Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 5 Issue 3(2014), Pages 1-8.

DOUBLE DIFFERENCE SEQUENCE SPACES $m^2(\mathcal{F}, \phi, u, p, \Delta)$

SEEMA JAMWAL, KULDIP RAJ

Abstract. In this paper we construct some double difference sequence spaces by means of sequence of modulus functions. We make an effort to examine the topological and algebraic properties of this sequence space.

1. Introduction

The initial work on double sequences was found in Bromwich [3]. Hardy [14] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [28] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [19] have recently introduced the statistical convergence and Cauchy convergence for double sequences. By the convergence of a double sequence we mean the convergence in the Pringsheim sense, i.e. a double sequence $x = (x_{i,j})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{i,j} - L| < \epsilon$ whenever $i, j > n$, (see [20]). We shall write more briefly as P-convergent. The double sequence $x = (x_{i,j})$ is bounded if there exists a positive number M such that $|x_{i,j}| < M$ for all i and j. The double sequence spaces in the various forms were introduced and studied by Khan and Tabassum in ([16], [17]).

By w^2 we denote the set of double complex sequences. A double series is infinity sum $\sum_{i,j=1}^{\infty} x_{i,j}$ and its convergence implies the convergence by $|.|$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j}$ (see[4]). For $1 \leq p < \infty$, $l_p^{(2)}$ denote the space of sequences $x = (x_{i,j})$ such that $\sum_{i,j=1}^{\infty} |x_{i,j}|^p < \infty$ (see [5]). The space of all bounded double sequences is denoted by $l_\infty^{(2)}$ (see [18]). Let $x = \{x_{i,j}\}$ be a double sequence. A set $S(x)$ is defined by

$$S(x) = \Big\{ \{x_{\pi_1(i), \pi_2(i)}\} : \pi_1 \text{ and } \pi_2 \text{ are permutations of } \mathbb{N} \Big\}.$$

A double sequence space E is said to be normal if $(y_{i,j}) \in E$ whenever $|y_{i,j}| \leq |x_{i,j}|$ for all $i, j \in \mathbb{N}$ and $(x_{i,j}) \in E$. A double sequence space E is said to be symmetric if $s = (s_{i,j}) \in E$ and $\|s\| = \|x\|$ whenever $x = (x_{i,j}) \in E$ and $s \in S(x)$. A $BK-$space

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2000 *Mathematics Subject Classification.* 40A05, 46A45.

Key words and phrases. sequence space, modulus function, completeness, normal space, Banach space, symmetric space.

Submitted March 14, 2014. Published Jul 22, 2014.

is a Banach sequence space E in which the coordinate maps are continuous.
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

1. $f(x) = 0$ if and only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([1],[21]) and references therein.
The notion of difference sequence spaces was introduced by Kızmaz [15], who defined the sequence spaces

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\} \text{ for } Z = c, c_0 \text{ and } l_\infty$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. The notion was further generalized by Et and Çolak [9] by introducing the spaces. Let r be a non-negative integer, then

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\} \text{ for } Z = c, c_0 \text{ and } l_\infty$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. The generalized difference sequence has the following binomial representation

$$\Delta^r x_k = \sum_{m=0}^{r} (-1)^m \begin{pmatrix} r \\ m \end{pmatrix} x_{k+m}.$$

Later the concept have been studied by Bektaş et al. [2] and Et et al. [10]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [26] who studied the spaces $l_\infty(\Delta_v)$, $c(\Delta_v)$ and $c_0(\Delta_v)$. Recently, Esi et al. [11] and Tripathy et al. [25] have introduced a new type of generalized difference operators and unified those as follows:
Let r, v be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_v^r) = \{x = (x_k) \in w : (\Delta_v^r x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_v^r x = (\Delta_v^r x_k) = (\Delta_v^{r-1} x_k - \Delta_v^{r-1} x_{k+v})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$. For more details about sequence spaces one can refer to ([6], [7], [12], [13]).
Let $\varphi_{s,t}$ denote the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $\mathbb{N} \times \mathbb{N}$ such that the elements of σ_1 and σ_2 are most s and t, respectively. Besides $\{\phi_{i,j}\}$ is taken as a nondecreasing double sequences of the positive real numbers such that

$$i\phi_{i+1,j} \leq (i+1)\phi_{i,j}, \qquad j\phi_{i,j+1} \leq (j+1)\phi_{i,j} \qquad \text{(see[8])}.$$

The $BK-$space $m(\phi)$, introduced by Sargent [23] in the form

$$m(\phi) = \left\{ x = \{x_i\} \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{i \in \sigma} |x_i| < \infty \right\}.$$

Sargent studied some properties of this space and examined relationship between this space and l_p- space.

The space $m(\phi)$ was extended to $m(\phi,p)$ by Tripathy and Sen [27] as follows:

$$m(\phi,p)=\left\{x=\{x_i\}\in w:\|x\|_{m(\phi,p)}=\sup_{s\geq 1,\sigma\in\varphi_s}\frac{1}{\phi_s}\sum_{i\in\sigma}\Big(|x_i|^p\Big)^{\frac{1}{p}}<\infty\right\}.$$

Further, Raj et al. [22] introduced and studied the following sequence space $m(\mathcal{F},\phi,p)$

$$m(\mathcal{F},\phi,p)=\Bigg\{x=\{x_i\}\in w:\|x\|_{m(\mathcal{F},\phi,p)}$$

$$=\sup_{s\geq 1,\sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{i\in\sigma}\left[f_i\left(\frac{|x_i|}{\rho}\right)\right]^p\right)^{\frac{1}{p}}<\infty,\quad\text{for some }\ \rho>0\Bigg\},$$

Recently, Sağir et al. [24] introduce double sequence spaces $m^2(\mathcal{F},\phi,p)$ as follows:

$$m^2(\mathcal{F},\phi,p)=\Bigg\{x=\{x_{i,j}\}\in w^2:\sup_{(s,t)\geq(1,1)}\ \sup_{\sigma_1\times\sigma_2\in\varphi_{st}}\frac{1}{\phi_{st}}$$

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{|x_{i,j}|}{\rho}\Big)\Big]^p\Big\}^{\frac{1}{p}}<\infty,\quad\text{for some }\ \rho>0\Bigg\},$$

Let $\mathcal{F}=(f_{i,j})$ be a double sequence of modulus functions, $p=(p_{i,j})$ be a double bounded sequence of positive real numbers, $u=(u_{i,j})$ be a double sequence of strictly positive real numbers. In the present paper we defined the following double sequence space:

$$m^2(\mathcal{F},\phi,u,p,\Delta)=\Bigg\{x=\{x_{i,j}\}\in w^2:\sup_{(s,t)\geq(1,1)}\ \sup_{\sigma_1\times\sigma_2\in\varphi_{st}}\frac{1}{\phi_{st}}$$

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}}<\infty,\quad\text{for some }\ \rho>0\Bigg\},$$

If we take $u=(u_{i,j})=1$, $(p_{i,j})=p$ for all $i,j\in\mathbb{N}$ and $\Delta=I$, we get the double sequence space defined by Sağir et al. [24]
The following inequality will be used throughout the paper.
Let $p=(p_{i,j})$ be a sequence of positive real numbers with $0<p_{i,j}\leq\sup_{i,j}p_{i,j}=H$, and let $K=\max\big\{1,2^{H-1}\big\}$. Then, for the factorable sequences $(a_{i,j})$ and $(b_{i,j})$ in the complex plane, we have

$$|a_{i,j}+b_{i,j}|^{p_{i,j}}\leq K(|a_{i,j}|^{p_{i,j}}+|b_{i,j}|^{p_{i,j}}).\tag{1.1}$$

The main aim of this paper is to study new difference double sequence spaces $m^2(\mathcal{F},\phi,u,p,\Delta)$ by means of sequence of modulus functions. We shall study some topological, algebraic properties and inclusion relations of the sequence spaces $m^2(\mathcal{F},\phi,u,p,\Delta)$.

2. Main Results

Theorem 2.1. *The sequence space $m^2(\mathcal{F},\phi,u,p,\Delta)$ is a linear space over the complex field $\mathbb{C}$.*

Proof. The proof is easy so we omit it. □

Theorem 2.2. *The sequence space* $m^2(\mathcal{F}, \phi, u, p, \Delta)$ *is complete.*

Proof. Let $\{x^{(n)}\}$ be a double Cauchy sequence in $m^2(\mathcal{F}, \phi, u, p, \Delta)$ such that $x^{(n)} = \{x^{(n)}_{i,j}\}_{i,j=1}^{\infty}$ for all $n \in \mathbb{N}$. Then

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x^{(n)}_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \infty$$

for some $\rho > 0$ and for all $n \in \mathbb{N}$. For each $\epsilon > 0$, there exists a positive integer n_0 such that

$$\|x^{(n)} - x^{(m)}\|_{m^2(\mathcal{F},\phi,u,p,\Delta)} < \epsilon, \quad \text{for all } \ n, m \geq n_0.$$

This implies that

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta(x^{(n)}_{i,j} - x^{(m)}_{i,j})|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \epsilon \tag{2.1}$$

for some $\rho > 0$ and for all $n, m \geq n_0$. Hence

$$|x^{(n)}_{i,j} - x^{(m)}_{i,j}| < \epsilon\phi_{1,1}, \quad \text{for all } \ n, m \geq n_0$$

and for each fixed $(i, j) \in \mathbb{N} \times \mathbb{N}$. Hence $\{x^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$. Then, there exists $x_{i,j} \in \mathbb{C}$ such that $x^{(n)}_{i,j} \to x_{i,j}$ as $n \to \infty$ and let us define $x = (x_{i,j})$. From (2.1), for each fixed (s, t)

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta(x^{(n)}_{i,j} - x^{(m)}_{i,j})|}{\rho}\Big)\Big]^{p_{i,j}}\Big\} < \epsilon^{p_{i,j}}\phi_{st}^{p_{i,j}}$$

for some $\rho > 0$, for all $n, m \geq n_0$ and $\sigma_1 \times \sigma_2 \in \varphi_{st}$. Taking $m \to \infty$, we get

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta(x^{(n)}_{i,j} - x_{i,j})|}{\rho}\Big)\Big]^{p_{i,j}}\Big\} < \epsilon^{p_{i,j}}\phi_{st}^{p_{i,j}}$$

for some $\rho > 0$, for all $n, m \geq n_0$ and $\sigma_1 \times \sigma_2 \in \varphi_{st}$. Thus we obtain

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta(x^{(n)}_{i,j} - x_{i,j})|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \epsilon \tag{2.2}$$

for some $\rho > 0$, for all $n, m \geq n_0$. This implies that $x^{(n)} - x \in m^2(\mathcal{F}, \phi, u, p, \Delta)$, for all $m, n \geq n_0$. Hence $x = x^{(n_0)} + x - x^{(n_0)} \in m^2(\mathcal{F}, \phi, u, p, \Delta)$. By equation (2.2)

$$\|x^{(n)} - x\|_{m^2(\mathcal{F},\phi,u,p,\Delta)} < \epsilon, \quad \text{for all } \ n \geq n_0.$$

This means that $x^{(n)} \to x$ as $n \to \infty$. Thus the sequence space $m^2(\mathcal{F}, \phi, u, p, \Delta)$ is complete. □

Theorem 2.3. *The sequence space* $m^2(\mathcal{F}, \phi, u, p, \Delta)$ *is* $BK-$*space.*

Proof. Suppose $\{x^{(n)}\} \in m^2(\mathcal{F}, \phi, u, p, \Delta)$ with $\|x^{(n)} - x\|_{m^2(\mathcal{F},\phi,u,p,\Delta)} \to 0$, as $n \to \infty$. For each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x^{(n)} - x\|_{m^2(\mathcal{F},\phi,u,p,\Delta)} < \epsilon$ for all $n \geq n_0$. Thus

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta(x^{(n)}_{i,j} - x_{i,j})|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \epsilon$$

for some $\rho > 0$, for all $n \geq n_0$. Hence we obtain $|x_{i,j}^{(n)} - x_{i,j}| < \epsilon\phi_{1,1}$ for all $n \geq n_0$ and for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. This implies $|x_{i,j}^{(n)} - x_{i,j}| \to 0,$ as $n \to \infty$. This completes the proof. ☐

Corollary 2.4. *The sequence space $m^2(\mathcal{F}, \phi, u, p, \Delta)$ is symmetric space.*

Proof. Let $x = \{x_{i,j}\} \in m^2(\mathcal{F}, \phi, u, p, \Delta)$ and let $y = \{y_{i,j}\} \in S(x)$. Then we can write $y_{i,j} = x_{m_i,m_j}$. Thus we obtain $\|x\|_{m^2(\mathcal{F},\phi,u,p,\Delta)} = \|y\|_{m^2(\mathcal{F},\phi,u,p,\Delta)}$. ☐

Corollary 2.5. *The sequence space $m^2(\mathcal{F}, \phi, u, p, \Delta)$ is normal space.*

Proof. It is obvious. ☐

Theorem 2.6. $m^2(\mathcal{F}, \phi, p) \subset m^2(\mathcal{F}, \phi, u, p, \Delta)$.

Proof. Suppose that $x \in m^2(\mathcal{F}, \phi, p)$. Then we have

$$\begin{aligned} \|x\|_{m^2(\mathcal{F},\phi,p)} &= \sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \Big\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \Big[f_{i,j}\Big(\frac{|x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}} \Big\}^{\frac{1}{p_{i,j}}} \\ &= G < \infty. \end{aligned}$$

Thus for each fixed (s,t) and for $\sigma_1 \times \sigma_2 \in \varphi_{st}$, $\Big\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \Big[f_{i,j}\Big(\frac{|x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}} \Big\}^{\frac{1}{p_{i,j}}} \leq G\phi_{st}$ for some $\rho > 0$. Hence

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \Big\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}} \Big\}^{\frac{1}{p_{i,j}}} \leq G$$

for some $\rho > 0$. This implies that $x \in m^2(\mathcal{F}, \phi, u, p, \Delta)$. Thus $m^2(\mathcal{F}, \phi, p) \subset m^2(\mathcal{F}, \phi, u, p, \Delta)$. ☐

Theorem 2.7. $m^2(\mathcal{F}, \phi, u, p, \Delta) \subseteq m^2(\mathcal{F}, \psi, u, p, \Delta)$ *if and only if* $\sup_{(s,t)\geq(1,1)} \Big(\frac{\phi_{st}}{\psi_{st}}\Big) < \infty$.

Proof. Let $D = \sup_{(s,t)\geq(1,1)} \Big(\frac{\phi_{st}}{\psi_{st}}\Big) < \infty$. Then $\phi_{st} \leq D\psi_{st}$, for all $(s,t) \geq (1,1)$. If $x \in m^2(\mathcal{F}, \phi, u, p, \Delta)$. Then

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}} \Big\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}} \Big\}^{\frac{1}{p_{i,j}}} < \infty$$

for some $\rho > 0$. Thus,

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{D\psi_{st}} \Big\{ \sum_{i\in\sigma_1} \sum_{j\in\sigma_2} \Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}} \Big\}^{\frac{1}{p_{i,j}}} < \infty$$

for some $\rho > 0$. This shows that $m^2(\mathcal{F}, \phi, u, p, \Delta) \subseteq m^2(\mathcal{F}, \psi, u, p, \Delta)$.
Conversely, Let $m^2(\mathcal{F}, \phi, u, p, \Delta) \subseteq m^2(\mathcal{F}, \psi, u, p, \Delta)$. We define $\eta_{s,t} = \frac{\phi_{st}}{\psi_{st}}$. Let $\sup_{(s,t)\geq(1,1)} \eta_{s,t} = \infty$. Then there exists a subsequence $\{\eta_{s_l t_l}\}$ of $\{\eta_{st}\}$ such that $\eta_{s_l t_l} \to$

∞ as $l \to \infty$. For $x \in m^2(\mathcal{F}, \phi, u, p, \Delta)$ we have

$$\begin{aligned}
&\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\psi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}}\\
&\quad\geq \sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{\eta_{st}}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}}\\
&\quad= \infty
\end{aligned}$$

for some $\rho > 0$. This is a contradiction as $x \notin m^2(\mathcal{F}, \psi, u, p, \Delta)$ and this completes the proof. □

Theorem 2.8. $l_p^{(2)} \subseteq m^2(\mathcal{F}, \phi, u, p, \Delta) \subseteq l_\infty^{(2)}$.

Proof. Clearly, $l_p^{(2)} = m^2(\mathcal{F}, \psi, u, p, \Delta)$, where $\psi_{st} = 1$ for $s, t = 1, 2, ...$ when $f_{i,j}(x) = x$, $u = (u_{ij}) = 1$ for all $i, j \in \mathbb{N}$, $\Delta = 1$ and $\sup_{(s,t)\geq(1,1)} \Big(\frac{\psi_{st}}{\phi_{st}}\Big) < \infty$ by nondecreasing (ϕ_{st}). Then by Theorem (2.7), first inclusion is obtained. Suppose $x \in m^2(\mathcal{F}, \phi, u, p, \Delta)$ with $\Delta = 1$ and $u = (u_{ij}) = 1$ for all $i, j \in \mathbb{N}$. Then we have

$$\sup_{(s,t)\geq(1,1)} \sup_{\sigma_1\times\sigma_2\in\varphi_{st}} \frac{1}{\phi_{st}}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{|x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} = L < \infty$$

for some $\rho > 0$. Hence we obtain

$$|x_{i,j}| \leq L\phi_{1,1}$$

for all $i, j \in \mathbb{N}$. Thus $x \in l_\infty^{(2)}$ and proof is completed. □

Theorem 2.9. *Let $\mathcal{F} = (f_{i,j})$ be a sequence of modulus functions and $\alpha = \lim_{t\to\infty} \frac{f_{i,j}(t)}{t} > 0$. Then $m^2(\mathcal{F}, \phi, u, p, \Delta) \subset m^2(\phi, u, p, \Delta)$.*

Proof. Let $\alpha > 0$. By definition of α, we have $f_{i,j}(t) \geq \alpha.t$, for all $t \geq 0$. Since $\alpha > 0$, we have $t \leq \frac{1}{\alpha} f_{i,j}(t)$ for all $t \geq 0$. Let $x = (x_{i,j}) \in m^2(\mathcal{F}, \phi, u, p, \Delta)$ Thus we have

$$\begin{aligned}
\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} &\leq \frac{1}{\alpha}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}}\\
&< \infty.
\end{aligned}$$

Which implies that $x = (x_{i,j}) \in m^2(\phi, u, p, \Delta)$. This completes the proof. □

Theorem 2.10. *Let $\mathcal{F} = (f_{i,j})$ and $\mathcal{F}' = (f'_{i,j})$ be two sequences of modulus functions. Then*

$$m^2(\mathcal{F}, \phi, u, p, \Delta) \cap m^2(\mathcal{F}', \phi, u, p, \Delta) \subseteq m^2(\mathcal{F} + \mathcal{F}', \phi, u, p, \Delta).$$

Proof. Let $x = (x_{i,j}) \in m^2(\mathcal{F}, \phi, u, p, \Delta) \cap m^2(\mathcal{F}', \phi, u, p, \Delta)$. Then

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho_1}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f'_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho_2}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. Using (1.1), the result follows from the inequality

$$\begin{aligned}\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[(f_{i,j}+f_{i,j}^{'})\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} \\ \le\ K\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho_1}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} \\ +K\Big\{\sum_{i\in\sigma_1}\sum_{j\in\sigma_2}\Big[f_{i,j}^{'}\Big(\frac{u_{i,j}|\Delta x_{i,j}|}{\rho_2}\Big)\Big]^{p_{i,j}}\Big\}^{\frac{1}{p_{i,j}}} \\ < \infty.\end{aligned}$$

Therefore, $x \in m^2(\mathcal{F}+\mathcal{F}^{'}, \phi, u, p, \Delta)$. This completes the proof. □

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J&K, India.
E-mail address: `seemajamwal8@gmail.com`
E-mail address: `kuldipraj68@gmail.com`