||_{\infty,J_{k,s_1,n}} + ||b||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n}}].\varrho(z,[-1,1]) \\ &\leq \Lambda(s_1)\frac{1}{\lambda}s_1(n+1)^{-\frac{1}{k}} \\ &< \Lambda(s_1)s_1(n+1)^{\frac{-1}{k}} \end{aligned}$$

It follows from **??** that

$$\Lambda(s_1) < 1$$

Thence

$$\varrho(F_{n+1}(z), J) < s_1(n+1)^{\frac{-1}{k}}$$

It follows that the assertion $A(n+1)$ is true.

Consequently $A(n)$ is true for all $n \in \mathbb{N}^*$.□

We have for all $n \in \mathbb{N}^*\backslash\{1\}$ and $z \in [-1,1]_{k,\frac{1}{\lambda}s_1,n+1}$

$$\begin{aligned}
|F_{n+1}(z)-F_n(z)| &\leq \int_{\overrightarrow{d,z}} |a(\zeta)|.|\Phi(F_n(\psi(\zeta))-\Phi(F_{n-1}(\psi(\zeta))||d\zeta| \\
&\leq \int_{\overrightarrow{d,z}} |a(\zeta)|.||\Phi'||_{\infty,J_{k,\frac{1}{\lambda}s_1,n+1}}|F_n(\psi(\zeta))-F_{n-1}(\psi(\zeta))||d\zeta| \\
&\leq [(\int_{\overrightarrow{d,z}} |a(\zeta)|.|d\zeta|)||\Phi'||_{\infty,J_{k,\frac{1}{\lambda}s_1,n+1}}].||F_n-F_{n-1}||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n}} \\
&\leq \Lambda(s_1)||F_n-F_{n-1}||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n}}
\end{aligned}$$

It follows that

$$||F_{n+1}-F_n||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n+1}} \leq \Lambda(s_1)||F_n-F_{n-1}||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n}}$$

Thence we have for all $n \in \mathbb{N}^*$

$$||F_{n+1}-F_n||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n+1}} \leq (\Lambda\ (s_1))^n \frac{||F_2-F_1||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,2}}}{\Lambda\ (s_1)}$$

Let us set for all $n \in \mathbb{N}^*$

$$\begin{aligned}
g_0 &: = Q \\
g_n &: = F_{n+1}-F_n\ ,\ n \in \mathbb{N}^* \\
C &: = \frac{||F_2-F_1||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,2}}}{\Lambda\ (s_1)} + ||Q||_{\infty,[-1,1]_{s_1}} \\
\delta &: = \Lambda\ (s_1)
\end{aligned}$$

Thence the following relations hold for every $n \in \mathbb{N}$

$$\begin{aligned}
g_n &\in O\ ([-1,1]_{k,s_1,n\ +1}) \\
||g_n||_{\infty,[-1,1]_{k,\frac{1}{\lambda}s_1,n+1}} &\leq C\ \delta^n, n \in \mathbb{N}
\end{aligned}$$

Since $\delta \in [0,1[$ it follows that the function series $\sum g_n|_{[-1,1]}$ is uniformly convergent on $[-1,1]$ to a function $h \in \overline{\Delta}_\infty(u_{d,c},s_1)$. But then direct computations show that h is a solution to the problem (3.6)-(3.7) which belongs to $\overline{\Delta}_\infty(u_{d,c},r_2)$ it follows that $h = u_{d,c}$.

Thence the function series $\sum g_{n|[-1,1]}$ is uniformly convergent on $[-1,1]$ to the function $u_{d,c}$. Furthermore in view of the Cauchy's inequalities we have for every $p \in \mathbb{N}$ and $n \in \mathbb{N}^*$

$$||g_n^{(p)}|_{[-1,1]}||_{\infty,[-1,1]} \leq Cp!(\frac{s_1 n^{\frac{-1}{k}}}{2})^{-p}\delta^n$$

It follows that

$$||g_n^{(p)}|_{[-1,1]}||_{\infty,[-1,1]} \leq C\ (\frac{2}{s_1})^p \sup_{x\ \geq 0}(\sqrt{\delta}^x x^{\frac{p}{k}}).\sqrt{\delta}^n p^p \tag{4.5}$$

Direct computations show that

$$\sup_{x\ \geq 0}(\sqrt{\delta}^x x^{\frac{p}{k}}) \leq (\frac{1}{\sqrt{e\ln(\frac{1}{\delta})}})^p p^{\frac{p}{k}}\ ,\ p \in \mathbb{N}$$

Thence the estimate 4.5 becomes

$$||g_n^{(p)}|_{[-1,1]}||_{\infty,[-1,1]} \leq C\Big(\frac{2}{s_1\sqrt{e\ln(\frac{1}{\delta})}}\Big)^p(\sqrt{\delta})^n p^{p(1+\frac{1}{k})} \tag{4.6}$$

The estimates 4.6 show that

$$\|(u_{d,c} - Q|_{[-1,1]})^{(p)}\|_{\infty,[-1,1]} \leq \frac{C\sqrt{\delta}}{1-\sqrt{\delta}}\Big(\frac{2}{s_1\sqrt{e\ln(\frac{1}{\delta})}}\Big)^p p^{p(1+\frac{1}{k})},\ p \in \mathbb{N} \tag{4.7}$$

From (4.7) it follows that $u_{d,c}$ belongs to the Gevrey class $G_k([-1,1])$.
The proof of the main result is then complete. $\square$

5. Examples

Proposition 1. *The function* sin *verifies the* $E(1)$ *property.*

Proof
Let $A \in]0,1[$, $p \in \mathbb{N}^*$and $z \in [-1,1]_{1,p+1,A}$.Let $\widehat{z}$ be the closest point of $[-1,1]$ to z. We have the following inequalities

$$\begin{aligned}
\varrho(\sin z, [-1,1]) &\leq |\sin z - \sin\widehat{z}| \\
&\leq \sum_{j=1}^{+\infty}\frac{|\sin^{(j)}(\widehat{z}))|}{j!}|z-\widehat{z}|^j \\
&\leq \sum_{j=1}^{+\infty}\varrho(z,[-1,1])^j \\
&\leq \frac{A}{p+1-A} \\
&< \frac{A}{p}
\end{aligned}$$

It follows that

$$\sin([-1,1]_{1,p+1,A}) \subset [-1,1]_{1,p,A}\ , p \in \mathbb{N}^*$$

Since the function sin is entire and stabilizes the interval $[-1,1]$ it follows that sin verifies the $E(1)$ property. $\square$

Example 1.

Let be $\alpha > 0, \beta > 0, \gamma \in \mathbb{R}, N \in \mathbb{N}$. We assume that

$$\alpha\beta e^{2\beta|\gamma|\sinh 1} < \frac{N+1}{e} \tag{5.1}$$

$$\frac{\alpha e^\beta}{N+1} + |\gamma|\cosh 1 < 1 \tag{5.2}$$

Consider the FDE

$$y'(t) = -\alpha t^{2N+1}e^{\beta y(\sin t)} + \gamma\cosh t \tag{5.3}$$

with the initial condition

$$y(0) = 0 \tag{5.4}$$

Consider then the functions

$$\Phi_1 : z \to \alpha e^{\beta z}, a_1 : z \to -z^{2N+1}, b_1 : z \to \gamma\cosh z$$

The function Φ_1 is an entire function bounded on $]-\infty, 0]$ and we have

$$\begin{array}{rcl} |\Phi_1(z)| & \leq & \alpha e^{\beta|z|}, z \in \mathbb{C} \\ \Phi_1(\mathbb{R}) & \subset & \mathbb{R}^+ \\ sgn\ (t-0\ ).a_1\ (t\ ) & = & sgn\ (-t^{2N+2}) \in \mathbb{R}^-, t \in [-1,1] \end{array}$$

Furthermore the conditions (5.1) and (5.2) entail that

$$\begin{array}{rcl} ||\Phi_1'||_{\infty,[0-||b_1||_1-||a||_1||\Phi||_{\infty,]-\infty,0+||b||_1},0+||b_1||_1]} & = & \alpha\beta e^{2\beta|\gamma|\sinh 1)} < \dfrac{1}{||a_1||_1} \\ ||b_1||_1+|0| & = & 2|\gamma|\sinh 1 < \dfrac{\ln(\frac{1}{\alpha\beta e\|a_1\|_1})}{\beta} \\ ||a_1||_{\infty,[-1,1]}||\Phi_1||_{\infty,]-\infty,0+||b||_1} + ||b_1||_{\infty,[-1,1]} & = & \alpha e^{\frac{\beta}{N+1}} + |\gamma|\cosh 1 < 1 \end{array}$$

Thence according to the last proposition and to the main result of the paper, the problem (5.3)-(5.4) has a solution which belongs to the Gevrey class $G_1([-1,1])$.

Example 2.

Let be $\alpha > 0, \nu > 0, N \in \mathbb{N}$. We assume that

$$\alpha < \frac{1}{2} \tag{5.5}$$

$$\alpha\nu < \frac{N+1}{2e} \tag{5.6}$$

Consider the FDE

$$y'(t) = -\alpha t^{2N+1}(1+\cos[\nu y(\sin t)]) \tag{5.7}$$

with the initial condition

$$y(0) = 0 \tag{5.8}$$

Consider then the functions

$$\Phi_2 : z \to \alpha(1+\cos(\nu z)), a_2 : z \to -z^{2N+1}, b_2 : z \to 0$$

The function Φ_2 is an entire function bounded on $\mathbb{R}$. We have also

$$\begin{array}{rcl} |\Phi_2(z)| & \leq & 2\alpha e^{\nu|z|}, z \in \mathbb{C} \\ \Phi_2(\mathbb{R}) & = & [0, 2\alpha] \subset \mathbb{R}^+ \\ sgn\ (t-0\ ).a_2\ (t\ ) & = & sgn\ (-t^{2N+2}) \in \mathbb{R}^-, t \in [-1,1] \end{array}$$

Furthermore the conditions (5.5) and (5.6) entail that

$$\begin{array}{rcl} ||\Phi_2'||_{\infty,[0-||b_2||_1-||a||_1||\Phi_2||_{\infty,]-\infty,0+||b_2||_1},0+||b_2||_1]} & \leq & \alpha\nu < \dfrac{1}{||a_2||_1} \\ ||b_2||_1+|0| & < & \dfrac{\ln(\frac{1}{2\alpha\nu e\|a_2\|_1})}{\nu} \\ ||a_2||_{\infty,[-1,1]}||\Phi_2||_{\infty,]-\infty,0+||b_2||_1} + ||b_2||_{\infty,[-1,1]} & = & 2\alpha < 1 \end{array}$$

Thence according to the last proposition and to the main result of the paper, the problem (5.7)-(5.8) has a solution which belongs to the Gevrey class $G_1([-1,1])$.

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